Chapter 2

Discrete IIR Filters Design

Chapter content:

2. 1 Introduction

A digital filter is a discrete linear system (DLS) invariant in time and modifying the time and frequency representation of the input signals. As examples, we have:

- Noise reduction for radio, audio and images signals from sensors.
- Modification of certain frequency regions in audio or image signal.
- Limitation to a predefined frequency band.
- Use in telephony for example the DTMF code (Digital Tone Multiple Frequency).

The realization of a discrete linear filter invariant in time requires three steps:

- Specifications of desired system properties (frequency band gaits).
- Approximation of these specifications using a causal discrete system, $H(z)$.
- Implementation of this system using finite arithmetic, $y(n)$ =….

The problem is therefore to determine an appropriate set of specifications on the digital filter. In the case of a low-pass filter, the specifications take the form of a tolerance diagram as shown in Fig. 2. 1. The dotted represents the frequency response of the system (filter) which satisfies the following predefined specifications [1] :

- Pass-band : $1-\delta_1 < |H(e^{j\omega})| < 1+\delta_1 \quad |\omega| < \omega_p$
- Stop-band : $|H(e^{j\omega})| < \delta_2 \quad \omega_s < |\omega| < \pi$
- Cut-off frequency of the pass-band is: ω_p
- Cut-off frequency of the stop-band is: ω_s

The next step is therefore to find a discrete linear system whose response corresponds to the predefined tolerance. IIR filter design will be considered in the following section.

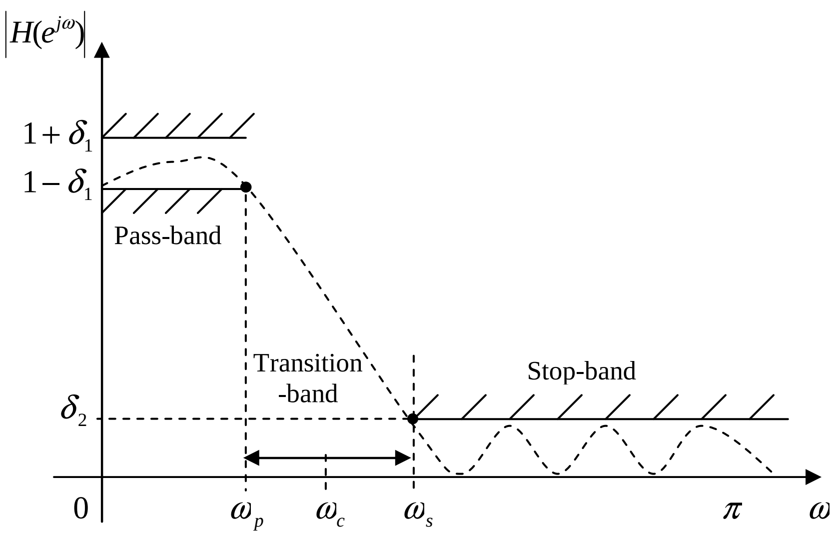


Fig. 2. 1 Tolerance limits for the approximation of low-pass ideal filter.

2. 2 Representation of IIR discrete filters

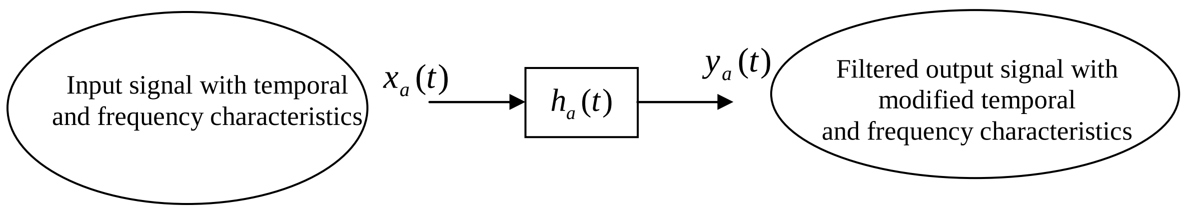
The traditional approach is to use the transformation of an analog filter into a digital filter. This is due to:

- The design of analog filters includes very advanced techniques.
- Many analog design methods have simple formulas.
- In many applications, it is interesting to use digital filters to simulate analog filters.

Consequently, the general TF of the linear analog system is given by.

$$H_a(s) = \frac{\sum_{k=0}^{M} d_k s^k}{\sum_{k=0}^{N} c_k s^k} = \frac{Y_a(s)}{X_a(s)} \tag{2.1}$$

where $x_a(t)$ is the input signal, $X_a(s)$ its 'Laplace' transform (LT), $y_a(t)$ is the output signal and $Y_a(s)$ its LT.



