حلول السلسلة رقم 04

Exercice 01 :

$$\boxed{\text{N.B.} \quad \cos\theta = \pm\sqrt{1-\sin^2\theta}\ , \quad \sin\theta = \pm\sqrt{1-\cos^2\theta}}$$

Soit $x \in [-1, 1]$.

* On pose $\theta = \text{Arccos}\, x \in [0, \pi]$ $\left(\text{car Arccos} : [-1,1] \longrightarrow [0,\pi]\right)$

Donc $\sin \text{Arccos}\, x = \sin\theta = +\sqrt{1-\cos^2\theta}$ $\left(\text{car } \sin\theta \geq 0, \forall \theta \in [0,\pi]\right)$

$$= \sqrt{1-(\cancel{\cos \text{Arccos}}\, x)^2} = \boxed{\sqrt{1-x^2}}$$

* On pose $\theta = \text{Arcsin}\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\left(\text{car Arcsin} : [-1,1] \longrightarrow \left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$

Donc $\cos \text{arcsin}\, x = \cos\theta = +\sqrt{1-\sin^2\theta}$ $\left(\text{car } \cos\theta \geq 0, \forall \theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$

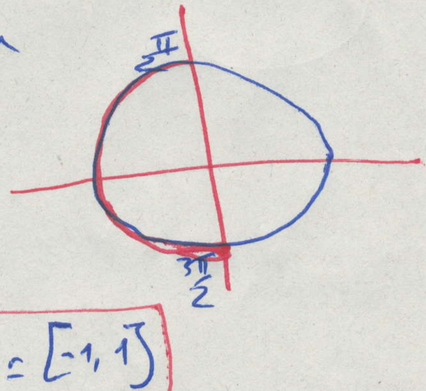
$$= \sqrt{1-(\sin \text{Arcsin}\, x)^2} = \boxed{\sqrt{1-x^2}}$$

Exercice 02 :

$$f : D \longrightarrow [-1, 1] \quad , \quad D = \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$
$$x \longmapsto f(x) = \sin x$$

1) On a f est continue et dérivable et on a

$$\forall x \in D : \quad f'(x) = \cos x \leq 0$$



Comme f est continue et décroissante sur D alors elle est bijective de $\boxed{D \text{ vers } f(D) = \left[f\left(\frac{3\pi}{2}\right), f\left(\frac{\pi}{2}\right)\right] = [-1, 1]}$

Donc f^{-1} existe. Calculons f^{-1}.

On pose $f^{-1}(y) = x$, $y \in [-1, 1]$ et $x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$

D'où $y = f(x) = \sin x = -\sin(x - \pi)$

Donc $-y = \sin(x - \pi)$, $x - \pi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

D'où $\text{Arcsin}(-y) = x - \pi$ et donc $\boxed{x = -\text{Arcsin}\, y + \pi}$

D'où $\boxed{f^{-1}(y) = -\text{Arcsin}\, y + \pi}$

2) $f(x) = \cos x$, $D = [2022\pi, 2023\pi]$

De la même manière f^{-1} existe et $f^{-1} : [-1, 1] \longrightarrow D$

On pose $f^{-1}(y) = x$

D'où $y = f(x) = \cos x$, $x \in [2022\pi, 2023\pi]$

$$y = \cos(x - 2022\pi), \quad x - 2022\pi \in [0, \pi]$$

Donc $\text{Arccos}\, y = \cancel{\text{Arccos}}\, \cancel{\cos}(x - 2022\pi) = x - 2022\pi$

D'où $x = \text{Arccos}\, y + 2022\pi$

Donc $\boxed{f^{-1}(y) = \text{Arccos}\, y + 2022\pi}$

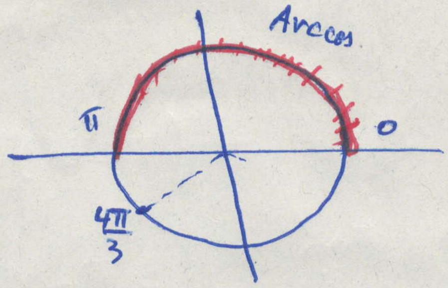
Exercice 03 : $\boxed{\text{N.B. } \text{Arcsin} \sin x = x \Leftrightarrow x \in [-\frac{\pi}{2}, \frac{\pi}{2}]}$

