Exercice 01(6 pts)
Le système caractéristique est

$$\begin{cases} \dfrac{\partial x}{\partial s} = x \\ \dfrac{\partial t}{\partial s} = 1 \\ \dfrac{\partial u}{\partial s} = 1 \end{cases} \quad ...(0.75) \quad \Rightarrow \begin{cases} x = c_1 e^s \\ t = s + c_2 \\ u = s + c_3 \end{cases} \quad ...(0.75)$$

de la condition initiale

$$\begin{cases} x(0,\tau) = 1 \\ t(0,\tau) = \tau \\ u(0,\tau) = 2\cosh(\tau) \end{cases} \quad ...(0.75) \quad \Rightarrow \begin{cases} x(s,\tau) = e^s \\ y(s,\tau) = s + \tau \\ u(s,\tau) = s + 2\cosh(\tau) \end{cases} \quad ...(0.75)$$

puisque $J = \begin{vmatrix} \frac{dx}{dS} & \frac{dx}{d\tau} \\ \frac{dy}{dS} & \frac{dy}{d\tau} \end{vmatrix} = \begin{vmatrix} e^s & 0 \\ 1 & 1 \end{vmatrix} =_{(0,\tau)} 1 \neq 0$ ...(1)
donc, on peut écrire s et τ en fonction de x et y

$$\begin{cases} s = \ln(x) \\ \tau = t - \ln(x) \end{cases} \quad ...(1)$$

d'où

$$u(x,y) = \ln(x) + 2\cosh(t - \ln(x))...(1)$$

Exercice 02(10 pts)
1)

$$u_{xx} - 2\sin(x)u_{xy} - \cos^2(x)u_{yy} - \cos(x)u_y = 0, \quad y > 0$$

$\Delta = (-2\sin(x))^2 - 4(1)(-\cos^2(x)) = 4 > 0, ...(0.5)$
alors l'équation est hyperbolique sur tout $R \times R^+$...(0.5)
L'équation caractéristique est

$$\frac{dy}{dx} = \frac{B \pm \sqrt{\Delta}}{2A} = \frac{-2\sin(x) \pm 2}{2(1)} = -\sin(x) \pm 1...(0.5) \Rightarrow \cos(x) \pm x - y = c...(0.5)$$

Les courbes caractéristiques associées est donc

$$\begin{cases} \zeta(x,y) = \cos(x) + x - y \\ \eta(x,y) = \cos(x) - x - y \end{cases} \quad ...(0.5)$$

Par la règle de chaînes, on a

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = (1 - \sin(x))\frac{\partial u}{\partial \zeta} - (1 + \sin(x))\frac{\partial u}{\partial \eta}...(0.5)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u_x}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial x} = -\cos(x)\frac{\partial u}{\partial \zeta} + (1 - \sin(x))\frac{\partial u_\zeta}{\partial x} - \cos(x)\frac{\partial u}{\partial \eta} - (1 + \sin(x))\frac{\partial u_\eta}{\partial x}$$

$= -\cos(x)(\frac{\partial u}{\partial \zeta}+\frac{\partial u}{\partial \eta})+(1-\sin(x))^2 u_{\zeta\zeta}-2(1-\sin^2(x))u_{\zeta\eta}+(1+\sin(x))^2 u_{\eta\eta}$...(0.75)

$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}=-\frac{\partial u}{\partial \zeta}-\frac{\partial u}{\partial \eta}$...(0.5)

$\frac{\partial^2 u}{\partial y^2}=\frac{\partial u_y}{\partial \zeta}\frac{\partial \zeta}{\partial y}+\frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y}=\frac{\partial^2 u}{\partial \zeta^2}+2\frac{\partial^2 u}{\partial \zeta\partial \eta}+\frac{\partial^2 u}{\partial \eta^2}$...(0.5)