So, $h_a(t)$ is the impulsive response of the system (analog filter). Alternatively, an analog system being $H_a(s)$ as a transfer function can be described by the following differential equation [1]:

$$\sum_{k=0}^{N} c_k \frac{d^k y_a(t)}{d^k t} = \sum_{k=0}^{M} d_k \frac{d^k x_a(t)}{d^k t} \tag{2.2}$$

(2.2) is the inverse 'Laplace' transform (ILT) of (2.1). The rational function of the corresponding system (2.2) for a digital filter has the form

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{Y(z)}{X(z)} \qquad \text{(TLI)} \tag{2.3}$$

The IZT of (2.3) is given by

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad \text{(TZI)} \tag{2.4}$$

The input and the output are correlated by the following convolution product

$$y(n) = \sum_{k=-\infty}^{+\infty} x(k)h(n-k) \tag{2.5}$$

When converting an analog system into a digital system, one must obtain $H(z)$ or $h(n)$ from the digital filter. Then, we have to preserve the frequency response. In this transformation, the essential properties of the frequency response must be retained. Also, the stability of the analog filter ensures the stability of the digital filter. Depending on the form of the transfer function of the IIR system, four classes of its structure can be considered: direct form, cascaded form, parallel form and transposed form. For example, the direct structure is considered if the rational function of the IIR system is written in the following form:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \tag{2.6a}$$

The input and the output of the system are related by

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k) \tag{2.6b}$$

The differential equation can be expressed by the structure of Fig. 2. 2.

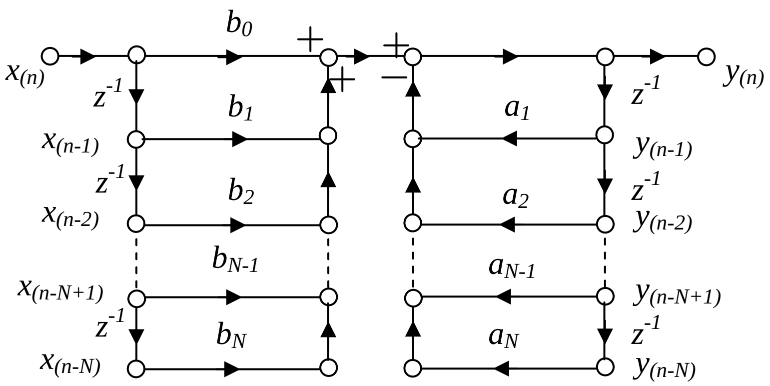


Fig. 2. 2 Structure of IIR filter given by (2.6) [1]

2. 3 Design of discrete IIR filters

In this section, three different methods for the design of RII filters are presented.

2. 3. 1 Impulse invariance method

This method selects the digital filter pulse response as equal samples spaced from the analog filter pulse response, $h(n) = h_a(nT)$, where T is the sampling period. It has been shown that the ZT of $h(n)$ is related to LT by the following equation [1]:

$$H(z)\big|_{z=e^{sT}} = \frac{1}{T}\sum_{k=-\infty}^{+\infty} H_a\left(s + j\frac{2\pi}{T}k\right) \tag{2.7}$$

The frequency response (FR) of the digital filter is also related to the FR of the analog filter by (Shannon's theorem)

$$H\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} H_a\left(j\frac{\omega}{T} + j\frac{2\pi}{T}k\right) \tag{2.8}$$

From the sampling theorem, we have

$$\begin{cases} H_a(j\Omega) = 0, & |\Omega| \geq \dfrac{\pi}{T} \\ H\left(e^{j\omega}\right) = \dfrac{1}{T} H_a\left(j\dfrac{\omega}{T}\right), & |\omega| \leq \pi \end{cases} \tag{2.9}$$

Unfortunately, there is no analog low-pass filter that is limited. This phenomenon is called "Aliasing" or "overlap", Fig. 2.3.