$\boxed{\text{Arccos} \cos x = x \Leftrightarrow x \in [0, \pi]}$

1) On a $\cancel{\text{Arcsin}}\, \cancel{\sin} \frac{\pi}{3} = \frac{\pi}{3}$ car $\frac{\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cancel{\text{Arccos}}\, \cancel{\cos} \frac{\pi}{3} = \frac{\pi}{3}$, $\frac{\pi}{3} \in [0, \pi]$

$\text{Arccos} \sin \frac{\pi}{3} = \text{Arccos} \cos(-\frac{\pi}{3} + \frac{\pi}{2}) = \cancel{\text{Arccos}}\, \cancel{\cos} \frac{\pi}{6} = \frac{\pi}{6}$, car $\frac{\pi}{6} \in [0, \pi]$

2) • $\text{Arccos}\left(\cos\frac{4\pi}{3}\right) = \,??$



On a $\pi < \frac{4\pi}{3} < 2\pi$

D'où $-\pi < \frac{4\pi}{3} - 2\pi < 0$ et donc $\boxed{0 < 2\pi - \frac{4\pi}{3} < \pi}$

et on a $\text{Arccos}\cos\frac{4\pi}{3} = \cancel{\text{Arccos}}\,\cancel{\cos}\left(2\pi - \frac{4\pi}{3}\right) = 2\pi - \frac{4\pi}{3} = \frac{2\pi}{3}$

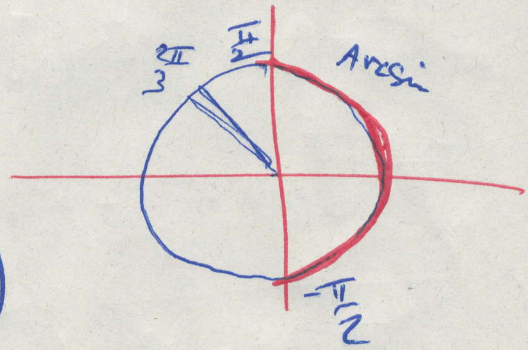
• $\text{Arccos}\cos\frac{7\pi}{3} = \text{Arccos}\cos\left(\frac{7\pi}{3} - 2\pi\right)$

$= \cancel{\text{Arccos}}\,\cancel{\cos}\left(\frac{\pi}{3}\right) = \frac{\pi}{3} \quad \left(\text{car } \frac{\pi}{3} \in [0, \pi]\right)$

• $\text{arcsin}\sin\frac{2\pi}{3} = \text{Arcsin}\left(-\sin\left(\frac{2\pi}{3} - \pi\right)\right)$

$= -\cancel{\text{Arcsin}}\,\cancel{\sin}\left(-\frac{\pi}{3}\right)$

$= \frac{\pi}{3} \quad \left(\text{car } -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$



• $\text{Arc}\sin\sin\frac{7\pi}{3} = \cancel{\text{Arcsin}}\,\cancel{\sin}\Big(\underbrace{\frac{7\pi}{3} - 2\pi}_{\in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}\Big)$

$= \frac{7\pi}{3} - 2\pi = \boxed{\frac{\pi}{3}}$

Exercice 04 = $\boxed{\text{Arctg} : \mathbb{R} \longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[}$ est croissante et impaire

$\boxed{\text{Arctg}\,\cancel{tg}\,x = x \iff x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[}$

(*)

• Montrons que $\boxed{\text{Arctg}\,a + \text{Arctg}\,b = \text{Arctg}\frac{a+b}{1-ab}}$, $\boxed{\forall a, b \in \mathbb{R} : ab < 1}$

On pose $x = \text{Arctg}\,a \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, $y = \text{Arctg}\,b \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

•

On a $\circledast \iff x + y = \operatorname{Arctg} \dfrac{a+b}{1-ab}$

et on a $tg(x+y) = \dfrac{tg x + tg y}{1 - tg x\, tg y} = \dfrac{a+b}{1-ab}$ $\left(tg x = a, tg y = b \right)$

D'où $\operatorname{Arctg}\, tg\, (\underbrace{x+y}_{\overset{??}{\in}]-\frac{\pi}{2}, \frac{\pi}{2}[}) = \operatorname{Arctg} \dfrac{a+b}{1-ab}$

Pour déduire $\circledast$, il suffit de montrer que $\boxed{-\frac{\pi}{2} < x+y < \frac{\pi}{2}}$.

• Si $a = 0$ ou $b = 0$ alors $\circledast$ est evidente.