$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u_y}{\partial x}=\frac{\partial u_y}{\partial \zeta}\frac{\partial \zeta}{\partial x}+\frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial x}=-(1-\sin(x))(\frac{\partial^2 u}{\partial \zeta^2}+\frac{\partial^2 u}{\partial \zeta\partial \eta})+(1+\sin(x))(\frac{\partial^2 u}{\partial \zeta\partial \eta}+\frac{\partial^2 u}{\partial \eta^2})$ ...(0.5)

d'où la forme canonique de cette équation est donnée par

$$\frac{\partial^2 u}{\partial \zeta\partial \eta}=0...(1)$$

4) On a $u_{\zeta\eta}=0 \Rightarrow u(\zeta,\eta)=F(\zeta)+G(\eta)$...(0.5) où $F,G\in C^2(R)$, Par conséquent, la solution générale de l'équation est

$$u(x,y)=F(\cos(x)+x-y)+G(\cos(x)-x-y)...(0.5)$$

5) On a $u(0,y)=F(1-y)+G(1-y)=e^y$...(0.5) et $u_x(0,y)=(1-\sin(0))F'(1-y)+(-1-\sin(0))G'(1-y)=4$...(0.5), on obtient

$$\begin{cases} F'+G'=-e^y \\ F'-G'=4 \end{cases} ...(0.25) \quad \Rightarrow \begin{cases} 2F'=4-e^y \\ F'-G'=4 \end{cases}$$
$$\Rightarrow \begin{cases} F'=\dfrac{4-e^y}{2}=2-\frac{1}{2}e^y \\ G'=\dfrac{4-e^y}{2}-4=-2-\frac{1}{2}e^y \end{cases} ...(0.25)$$
$$\Rightarrow \begin{cases} F(1-y)=2y-\frac{1}{2}e^y \\ G(1-y)=-2y-\frac{1}{2}e^y \end{cases} ...(0.5)$$

on effectuant le changement de variable ($a=1-y \Rightarrow y=1-a$)...(0.25)
on obtient

$$\begin{cases} F(a)=2(1-a)-\frac{1}{2}e^{(1-a)} \\ G(a)=-2(1-a)-\frac{1}{2}e^{(1-a)} \end{cases} ...(0.5)$$

avec $c_1+c_2=0$,d'où la solution est

$$\begin{aligned} u(x,y) &= 2(1-\cos(x)+x-y)-\frac{1}{2}e^{(1-\cos(x)+x-y)}-2(1-\cos(x)-x-y)-\frac{1}{2}e^{(1-\cos(x)-x-y)} \\ &= 4x-\frac{1}{2}(e^{(1-\cos(x)+x-y)}+e^{(1-\cos(x)-x-y)}).....(0.5) \end{aligned}$$

$$\begin{cases} u_{xx}+u_{yy}=0,\ (x,y)\in\Omega, \\ u(0,y)=u(1,y)=0,\ 0\le y\le 1, \\ u(x,0)=x(x^2-2x+1),\ 0\le x\le 1, \\ u(x,1)=1-\cos(\pi x),\ 0\le x\le 1. \end{cases}$$

Déterminer les réels α et β tels que:

$$\alpha \leq u(x,y) \leq \beta,\ \forall (x,y) \in \bar{\Omega}.$$

Exercice 03 (10 pts)

D'après le principe du maximum la solution u atteint le max et le min sur la frontière de Ω ; ce qui implique

$$\max_{(x,y)\in\bar{\Omega}} u(x,y) \leq u(x,y) \leq \min_{(x,y)\in\bar{\Omega}} u(x,y),\ \forall (x,y) \in \bar{\Omega}....(1)$$

$$\min_{x\in[0,1]} u(x,y) = \min_{x\in[0,1]} \{0, x(x^2-2x+1), 1-\cos(\pi x)\} = 0...(0.5)$$

$$\max_{x\in[0,1]} u(x,y) = \max_{x\in[0,1]} \{0, x(x^2-2x+1), 1-\cos(\pi x)\} = \max_{x\in[0,1]} \{0, x(x^2-2x+1), 1\}...(0.5)$$

Calcule de $\max_{x\in[0,1]}\{x^3-2x^2+x\}$
$f(x) = x^3 - 2x^2 + x$
$f\prime(x) = 3x^2 - 4x + 1 = 0 \Rightarrow x = 1/3....(1)$
$f(1/3) = 4/27....(0.5)$
d'où

$$\max_{x\in[0,1]} u(x,y) = \max_{x\in[0,1]} \{0, x(x^2-2x+1), 1-\cos(\pi x)\} = 4/27...(0.5)$$