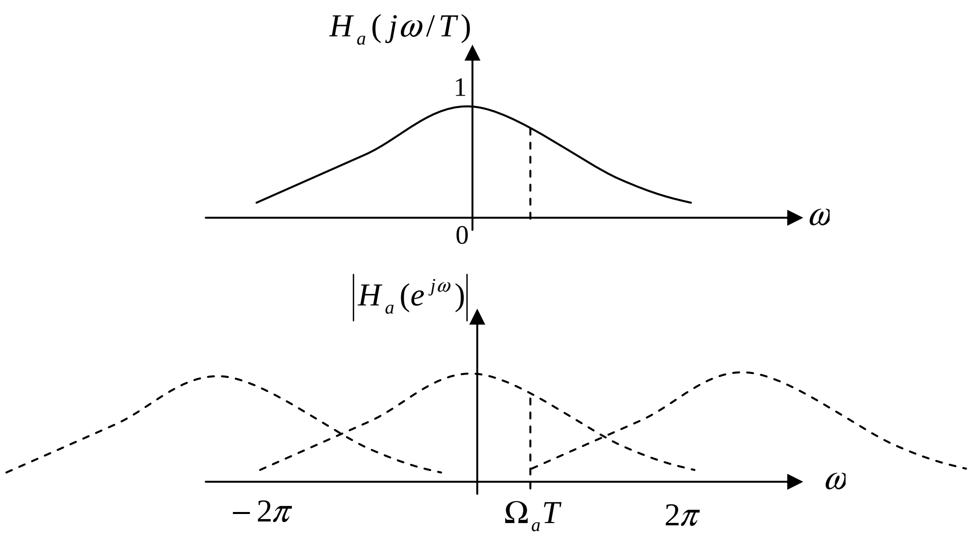


Fig. 2.3 Graphical representation of the overlap effect in the technique of design by impulsive invariance.

Taking the TF from the following analog filter:

$$H_a(s) = \frac{\sum_{k=1}^{M} d_k s^k}{\sum_{k=1}^{N} c_k s^k} = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \tag{2.10}$$

The corresponding impulse responses are

$$\begin{cases} h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t) \\ h(n) = h_a(nT) = \sum_{k=1}^{N} A_k e^{s_k nT} u(n) = \sum_{k=1}^{N} A_k \left(e^{s_k T}\right)^n u(n) \end{cases} \tag{2.11}$$

Calculating the ZT of the 2nd equation of (2.11), we obtain

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}} \tag{2.12}$$

By identification of (2.12) and (2.10), we notice that the pole in the S plane corresponds to the pole $z_k = e^{s_k T}$ in the Z plane.

Example 2. 1:

We want to use the previous design approach in order to obtain a discrete filter from the analog filter, $H_a(s) = \dfrac{s+a}{(s+a)^2 + b^2}$

Solution:

This analog filter is written as a sum of partial-fractions as

$$H_a(s) = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2}$$

So, the poles calculated to be

$s_1 = a$+jb and s_2=a-jb

$$A_1 = \lim_{s \to s_1} (s - s_1) H_a(s) = 0.5$$

$$A_2 = \lim_{s \to s_2} (s - s_2) H_a(s) = 0.5$$

$$H_a(s) = \frac{1/2}{s + a + jb} + \frac{1/2}{s + a - jb}$$

Applying (2.12), we calculate the transfer function of the digital filter as follows

$$H(z) = \frac{1/2}{1 - e^{-aT} e^{-jbT} z^{-1}} + \frac{1/2}{1 - e^{-aT} e^{jbT} z^{-1}} = \frac{1 - \left(e^{-aT} \cos bT\right) z^{-1}}{\left(1 - e^{-aT} e^{-jbT} z^{-1}\right)\left(1 - e^{-aT} e^{jbT} z^{-1}\right)}$$

$H(z)$ has two zeros at $z = 0$ and $z = e^{-aT} \cos bT$. The corresponding FR is shown in Fig. 2.4.

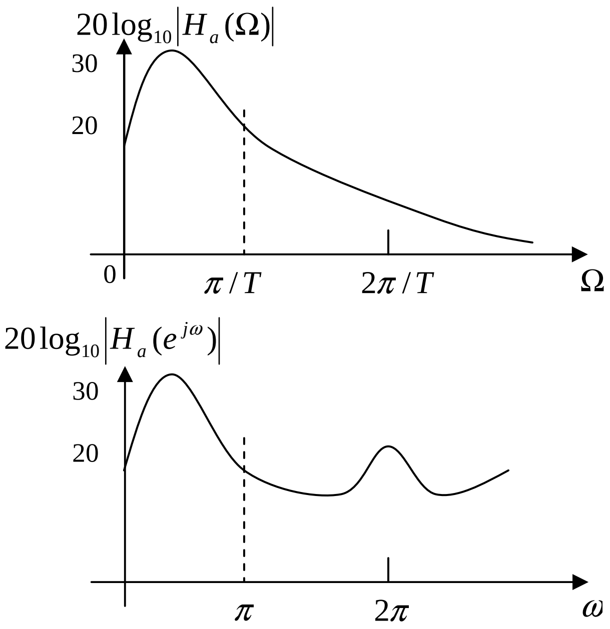


Fig. 2. 4 Frequency response of analog system of 2nd order.