• Si $a \neq 0$ et $b > 0$, alors :

$$ab < 1 \iff a < \frac{1}{b} \iff tg x < \frac{1}{tg y} = tg\left(\frac{\pi}{2} - y\right)$$

$$\iff \operatorname{Ar}\cancel{\text{ctg}}\ \cancel{tg}\ x < \cancel{\operatorname{Arctg}}\ \cancel{tg}\left(\frac{\pi}{2} - y\right)$$

$$\iff x < \frac{\pi}{2} - y \quad \left(\text{car } 0 < y = \operatorname{Arctg} b < \frac{\pi}{2} \right.$$

$$\Rightarrow \boxed{0 < \frac{\pi}{2} - y < \frac{\pi}{2}}$$

$$\iff \boxed{x + y < \frac{\pi}{2}}$$

et on a $-\dfrac{\pi}{2} < x$ et $y > 0$, Donc $\boxed{-\dfrac{\pi}{2} < x + y}$

Donc $\boxed{-\dfrac{\pi}{2} < x + y < \dfrac{\pi}{2}}$

• Si $a \neq 0$ et $b < 0$ alors $ab < 1 \iff -a < -\dfrac{1}{b}$

et de la même manière on montre que $-\dfrac{\pi}{2} < -x - y < \dfrac{\pi}{2}$

et donc $-\dfrac{\pi}{2} < x + y < \dfrac{\pi}{2}$

2) Calculons $\operatorname{Arctg} \frac{1}{2} + \operatorname{Arctg} \frac{1}{3}$. On a $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6} < 1$

Donc $\operatorname{Arctg} \frac{1}{2} + \operatorname{Arctg} \frac{1}{3} = \operatorname{Arctg} \dfrac{\frac{1}{2} + \frac{1}{3}}{1 - 1/6} = \operatorname{Arctg} 1 = \dfrac{\pi}{4}$

EXOS : 1) $C = \sum_{k=0}^{n} \mathrm{ch}\, kx, \quad S = \sum_{k=0}^{n} \mathrm{sh}\, kx$

- Si $x = 0$ alors $C = \sum_{k=0}^{n} \overset{=1}{\mathrm{ch}\,0} = n+1, \quad S = \sum_{k=0}^{n} \overset{=0}{\mathrm{sh}\,0} = 0$

- Si $x \neq 0$. Alors

$$C + S = \sum_{k=0}^{n} (\mathrm{ch}\, kx + \mathrm{sh}\, kx) = \sum_{k=0}^{n} e^{kx} = \frac{1 - e^{(n+1)x}}{1 - e^{x}}$$

$$= \frac{e^{(n+1)\frac{x}{2}}}{e^{\frac{x}{2}}} \; \frac{e^{-(n+1)\frac{x}{2}} - e^{(n+1)\frac{x}{2}}}{e^{-\frac{x}{2}} - e^{\frac{x}{2}}} = \boxed{e^{n\frac{x}{2}} \frac{\mathrm{sh}\,(n+1)x/2}{\mathrm{sh}\, x/2}}$$

$$C - S = \sum_{k=0}^{n} (\mathrm{ch}\, kx - \mathrm{sh}\, kx) = \sum_{k=0}^{n} e^{-kx} = \frac{1 - e^{-(n+1)x}}{1 - e^{-x}}$$

$$= \frac{e^{-(n+1)\frac{x}{2}}}{e^{-\frac{x}{2}}} \; \frac{e^{+\frac{(n+1)x}{2}} - e^{-\frac{n+1}{2}x}}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} = \boxed{e^{-n\frac{x}{2}} \frac{\mathrm{sh}\,(n+1)\frac{x}{2}}{\mathrm{sh}\,\frac{x}{2}}}$$

D'où $C = \frac{1}{2}\left[(C+S) + (C-S)\right] = \dfrac{\mathrm{ch}\, n\frac{x}{2}\; \mathrm{sh}\,(n+1)\frac{x}{2}}{\mathrm{sh}\,\frac{x}{2}}$

$S = \frac{1}{2}\left((C+S) - (C-S)\right) = \dfrac{\mathrm{sh}\,\frac{nx}{2}\; \mathrm{sh}\,(n+1)\frac{x}{2}}{\mathrm{sh}\,\frac{x}{2}}$

