Chapter 1

Sampling and Z Transform

1. 1 Introduction

Digital signal processing by means of a computer requires that the signal be converted into a sequence of numbers. This conversion is obtained after the execution of the following three steps:

1. Sampling the analog signal where their values are usually taken at regular intervals of time.
2. Conditioning or quantification the sampled signal to valid values for the digital system for example between, 0V and 5V.
3. Assigning a binary code for each conditioned value. Generally, 8, 10 and 16 binary bits can be used for this conversion.

Because the purpose of sampling of signals is to treat and transmit signal information, the question of choosing the sampling frequency is very important:

- If the sampling frequency is too low, the acquisition will be too spaced and if the signal has pertinent details between two capture positions, these will be lost.
- The higher the sampling frequency, the more expensive in processing power, transmission capacity and storage space.

In this Chapter, we state at first the sampling theorem which indicates the minimum value of the sampling frequency. Then, the Z transform is presented as a function of discrete signals which is defined by a series as an approximation of de 'Laplace' transform. Because the inverse Z transform can not be obtained by a specific formula, but it is simply calculated from specific methods. Finally, some useful Matlab codes are presented to calculate Z transform (ZT) and inverse Z transform (IZT).

1. 2 Sampling theorem

Sampling involves taking the values of a given signal at defined time intervals, usually regular. It produces a series of discrete values called samples. To select a sufficient sampling frequency, sufficient sample knowledge is required to calculate the signal value at all intermediate points. Mr. Claude Shannon showed how this was possible, knowing the bandwidth of the information encoded in the signal to be transmitted. The sampling theorem indicates that if all the frequencies of the signal are less than half the sampling frequency, it can be perfectly reconstructed. In general, frequencies above half the sampling frequency introduce a spectral overlay also called Spectrum aliasing.

In the following, we want to study the sampling of any deterministic signal g(t) with a limited band of maximum frequency f_m (see Fig. 1.1). Then, the Fourier transform (FT), $G(f) = 0$ for $|f| > f_m$ [1].

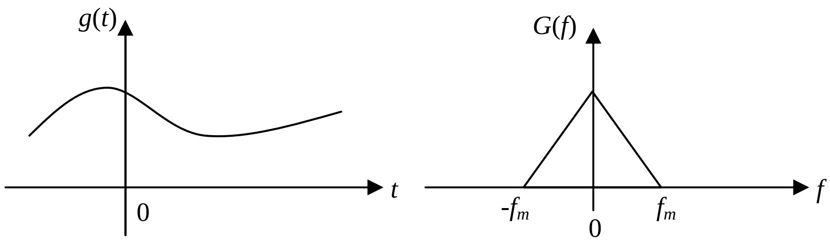


Fig. 1. 1 Analog signal with a limited band spectrum.

Ideally, sampling is obtained by multiplying the signal $g(t)$ by a train of pulses $p(t)$ (see Fig. 1.2). Let $g_s(t) = g(t).p(t)$

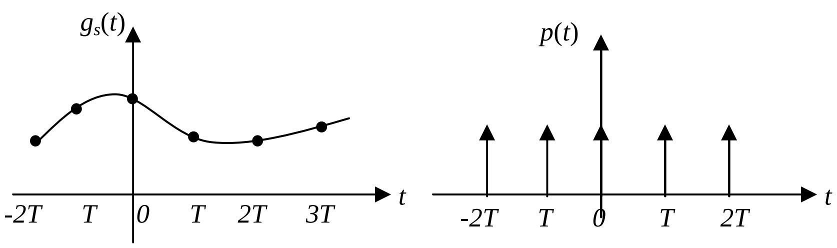


Fig. 1. 2 Sampling with a train of pulses $p(t)=\delta(t)$.

