Université de M'sila
Faculté MI
Département de Math.
Master EDP, (M1)

Analyse fonctionnelle pour EDP 2022/2023

Examen Final

La clarté des raisonnements et la qualité de la rédaction seront prises en compte à la correction. Veuillez justifier vos réponses

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6,5

Exercice 1. On considère $L^2(0,1)$ équipé par son produit scalaire $(f,g) := \int_0^1 f(t)g(t)dt$. On défini $T : L^2(0,1) \to L^2(0,1)$ par

$$(Tf)(x) = \int_0^1 x^2 y f(y) dy, \qquad f \in L^2(0,1), \quad x \in [0,1].$$

1. Démontrer que T est continue et calculer sa norme. (2,5) (1)
2. Quelle est la dimension du $R(T)$. (1)
3. Est-ce-que T est compact ? (1)
4. Même question pour $T \circ T$. ($T \circ T$ est la composition de T et T). (1)

Dans la suite H est un espace de Hilbert.

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Exercice 2. Soit A un opérateur linéaire continue dans H.

a) Rappeler la définition de A^* (l'adjoint de A). (1)

b) Calculer $(A^2)^*$, i.e. l'adjoint de $A^2 = A \circ A$. (1,5)

d) Démontrer que (A est compact) $\Longleftrightarrow$ (A^*A est compact).
(Il faut démontrer le deux implications $\Longrightarrow$ et $\Longleftarrow$). (1,5) (1,5)

4

Exercice 3. Les question suivantes sont indépendantes.

a) Supposons que $y_n \to y$ (fortement dans H) et $x_n \rightharpoonup x$ (faiblement dans H). Démontrer qu'on a (2)

$$(y_n, x_n)_H \to (y, x)_H.$$

b) Supposons que $y_n \rightharpoonup y$ (faiblement dans H) et $x_n \rightharpoonup x$ (faiblement dans H). Est-ce-qu'on a (1)

$$(y_n, x_n)_H \to (y, x)_H, \quad \text{(Justifier votre réponse).}$$ (1)

5

Exercice 4. On considère l'espace $E = C([0,1])$, muni de la norme $\|\cdot\|_\infty$, et la suite d'applications linéaires

$$A_n : E \to \mathbb{R}, \quad A_n\varphi = n\int_0^{1/n} \varphi(s)\,ds, \quad \text{pour } n = 1, 2, 3, \cdots$$

1. Rappeler la définition de la convergence *-faible sur E^*. (1)
2. Démontrer que la suite A_n est *-faible convergente vers l'application $A : \varphi \mapsto A\varphi = \varphi(0)$. (1)
   (Aide: Utiliser la definition de continuité de φ).
3. Considérer la suite de fonctions

$$f_n(t) = \begin{cases} 1, & t \in \left[\frac{1}{5n}, \frac{2}{5n}\right], \\ 0, & \text{si non.} \end{cases} \qquad n = 1, 2, 3, \cdots$$

   Calculer $|A_n(f_n) - A(f_n)|$. (2)

4. Est-ce-que la suite A_n est convergente en norme ? (1)

A. Sengouga.

Functional Analysis for PDE 2022/2023

Exam Correction.

$\boxed{\text{Exercice 1}}$

1) $|Tf(x)| = \left| \int_0^1 x^2 y f(y)\,dy \right| \leq x^2 \int_0^1 |y f(y)|\,dy.$

(by C-S) $\leq x^2 \left(\int_0^1 y^2\,dy \right)^{1/2} \left(\int_0^1 |f(y)|^2\,dy \right)^{1/2}$

$$\int_0^1 |Tf(x)|^2\,dx \leq \int_0^1 x^4\,dx \cdot \int_0^1 y^2\,dy \cdot \int_0^1 |f(y)|^2\,dy$$

$$\|Tf\|_{L^2(0,1)} \leq \|x^2\|_{L^2} \cdot \|y\|_{L^2} \cdot \|f(y)\|_{L^2}$$

we have $\|y\|_{L^2(0,1)} = \left(\int_0^1 y^2\,dy \right)^{1/2} = \left(\frac{y^3}{3} \Big]_0^1 \right)^{1/2} = \frac{1}{\sqrt{3}}$

$\|x^2\|_{L^2(0,1)} = \left(\int_0^1 x^4\,dx \right)^{1/2} = \left(\frac{x^5}{5} \Big]_0^1 \right)^{1/2} = \frac{1}{\sqrt{5}}$

then $\|Tf\|_{L^2(0,1)} \leq \frac{1}{\sqrt{15}} \cdot \|f\|_{L^2(0,1)}.$

This chows that T ~~f~~ is continuous from $L^2(0,1) \to L^2(0,1)$.

and $\|T\| \leq \frac{1}{\sqrt{15}}$. — ①

To show that $\|T\| = \frac{1}{\sqrt{15}}$, consider $f(y) = \frac{y}{\|y\|_{L^2(0,1)}} = \sqrt{3}\,y$

we have $\|f\|_{L^2(0,1)} = 1$ and

$$Tf(y) = T(\sqrt{3}\,y) = x^2 \int_0^1 \sqrt{3}\,y^2\,dy = \sqrt{3}\,x^2 \cdot \frac{y^3}{3}\Big]_0^1 = \frac{x^2}{\sqrt{3}}.$$

$$\|Tf\|_{L^2(0,1)} = \left\| \frac{x^2}{\sqrt{3}} \right\|_{L^2(0,1)} = \left(\int_0^1 \frac{x^4}{3}\,dx \right)^{1/2} = \left(\frac{x^5}{15} \Big]_0^1 \right)^{1/2} = \frac{1}{\sqrt{15}}.$$

It follows that $\|T\| = \sup_{\|g\|=1} \|Tg\|_{L^2(0,1)} \geq \|Tf\|_{L^2(0,1)} = \frac{1}{\sqrt{15}}$ — ②

① and ② implies that $\|T\| = \frac{1}{\sqrt{15}}$.