2) $\cancel{\mathrm{sh}\, a\, \mathrm{sh}\, b = \mathrm{ch}(a+b) - \mathrm{ch}(a-b)}$

$$\boxed{\mathrm{sh}\, a\, \mathrm{ch}\, b = \frac{1}{2}\left[\mathrm{sh}(a+b) - \mathrm{sh}(a-b)\right]}$$

$$\mathrm{sh}\, x\, \mathrm{ch}\, 2x = \frac{1}{2}\left[\mathrm{sh}\,(x+2x) - \mathrm{sh}\,(x-2x)\right] = \boxed{\frac{1}{2}\mathrm{sh}\, 3x + \frac{1}{2}\mathrm{sh}\, x}$$

$$\mathrm{ch}\, a\, \mathrm{ch}\, b = \frac{1}{2}\left[\mathrm{ch}(a+b) + \mathrm{ch}(a-b)\right]$$

$$\mathrm{ch}\, x\, \mathrm{ch}^2 x = \mathrm{ch}\, x\; \frac{1}{2}\left(\mathrm{ch}(x+x) + \mathrm{ch}\,(x-x)\right) = \frac{1}{2}\mathrm{ch}\, x\, \mathrm{ch}\, 2x + \frac{1}{2}\mathrm{ch}\, x$$

$$= \frac{1}{2} \times \frac{1}{2}\left[\mathrm{ch}(x+2x) + \mathrm{ch}(x-2x)\right] + \frac{1}{2}\mathrm{ch}\, x = \boxed{\frac{1}{4}\mathrm{ch}\, 3x + \frac{1}{4}\mathrm{ch}\, x}$$

3) $sh(2x) = sh(x+x) = sh\,x\,ch\,x + ch\,x\,sh\,x = 2\,sh\,x\,ch\,x$

On a $ch\,x = \dfrac{sh\,2x}{2sh\,x}$, $ch\,\dfrac{x}{2^n} = \dfrac{sh\,\frac{x}{2^{n-1}}}{2ch\,\frac{x}{2^n}}$

Donc

$$P = \frac{sh\,2x}{\cancel{2\,sh\,x}} \times \frac{\cancel{sh\,x}}{\cancel{2sh\,\frac{x}{2}}} \times \frac{\cancel{sh\,\frac{x}{2}}}{\cancel{2\,sh\,\frac{x}{4}}} \times \cdots \times \frac{\cancel{sh\,\frac{x}{2^{n-1}}}}{2sh\,\frac{x}{2^n}}$$

$$= \boxed{\frac{1}{2^n}\,\frac{sh\,2x}{sh\,\frac{x}{2^n}}}$$

Exo6 : $f(x) = Argch\sqrt{1+x^2}$ $\left(Argch : [1, +\infty[\to [0, +\infty[\right)$

1) $x \in D_f \iff \sqrt{1+x^2} \geq 1$

On a $x^2 \geq 0$ donc $\sqrt{1+x^2} \geq 1$, $\forall x \in \mathbb{R}$

Donc $\boxed{D_f = \mathbb{R}}$

2) $Argch\,ch\,t = t \iff t \geq 0$

D'où $*$ Si $t \geq 0$ alors $Argch\,ch\,t = t$

$*$ Si $t \leq 0$ alors $Argch\,ch\,t = Argch\,ch(-t) = -t$

Donc $\boxed{Argch\,t = |t|}$

3) On pose ~~t = st~~ $x = sh\,t$ (Donc $t = Argsh\,x$)

Alors $f(x) = argch\,\underbrace{\sqrt{1+sh^2 t}}_{ch^2 t} = Argch\sqrt{ch^2 t} = Argch\,\overset{\geq 0}{ch\,t}$

$= |t| = |Argsh\,x| = Argsh\,|x|$

4) $f'(x) = \begin{cases} (\text{Argsh}\, x)' & \text{si } x > 0 \\ (-\text{Argsh}\, x)' & \text{si } x < 0 \end{cases}$

$= \begin{cases} \dfrac{1}{\sqrt{1+x^2}} & \text{si } x \geq 0 \\ \dfrac{-1}{\sqrt{1+x^2}} & \text{si } x < 0 \end{cases}$

5) dérivabilité en 0 :

$$\lim_{x \overset{>}{\to} 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\overbrace{\text{Argsh}\, x}^{= t} - 0}{x}$$

$$= \lim_{t \overset{>}{\to} 0} \frac{t - 0}{\text{sh}\, t - 0} = \frac{1}{\text{sh}'0} = 1$$

$$\lim_{x \overset{<}{\to} 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \overset{<}{\to} 0} \frac{-\text{Argsh}\, \cancel{t}}{\cancel{t}} = \lim \frac{-t}{\text{sh}\, t}$$

$$= -\frac{1}{\text{sh}'0} = -1$$

Donc f n'est pas dérivable en 0.