As $p(t)$ is a periodic function, it can be represented by the following Fourier series [1]:

$$p(t) = \sum_{n=-\infty}^{+\infty} C_n e^{j2\pi n\frac{t}{T}} \tag{1.1}$$

where $C_n = \frac{1}{T}\int_{-T/2}^{T/2} p(t)e^{-j2\pi n\frac{t}{T}}dt = \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-j2\pi n\frac{t}{T}}dt = \frac{1}{T}$

$\frac{1}{T}$ is the fundamental frequency of the periodic signal $p(t)$, which also represents the sampling frequency, $f_s = \frac{1}{T}Hz$. The $g_s(t)$ function becomes

$$g_s(t) = f_s g(t)\sum_{n=-\infty}^{+\infty} e^{j2\pi n f_s} \tag{1.2}$$

The FT of $g_s(t)$ is

$$\begin{aligned} G_s(f) &= \int_{-\infty}^{+\infty}[f_s g(t)\sum_{n=-\infty}^{+\infty} e^{j2\pi n f_s}]e^{-j2\pi f t}dt \\ &= f_s \sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(t)e^{-j2\pi(f-nf_s)t}dt \\ &= f_s \sum_{n=-\infty}^{+\infty} G(f-nf_s) \end{aligned} \tag{1.3}$$

Fig. 1.3 represents the plot of equation (1.3) for $f_s > 2f_m$

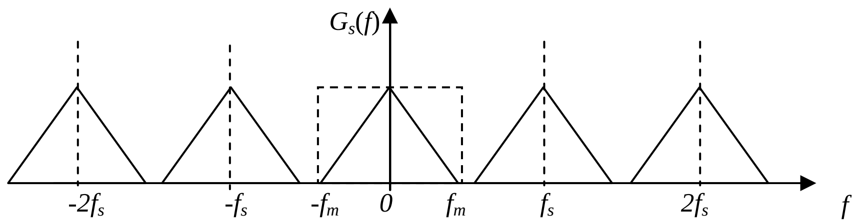


Fig. 1. 3 Sampled signal spectrum for $f_s > 2f_m$

It is observed that the original signal $g(t)$ can be reconstructed by applying the low-pass filter as shown by the dotted line. It can also be noted that the sampling rate becomes at least $2f_m$. Then the minimum sampling frequency is $f_s = 2f_m$ which is called the Nyquist rate. Sampling at a frequency below the Nyquist rate (see Fig. 1.4) produces the aliasing error and the original signal cannot be reconstructed [1, 2].

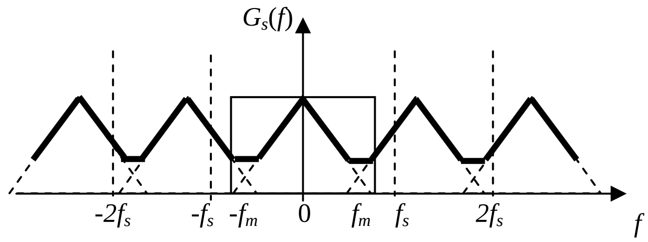


Fig. 1. 4 Sampled signal spectrum for $f_s < 2f_m$

1. 3 The Z transform

The Z-transform (ZT) is a mathematical tool used for signal processing. It is the discrete equivalent of the "Laplace transform". For instance, the ZT is used to design numerical filters with infinite impulse response (IIR), finite impulse response (FIR) and automatically to model dynamic systems in a discrete manner [2, 3].

1. 3. 1 Definition

We consider an analog signal defined by $x_a(t)$. This function defined on $\Re$ is causal if: for any $t < 0$, $x_a(t) = 0$ (see Fig. 1. 5). By taking only the values of the images of x at integers numbers, we build a numerical sequence, called a sampled signal of $x(t)$. This sequence $x(n)$ is called a discrete or digital causal signal as opposed to the $x_a(t)$ function which is a continuous or analog causal signal. If we call T the period between two measurements of the sample, we can construct the discrete causal signal: $x(n) = x_a(nT)$, $n = 0, 1, 2, \ldots+\infty$.