2. 3. 2 Differential equation method

Another approach for digital filter design is based on the differential equation solution which uses an approximation of the derivatives of (2.2) by finite differences [1].

$$\left.\frac{dy_a(t)}{dt}\right|_{t=nT} \rightarrow \nabla^{(1)}[y(n)] = \frac{y(n) - y(n-1)}{T} \tag{2.13}$$

where $y(n) = y_a(nT)$

The approximations of higher-order derivatives are obtained by repeating the application of (2.13).

$$\left.\frac{d^k y_a(t)}{dt^k}\right|_{t=nT} = \left.\frac{d}{dt}\left(\frac{d^{k-1} y_a(t)}{dt^{k-1}}\right)\right|_{t=nT} \rightarrow \nabla^{(k)}[y(n)] = \nabla^{(1)}\left[\nabla^{(k-1)} y(n)\right] \tag{2.14}$$

Using $\nabla^{[0]}[y(n)] = y(n)$ and (2.13), (2.2) becomes

$$\sum_{k=0}^{N} c_k \frac{d^k y_a(t)}{d^k t} = \sum_{k=0}^{M} d_k \frac{d^k x_a(t)}{d^k t} \rightarrow \sum_{k=0}^{N} c_k \nabla^{(k)}[y(n)] = \sum_{k=0}^{M} d_k \nabla^{(k)}[x(n)] \tag{2.15}$$

Where $y(n) = y_a(nT)$ and $x(n) = x_a(nT)$.

for $\begin{cases} Z\left[\nabla^{(1)}[y(n)]\right] = \left(\dfrac{1-z^{-1}}{T}\right)Y(z) \\ Z\left[\nabla^{(k)}[y(n)]\right] = \left(\dfrac{1-z^{-1}}{T}\right)^k Y(z) \end{cases}$ and $\begin{cases} Z\left[\nabla^{(1)}[x(n)]\right] = \left(\dfrac{1-z^{-1}}{T}\right)X(z) \\ Z\left[\nabla^{(k)}[x(n)]\right] = \left(\dfrac{1-z^{-1}}{T}\right)^k X(z) \end{cases}$

we obtain from the ZT of (2.15)

$$H(z) = \frac{\sum_{k=0}^{M} d_k \left(\frac{1-z^{-1}}{T}\right)^k}{\sum_{k=0}^{N} c_k \left(\frac{1-z^{-1}}{T}\right)^k} \tag{2.16}$$

Comparing (2.16) and (2.1), $H_a(s) = \sum_{k=0}^{M} d_k s^k \Big/ \sum_{k=0}^{N} c_k s^k$, we observe that the digital transfer function can be obtained directly from the analog transfer function by replacing only the variable, $s = \dfrac{1-z^{-1}}{T}$.

2. 3. 3 Bilinear transformation method

We have already seen that we can approximate the derivatives by differences. An alternative procedure is based on the integration of the differential equation and then we use an approximation by the integral. For example, we consider the following differential equation [1]:

$$c_1 \frac{dy_a(t)}{dt} + c_0 y_a(t) = d_0 x(t) \tag{2.17a}$$

The LT of (2.17a) has the form, $H_a(s) = \dfrac{d_0}{c_1 s + c_0}$

We can write $y_a(t)$ as an integral of $y_a^{'}(t)$ as, $y_a(t) = \int_{t_0}^{t} y_a^{'}(t)dt + y_a(t_0)$.

In particular, if $t = nT$ and $t_0 = (n-1)T$, $y_a(nT) = \int_{(n-1)T}^{nT} y_a^{'}(\tau)d\tau + y_a((n-1)T)$. If the integral is approximated by a trapezoidal law, we have

$$y_a(nT) = y_a((n-1)T) + \frac{T}{2}\left[y_a^{'}(nT) + y_a^{'}((n-1)T)\right] \tag{2.17b}$$

Based on (2.17a), we have

$$y_a^{'}(nT) = -\frac{c_0}{c_1} y_a(nT) + \frac{d_0}{c_1} x_a(nT) \tag{2.18}$$

Replacing (2.18) into (2.17b), we obtain

$$y(n) - y(n-1) = \frac{T}{2}\left[-\frac{c_0}{c_1}\big(y(n) + y(n-1)\big) + \frac{d_0}{c_1}\big(x(n) + x(n-1)\big)\right] \tag{2.19}$$

