Université Mohamed Boudiaf -M'sila-
Faculté de mathématiques et de l'informatique
Département de mathématiques
3ème année licence
Equation de la physique mathématique

TD 1: EDP d'ordre1-Méthodes des caractéristiques
Corrigé

Exercice 01:
1- Equation linéaire, non homogène,- d'ordre 1
2- Equation quasilinéaire, homogène, d'ordre 1
3- Equation linéaire, homogène, d'ordre 1
4- Equation non linéaire, non homogène, d'ordre 1
5- Equation quasi linéaire, non homogène, d'ordre 2
6- Equation non linéaire, homogène, d'ordre 1

Exercice 02:
1) Pour la fonction $u(x,y) = x^2 - y^2$
on a $u_{xx} = 2,\ u_{yy} = -2$, d'où $u_{xx} + u_{yy} = 0$
2) Pour la fonction $v(x,y) = e^x sin(y)$
on a $v_{xx} = e^x sin(y),\ v_{yy} = -e^x sin(y)$, d'où $v_{xx} + v_{yy} = 0$

Les fonctions $u + v$ et $2u + 3v$, sont aussi des solutions de l'équation de Laplace, car cette dernière est linéaire et homogène.

Exercice 03: 1) $\dfrac{\partial u^2}{\partial x^2} = 0$ 2) $\dfrac{\partial u^2}{\partial y^2} + u = 0$, où $u = u(x,y)$

1) $\dfrac{\partial u^2}{\partial x^2} = 0 \Rightarrow \dfrac{\partial u}{\partial x} = c_1(y)$
$\Rightarrow u(x,y) = c_1(y)x + c_2(y)$

2) $\dfrac{\partial u^2}{\partial y^2} + u = 0$

Exercice 04:

$$u_{xx} - u_{yy} = 0$$

Indication: utiliser les nouvelles variables $\zeta(x;y) = x + y,\ \eta(x,y) = x - y$
Par la règle de chaîne on a

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \zeta} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u_x}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u_x}{\partial \zeta} + \frac{\partial u_x}{\partial \eta} = \frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial^2 u}{\partial \zeta \partial \eta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \zeta} - \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial u_y}{\partial \zeta}\frac{\partial \zeta}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u_y}{\partial \zeta} - \frac{\partial u_y}{\partial \eta} = \frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial^2 u}{\partial \zeta \partial \eta}$$

d'où: $u_{xx} - u_{yy} = -4\dfrac{\partial^2 u}{\partial\zeta\partial\eta} = 0 \Rightarrow u(\zeta,\eta) = F(\zeta) + G(\eta)$

$$\Rightarrow u(x,y) = F(x+y) + G(x-y)$$

Exercice 05

Le système caractéristique est

$$\begin{cases} \dfrac{\partial x}{\partial s} = 1 \\ \dfrac{\partial y}{\partial s} = -2 \\ \dfrac{\partial u}{\partial s} = 0 \end{cases} \Rightarrow \begin{cases} x = s + c_1 \\ y = -2s + c_2 \\ u = c_3 \end{cases}$$

de la condition initiale

$$\begin{cases} x(0,\tau) = \tau \\ y(0,\tau) = 0 \\ u(0,\tau) = \sin(\tau) \end{cases} \Rightarrow \begin{cases} x(s,\tau) = s + \tau \\ y(s,\tau) = -2s \\ u(s,\tau) = \sin(\tau) \end{cases}$$

puisque $J = \begin{vmatrix} \frac{dx}{dS} & \frac{dx}{d\tau} \\ \frac{dy}{dS} & \frac{dy}{d\tau} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = 2 \neq 0$

donc, on peut écrire s et τ en fonction de x et y

$$\begin{cases} s = -\frac{1}{2}y \\ \tau = \frac{y+2x}{2} \end{cases}$$

d'où

$$u(x,y) = \sin(\frac{y+2x}{2})$$

Exercice 6

1) les courbes caractéristiques de (E)

$$\begin{cases} \dfrac{\partial x}{\partial s} = x \\ \dfrac{\partial y}{\partial s} = y \\ \dfrac{\partial u}{\partial s} = -u \end{cases} \Rightarrow \begin{cases} x = c_1 e^s \\ y = c_2 e^s \\ u = c3e^{-s} \end{cases}$$

2) la solution de (E) vérifiant $u = 1$ sur le cercle $x^2 + y^2 = 1$

$$\begin{cases} x(0,\tau) = \tau \\ y(0,\tau) = \sqrt{1-\tau^2} \\ u(0,\tau) = 1 \end{cases} \Rightarrow \begin{cases} x(s,\tau) = \tau e^s \\ y(s,\tau) = \sqrt{1-\tau^2}e^s \\ u(s,\tau) = e^{-s} \end{cases}$$

on a

$$\begin{cases} x = \tau e^s \\ y = \sqrt{1-\tau^2}e^s \end{cases} \Rightarrow \begin{cases} \tau = xe^{-s} \\ y^2 = (1-\tau^2)e^{2s} \end{cases}$$
$$\Rightarrow \begin{cases} \tau = xe^{-s} \\ y^2 = (1 - x^2e{-}^{2s})e^{2s} \end{cases}$$
$$\Rightarrow \begin{cases} \tau = xe^{-s} \\ e^{-s} = \frac{1}{\sqrt{x^2+y^2}} \end{cases}$$

d'où

$$u(x,y) = \frac{1}{\sqrt{x^2+y^2}}$$

Exercice 07

Le système caractéristique est

$$\left\{\begin{array}{l} \dfrac{\partial t}{\partial s} = 1 \\ \dfrac{\partial x}{\partial s} = 1 \\ \dfrac{\partial u}{\partial s} = u^2 \end{array}\right. \quad \Rightarrow \left\{\begin{array}{c} t = s + c_1 \\ x = s + c_2 \\ -\frac{1}{u} = s + c_3 \end{array}\right.$$

de la condition initiale

$$\left\{\begin{array}{c} t(0,\tau) = 0 = c_1 \\ x(0,\tau) = \tau = c_2 \\ u(0,\tau) = \ln(1+\tau^2) = -\frac{1}{c_3} \end{array}\right. \quad \Rightarrow \left\{\begin{array}{c} t(s,\tau) = s \\ x(s,\tau) = s + \tau \\ u(s,\tau) = -\frac{1}{s - \frac{1}{\ln(1+\tau^2)}} \end{array}\right.$$

puisque $J = \begin{vmatrix} \frac{dt}{dS} & \frac{dt}{d\tau} \\ \frac{dx}{dS} & \frac{dx}{d\tau} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0$

donc, on peut écrire s et τ en fonction de t et x

$$\left\{\begin{array}{c} s = t \\ \tau = x - t \end{array}\right.$$

d'où

$$u(x,y) = -\frac{1}{t - \frac{1}{\ln(1+(x-t)^2)}}$$