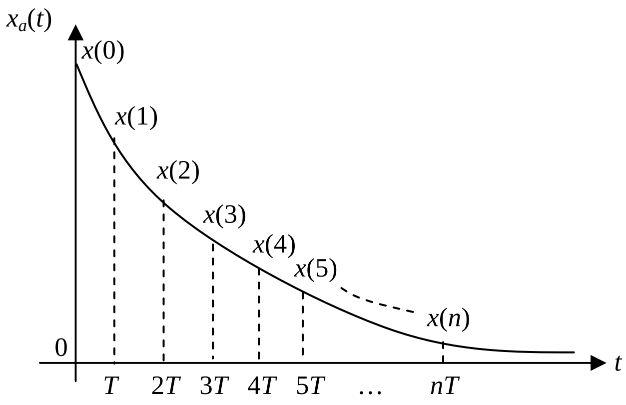


Fig. 1. 5 Discrete causal signal

Now we want to define for discrete causal signals a transformation analogous to the Laplace transform for continuous causal signals. Let a continuous causal signal $x_a(t)$ and the associated sampled discrete signal: For any $n \in N$, $x(n) = x_a(n)$. The Laplace transform of the continuous signal $x_a(t)$ is: $X(s) = \int_0^{+\infty} x_a(t)e^{-st}dt$. The associated sampled discrete signal is: $x_a(t)e^{-st}$ for any $n \in N$, $x(n)\ e^{-sn}$. This is to approximate the area defined by the integral $X(s)$ by a series, i.e., $\int_0^{+\infty} x_a(t)e^{-st}dt \approx \sum_{n=0}^{+\infty} x(n)e^{-sn}$ with $T = 1$ (see Fig. 1. 6).

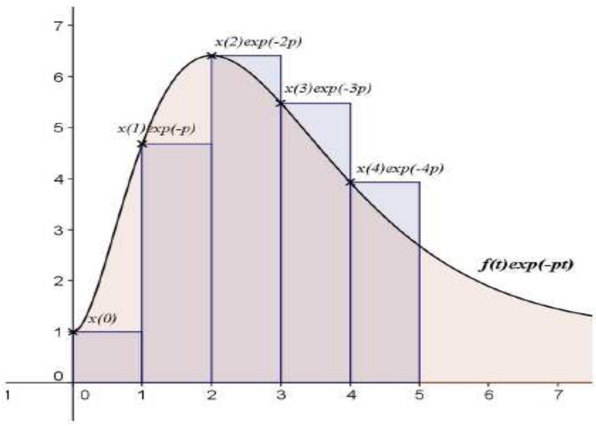


Fig. 1. 6 Approximated area defined by the integral of $X(s)$

Taking, $z = e^{j\omega} = e^{s}$, the ZT, $X(z)$ of the sequence $x(n)$ is defined by the following relation [2-4]:

$$X(z) = \sum_{n=0}^{+\infty} x(n) z^{-n} \tag{1.4}$$

Where Z is a complex variable and the function $X(z)$ has a convergence domain which is usually a ring with radiuses R_1 and R_2 centered on the origin (see Fig. 1. 7). That is, $X(z)$ is defined for $R_1 < z < R_2$. R_1 and R_2 values depend on the sequence $x(n)$.

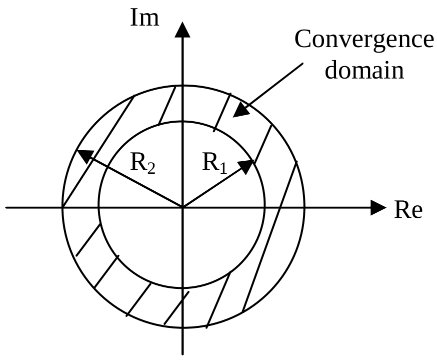


Fig. 1. 7 Convergence domain of $X(z)$

If $x(n)$ represents the sequence of samples of a signal taken at nT, $n = 0, 1, \ldots, +\infty$, then a discrete Fourier transform (DFT) of this sequence is written:

$$S(f) = \sum_{n=0}^{\infty} x(n) e^{-j2\pi f nT} \tag{1.5}$$

For $z = e^{j2\pi fT}$, the Z transform of $x(n)$ coincides with its FT. That is, we can carry out the frequency analysis of the discrete signal by the Z transform.