2) we have $Tf(x) = x^2 \cdot \left(\int_0^1 y f(y)\, dy \right) = \text{Constant} \cdot x^2$

so $R(T)$ is composed of polynoms of the form $c \cdot x^2$. This means that $R(T)$ is spaned by the unique vector x^2, i.e.

$\dim R(T) = 1$.

3) Yes. T is compact as an application of finit rang.

4) Yes. $T \circ T$ is also compact because it has also a finit rang. we can also see $T \circ T$ as a composition of two compact operator, so it is compact.

$\boxed{\text{Exercice 2}}$

a) The adjoint of an operator $A : H \to H$ is the operator $A^* : H \to H$ defined by $(Ax, y)_H = (x, A^* y)_H$, $\forall x, y \in H$.

b) $(A^2 x, y) = (A \circ A x, y) = (Ax, A^* y) = (x, A^* \circ A^* y) = (x, (A^*)^2 y)$

thus $(A^2)^* = (A^*)^2$

($\rightharpoonup$ weak conv. / $\rightarrow$ strong conv.)

c) ($\Rightarrow$) Assume that A is compact,

$\forall \{x_n\} \subset H$, $x_n \rightharpoonup 0 \Rightarrow Ax_n \to 0$ in H.

Since A^* is continuous (as the adjoint of a continuous operator)

we have $\boxed{A^* \circ A x_n \to 0}$ in H.

this means that $A^* A$ is compact.

($\Leftarrow$) Assume that $A^* A$ is compact and let $\{x_n\} \subset H$, $x_n \rightharpoonup 0$.

we have $\|Ax_n\|^2 = (Ax_n, Ax_n) = (x_n, A^* A x_n) \leq \|x_n\| \cdot \|A^* A x_n\|$

since $x_n \rightharpoonup 0$, then $\|x_n\|$ is bounded, i.e. $\|x_n\| \leq M$, $\forall n \in \mathbb{N}$.

since $A^* A$ is compact, then $\|A^* A x_n\| \to 0$

then we have $\|Ax_n\|^2 \leq M \cdot \|A^* A x_n\| \to 0$

this shows the $Ax_n \to 0$ and A is compact.

Exercise 3

a) $$|(y_n, x_n) - (y, x)| = |(y_n, x_n) - (y, x_n) + (y, x_n) - (y, x)|$$

$$\leq |(y_n - y, x_n)| + |(y, x_n - x)|$$

$$\leq \|x_n\| \, \|y_n - y\| + |(y, x_n - x)|$$

since $x_n \to x$, then $\|x_n\|$ is bounded $\|x_n\| \leq M$.

Then

$$|(y_n, x_n) - (y, x)| \leq M \|y_n - y\| + |(y, x_n - x)| \longrightarrow 0$$

since $\|y_n - y\| \to 0$ and $x_n - x \to 0$.

b) ~~If~~ $(y_n \rightharpoonup y$ and $x_n \rightharpoonup x) \not\Rightarrow (y_n, x_n) \longrightarrow (y, x)$.

For a counter example see Example 6.54 in Salsa's book.

Exercise 4

1) $A_n \xrightarrow{*} A$ in E^* $\iff$ $\langle A_n, x\rangle \to \langle A, x\rangle$, $\forall x \in E$.

2) Since $f \in C([0,1])$, i.e. f is a continuous function, then

$\forall \varepsilon > 0$, $\exists \delta \geq 0$, $|s - 0| \leq \delta \Rightarrow |f(s) - f(0)| \leq \varepsilon$.

If we choose $\delta = \frac{1}{n}$, $n \in \mathbb{N}^*$.

$\forall \varepsilon > 0$, $\exists n_0 > 0$, $0 \leq s \leq \frac{1}{n} \Rightarrow |f(s) - f(0)| \leq \varepsilon$.

then $$A_n(f) - f(0) = n\int_0^{1/n} f(s)\,ds - n\int_0^{1/n} f(0)\,ds \quad \Big/ \quad n\int_0^{1/n} dx = 1$$

$$= n\int_0^{1/n} f(s) - f(0)\,ds$$

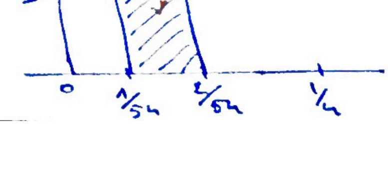
$$|A_n(f) - f(0)| \leq n\int_0^{1/n} |f(s) - f(0)|\,ds \leq \varepsilon . n \int_0^{1/n} ds = \varepsilon$$

So we have $\forall \varepsilon > 0$, $\exists n_0$, $n > n_0$, $|L_n(f) - f(0)| \leq \varepsilon$

this means that $A_n(f) \to f(0) = A(f)$.

3) $|A_n f_n - A f_n| = |A_n f_n - f_n(0)|$, $f_n(0) = 0$ by definition.

$$= n\left|\int_0^{1/n} f_n(s)\,ds\right| = \frac{1}{5}$$



4) If A_n converges in norm, it will converge to A. But we have

$$\|f_n\| = \sup_{s\in[0,1]} |f_n(s)| = 1 \quad \text{and} \quad |A_n(f_n) - A(f)| = \frac{1}{5}$$

Then $$\|A_n - A\| = \sup_{\|g\|=1} |A_n(g) - A(g)| \geq \frac{1}{5},$$

i.e. $\|A_n - A\| \not\to 0$. $\square$

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