Taking the ZT of (2.19), we find

$$H(z) = \frac{Y(z)}{X(z)} = \frac{d_0}{c_1 \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + c_0} \equiv \frac{d_0}{c_1 s + c_0} \tag{2.20}$$

From (2.20), it is clear that $H(z)$ is obtained from $H_a(s)$ by

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \tag{2.21}$$

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} \tag{2.22}$$

and

$$z = \frac{1+(T/2)s}{1-(T/2)s} \tag{2.23}$$

2. 4 Design examples : analog-digital transformation

In this section, we will consider two templates for the design of digital low-pass filters characterized by desired specifications; the Butterworth analog filter and the Chebyshev analog filter.

2. 4. 1 Butterworth analog filter

The transfer function (squared module) of the Butterworth filter that represents a template of an analog low-pass filter is given by [1, 2]

$$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}} \tag{2.24}$$

where $H_a(j\Omega) = \frac{1}{1+j(\Omega/\Omega_c)^N}$.

For different values of N, we will show that as N increases, the curve becomes sharper (closer to the response of an ideal low-pass filter). This characteristic is shown in Fig. 2.5.

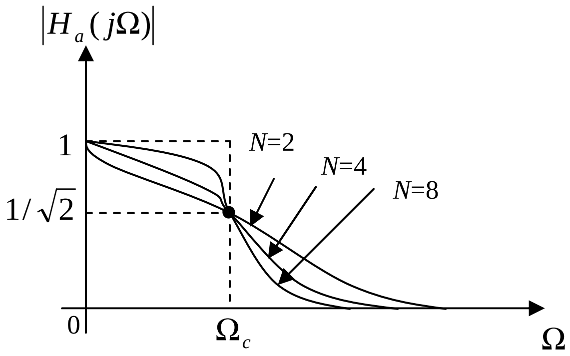


Fig. 2.5 Magnitude of analog Butterworth filter in terms of *N*

From (2.24) and with $s = j\Omega$, we can write

$$H_a(s)H_a(-s) = \frac{1}{1 + j(s/j\Omega_c)^N} \cdot \frac{1}{1 - j(s/j\Omega_c)^N} = \frac{1}{1 + (s/j\Omega_c)^{2N}} \tag{2.25}$$

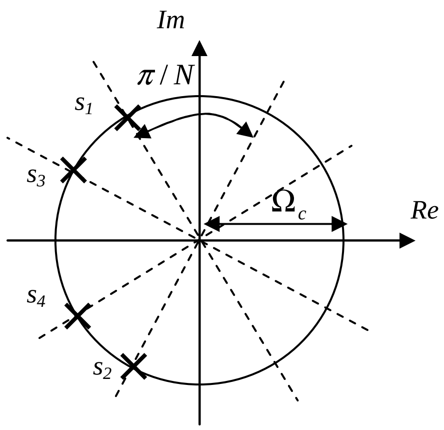
From (2.25), the poles are: $s_k = (-1)^{\frac{1}{2N}}(j\Omega_c)$. Taking Euler transformation, i.e., $-1 = e^{j\pi + j2k\pi}$ and $j = e^{j\pi/2}$, the roots of denominator (poles) of (2.25) become :

$$s_k = \Omega_c e^{j\frac{(N+1+2k)\pi}{2N}}, \quad \text{avec } k=0, 1, \ldots, 2N\text{-}1 \tag{2.26}$$

So, there are 2*N* poles also spaced in angle of π/N on a circle of radius, Ω_c in the S plane. The poles with negative real part ensuring the stability of the filter are given by the following explicit expression (analytic) [2]:

$$\begin{aligned} s_k &= \Omega_c \cos\left(\frac{\pi}{2} + \frac{2k+1}{2N}\pi\right) + j\Omega_c \sin\left(\frac{\pi}{2} + \frac{2k+1}{2N}\pi\right) \\ &= -\Omega_c \sin\left(\frac{2k+1}{2N}\pi\right) + j\Omega_c \cos\left(\frac{2k+1}{2N}\pi\right) \end{aligned}, \quad 0 \le k \le N-1 \tag{2.27}$$

After calculating the angle, π/N we can also directly determine the system poles on the circle in the following S-plane (graphical method) [1, 2]:



After the determination of poles s_k, $H_a(\Omega)$ is written in such a way that $H_a(j\Omega)=1$ for $\Omega=0$.