1. 3. 2 Properties

In order to reduce the calculation complexity of the ZT, the following properties can be used:

(i) Linearity:

$$Z\{a_1 x_1(n) \pm a_2 x_2(n)\} = a_1 Z\{x_1(n)\} \pm a_2 Z\{x_2(n)\} \tag{1.6}$$

(ii) Time delay:

$$Z\{x(n-k)\} = z^{-k} Z\{x(n)\} \tag{1.7}$$

(iii) Time forward:

$$Z\{x(n+k)\} = z^{k}\left[Z\{x(n)\} - \sum_{j=0}^{k-1} x(j) z^{-j} \right] \tag{1.8}$$

(iv) Convolution:

$$Z\{x * y\} = Z\{x\} Z\{y\} \tag{1.9}$$

where $x(n) * y(n) = \sum_{k=-\infty}^{+\infty} x(n-k) y(k)$

(v) Exponential multiplication :

$$Z\{a^n x(n)\} = X(z/a) \tag{1.10}$$

(vi) Initial value theorem :

$$x(0) = \lim_{z \to +\infty} X(z) \tag{1.11}$$

(vii) Final value theorem :

If the poles of $(z-1)X(z)$ are inside the unitary circle

$$\lim_{n \to +\infty} x(n) = \lim_{z \to 1, |z|<1} (z-1) X(z) \tag{1.12}$$

(viii) Multiplication by an evolutional variable:

$$Z\{n^k x(n)\} = \underbrace{-z\frac{d}{dz}\left\{ -z\frac{d}{dz}\left[-z\frac{d}{dz} ......X(z) \right.\right\}}_{k\ times} \tag{1.13}$$

In addition with the above properties, we can exploit the results of the geometric series, $\sum_{0}^{+\infty} z^n$ to calculate the ZT of some sequences. Hence

$$\sum_{n=0}^{+\infty} z^n = \frac{1}{1-z}, \ |z| < 1 \tag{1.14}$$

So, $\sum_{n=0}^{+\infty} z^n$ converges if $|z|<1$, i.e., R_1=0 and R_2=1.

Example 1. 1:

Calculate the ZT of the following sequence:

x(n) = (*n*+2)^2

Solution :

For, $y(n)= n^2$, we apply the multiplication by an evolutional variable property.

$$Y(z) = -z\frac{d\left(\frac{z}{(z-1)^2}\right)}{dz} = \frac{z(z+1)}{(z-1)^3}, \text{ where } Z\{n\} = \frac{z}{(z-1)^2} \text{ is used here (see Table. 1. 1)}$$

Then, we utilize also the shift property. Hence

$$X(z) = Z\{y(n+2)\} = z^2\left(Y(z) - \sum_{j=0}^{2-1} y(j)z^{-j}\right)$$

$$= z^2\left(\frac{z(z+1)}{(z-1)^3} - (0^2 z^{-0} + 1^2 z^{-1})\right)$$

$$= \frac{z^3(z+1)}{(z-1)^3} - z$$

$$= \frac{z(4z^2-3z+1)}{(z-1)^3}$$

1. 4 Inverse Z transform

In general, the inverse Z transform (IZT) of *X*(*z*) is performed by one of the four following methods:

(i) Table method:

The ZT of some functions mostly used in signal processing are presented in Table. 1. 1. The latter can be consulted to get the corresponding sequence *x*(*n*) from usual ZT functions.

Table. 1. 1 Habitual ZT of some sequences

***x*(*n*)**	***X*(*z*)**	**Convergence domain**
$\delta(n)$	1	C
$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\|>1$

$nu(n)$	$\dfrac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
$n^2u(n)$	$\dfrac{z^{-1}+z^{-2}}{(1-z^{-1})^3}$	$\lvert z\rvert > 1$
$n^3u(n)$	$\dfrac{z^{-1}+4z^{-2}+z^{-3}}{(1-z^{-1})^4}$	$\lvert z\rvert > 1$
$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > a$
$n^2a^n u(n)$	$\dfrac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\lvert z\rvert > a$
$\cos(\omega_0 n)u(n)$	$\dfrac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)u(n)$	$\dfrac{1-z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
$a^n\cos(\omega_0 n)u(n)$	$\dfrac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert > a$
$a^n\sin(\omega_0 n)u(n)$	$\dfrac{1-az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert > a$

(ii) Partial-fraction expansion method:

Here, $X(z)$ is decomposed into a sum of several simple rational functions as follows:

$$X(z) = X_1(z) + X_2(z) + X_3(z) + \ldots$$

The IZT may be obtained using the transformations in Table. 1. 1. Hence

$$x(n) = x_1(n) + x_2(n) + x_3(n) + \ldots$$

(iii) Power-series method:

$X(z)$ is transformed into a finite series (z^{-k}, $k = 1, \ldots, n$) of power using polynomial division.

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots. + x(k)z^{-k}$$

The general term $x(n)$ can be obtained from $x(0), x(1), x(2), \ldots., x(k)$

(vi) Inversion formula method (residues method):

This method is based on the calculation of the following contour integral:

$$x(n) = \frac{1}{2\pi j}\oint_C X(Z)Z^{n-1}dZ \tag{1.15}$$

Where C is a closed contour containing all the singular points or poles of $X(Z)$. The residue theorem is often used to determine $x(n)$ given by

$$x(n) = \sum_{Z_k = \text{poles of } Z^{n-1}X(Z)} \operatorname{Re}s\left\{X(Z)Z^{n-1}\right\}_{Z=Z_k} \tag{1.16}$$

The residue at a pole $z = a$ of order q of the function is formulated as [4]:

$$\operatorname{Re}s_a^q = \lim_{z\to a}\frac{1}{(q-1)!}\frac{d^{q-1}}{dz^{q-1}}\left[X(Z)Z^{n-1}(z-a)^q\right] \tag{1.17}$$

Example. 1. 2:

Calculate the IZT of the sequence using the partial-fraction expansion method:

$$X(z) = \frac{z^2}{6z^2 - 5z + 1}$$

Solution :

We can write

$$X(z) = \frac{z^2}{6z^2 - 5z + 1} = \frac{z^2}{6(z-1/2)(z-1/3)}$$

$$X(z) = \frac{z}{6}\left(\frac{A}{z-1/2} + \frac{B}{z-1/3}\right)$$

Coefficients A and B are simply given by

$$A = \lim_{z\to 1/2}\frac{z}{z-1/3} = 3$$

and

$$B = \lim_{z\to 1/3}\frac{z}{z-1/2} = -2$$

$$X(z) = \frac{1}{2}\frac{z}{z-1/2} - \frac{1}{3}\frac{z}{z-1/3}$$

If we consult Table. 1. 1, we get

$$x(n) = \left(\frac{1}{2}\right)^{n+1}u(n) - \left(\frac{1}{3}\right)^{n+1}u(n)$$

1. 5 Some useful definitions

(i) Causal system:

A causal signal is defined by $x(t)$ for $t > 0$. In other words, a signal is said to be causal if that signal is null when $t < 0$. A non causal signal is not null when at least a $t < 0$ value. This definition applies to both discrete and continuous systems. The system is causal if and only if its transfer function is clean. This means that the output at a given moment is not influenced by the future of the input. For example, the system $y(n) = x(n+1)$, where x denotes the input and y denotes the output, is not causal because the value of the output signal at the time n does not depend on the value of the input signal at a later time (n+1).

(ii) Stability of discrete systems:

A discrete transfer function time system $H(z)$ is stable if and only if its poles, p_1, p_2, … p_n, that is, the roots of the denominator of $H(z) = \frac{(z - z_1)(z - z_2)\ldots(z - z_m)}{(z - p_1)(z - p_2)\ldots(z - p_n)}$, are all located in the unit circle. Since, $z = e^{pT}$, we have therefore $|z| < 1$ if and only if $\Re(p) < 0$.

(iii) Systems with a minimum phase

For a discrete system, assuming that the transfer function $H(z)$ is rational, this system has a minimum phase if and only if all poles and zeros of $H(z)$ are inside the unit disk (circle).

1. 6 Related Matlab codes [5]

(i) Calculation of ZT and IZT:

The Matlab library offers the functions «ztrans» and «iztrans» for the calculation of ZT and IZT. To highlight these, the following instructions are given for two examples, $x(n) = \frac{1}{4^n}$ and

$$X(z) = \frac{6 - 9z^{-1}}{1 - 2.5z^{-1} + z^{-2}}$$

```
>> syms z n
>> ztrans(1/4^n)
ans =z/(z - 1/4)

>> syms z n
>> iztrans((6-9*z^-1)/(1-2.5*z^-1+z^-2))
ans =2*2^n + 4*(1/2)^n
```

(ii) Calculation of power series:

The "deconv" function is used to execute the polynomial division requested by the method into a power series. Given the transfer function $H(z)$ by

$$H(z)=\frac{b_0+b_1z^{-1}+...+b_nz^{-n}}{a_0+a_1z^{-1}+...+a_nz^{-m}}$$

Corresponding Matlab instruction is :

```
>> [q,r]=deconv(b,a)
```

For the case of $H(z)=\dfrac{1+2z^{-1}+z^{-2}}{1-z^{-1}+0.356z^{-2}}$, the Matlab code is as follows

```
>> b=[1 2 1];
>> a=[1 -1 0.356];
>> n=5
n = 5;
>> b=[b zeros(1,n-1)];
>> [x,r]=deconv(b,a);
>> disp(x)
    1.0000    3.0000    3.6440    2.5760    1.2787
```

(iii) Calculation of rational fractions:

The residue function is used to find the coefficients and poles of the partial fractions of the function $X(z)$.

$$H(z)=\frac{b_0+b_1z^{-1}+...+b_nz^{-n}}{a_0+a_1z^{-1}+...+a_nz^{-m}}=\frac{r_0}{1-p_1z^{-1}}+...+\frac{r_n}{1-p_nz^{-1}}+k_1+k_2z^{-1}+...+k_{m-n+1}z^{-(m-n)}$$

The Matlab instruction is :

```
>> [r,p,k]=residuez(b,a);
```

If $H(z)=\dfrac{1+2z^{-1}+z^{-2}}{1-z^{-1}+0.356z^{-2}}$, the Matlab code is

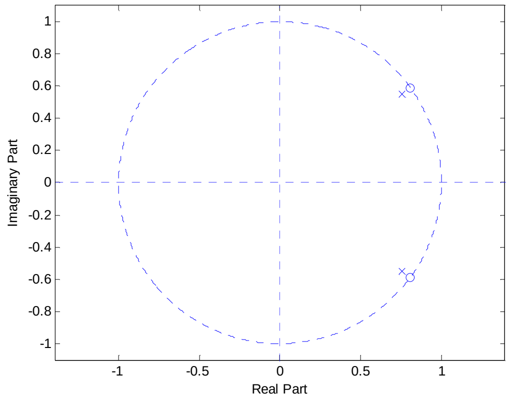
```
>> [r,p,k]=residuez([1,2,1],[1,-1,0.3561])
r =
  -0.9041 - 5.9928i
  -0.9041 + 5.9928i
p =
   0.5000 + 0.3257i
   0.5000 - 0.3257i
k =
    2.8082
```

(iv) Calculation of the pole/zeros diagram:

The command “zplane” calculates and race the location of poles and zeros in the complex plane. The command is: >> zplane(b,a)

If $H(z) = \dfrac{1+1.618z^{-1}+z^{-2}}{1-1.516z^{-1}+0.878z^{-2}}$, we can write

```
>> b=[1 -1.618 1];
>> a=[1 -1.5161 0.878];
>> roots(a)
ans =
   0.7581 + 0.5508i
   0.7581 - 0.5508i
>> roots(b)
ans =
   0.8090 + 0.5878i
   0.8090 - 0.5878i
>> zplane(b,a)
```



(v) Frequency response calculation:

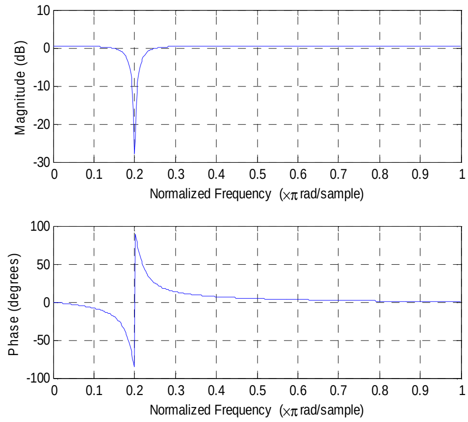
The ‘freqz’ command Matlab calculates and plots the frequency response of $H(z)$. The command is:

>> freqz(b,npt,Fs)

where f_s is the sampling frequency, npt is the number of points of the frequency between 0 and $f_s/2$.

If $H(z) = \dfrac{1+1.618z^{-1}+z^{-2}}{1-1.516z^{-1}+0.878z^{-2}}$, corresponding Matlab code is

```
>> b=[1 -1.618 1];
>> a=[1 -1.5161 0.878];
>> freqz(b,a)
```



Tutorial # 1

Exercise 1 :

1- For each analog signal x_a(t), calculate its Z transform (ZT).

$x_a(t) = u(t)$

$x_a(t) = e^{-at}u(t)$ with $u(t) = \begin{cases} 1 & t > 0 \\ 0 & otherwise \end{cases}$

$x_a(t) = t\, u(t)$

2- Now from every discrete signal $x(n)$, determine its ZT by means of ZT properties.

$x(n) = a^n$

$x(n) = n - 5$

$x(n) = n + 1$

$x(n) = (n + 2)^2$

$x(n) = 2^n n^2$

Exercise 2 :

We have a discrete system with the following differential relationship.

$$y(n) = -0.9y(n-5) + x(n)$$

Where $x(n)$ is a white Gaussian with a unit power.

1- Give the Z response and plot the localization of their poles and zeros.
2- Has this system a minimum phase and what can be said about its stability.
3- Deduce the frequency response of the system.
4- Calculate the system spectrum (i.e., magnitude and phase) and characterize it.

Exercise 3 :

1- Use the table method to calculate the sequence $x(n)$ from the following functions:

$$X(z) = \frac{2z(z+2)}{(z-2)^3} \quad \text{and} \quad X(z) = \frac{z^2 - 2z + 1}{(z-a)^2}$$

2- Determine the IZT of X(z) using the partial-fraction expansion method.

$$X(z) = \frac{z^2 - 2z + 1}{(z-a)^2}$$

3- Use the power-series method to compute $x(n)$ from $X(z)$.

$$X(z) = \frac{(1 - e^{-aT})z}{(z-1)(z - e^{-aT})}$$

4- Use the residues method to compute IZT of the function, $X(z) = \dfrac{6 - 9z^{-1}}{1 - 2.5z^{-1} + z^{-2}}$