$$H_a(s)=\prod_{k=0}^{N-1}\frac{-s_k}{s-s_k}=\frac{\Omega_c^N}{(s-s_0)(s-s_1)...} \tag{2.28}$$

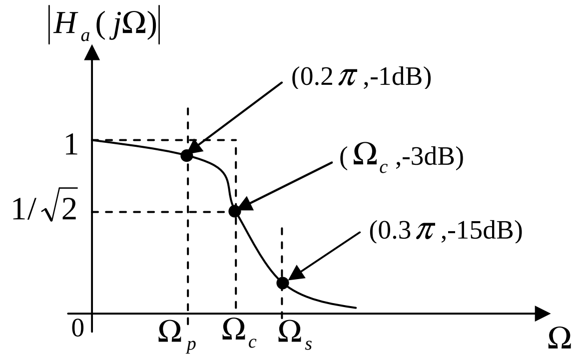
Example 2. 2:

A Butterworth digital low-pass filter can be designed with the following specifications:

- Attenuation in the pass-band: at 1dB we have $\Omega=\Omega_p=0.2\pi$.
- Attenuation in the stop-band: at 15dB we have $\Omega=\Omega_s=0.3\pi$

From the above specifications, we can write

$$\begin{cases}20\log_{10}\left(H_a(j\Omega)\right)\geq -1\\ 20\log_{10}\left(H_a(j\Omega)\right)\leq -15\end{cases}$$



(i) Impulse invariance method:

Based on the module $|H_a(j\Omega)|$ and specifications above, it is easy to write

$$\begin{cases}1+\left(\dfrac{0.2\pi}{\Omega_c}\right)^{2N}=10^{0.1}\\ 1+\left(\dfrac{0.3\pi}{\Omega_c}\right)^{2N}=10^{1.5}\end{cases} \tag{2.29}$$

The solution of these equations leads to N = 5.8858 and Ω_c =0.7047. Since N must be integer, we round up this value, N = 6. The specifications of the two bands are not met together because N = 6. To determine Ω_c, the 1st equation of (2.29) is used for this approach. In this case, Ω_c = 0.7032 where the pass-band specifications are met and the stop-band specifications are exceeded. To determine the poles of the Butterworth filter on a circle of radius Ω_c, the points on the radius circle must first be identified, also spaced at an angle with spacing π/N so that the points are symmetrically located relative to the imaginary axis. In this case, there are 3 pairs of poles marked in bold in the left part of the S-plane (Fig. 2. 6).

$s_{1,2}$= -0.1820 ± j0.6792

$s_{3,4}$= -0.4972 ± j0.4972

$s_{5,6}$= -0.6792 ± j0.1820

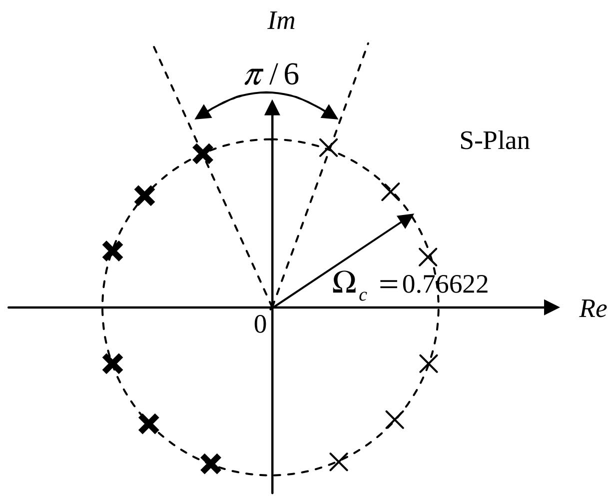


Fig. 2. 6 Location of 2*N* poles in the S- plan (*N* is even).

Hence

$$H_a(s) = \frac{\Omega_c^6}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}$$
$$= \frac{0.12093}{(s^2+0.3640s+0.4945)(s^2+0.9945s+0.4945)(s^2+1.3585s+0.4945)} \qquad (2.30)$$

If we express $H_a(s)$ by a sum of fractions i.e., $H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s-s_k}$ and we utilize the ZT, i.e., $H(z) = \sum_{k=1}^{N} \frac{A_k}{1-e^{s_kT}z^{-1}}$, we obtain

$$H(z) = \frac{0.2871-0.4466z^{-1}}{1-1.2971z^{-1}+0.6949z^{-2}} + \frac{-2.1428-1.1454z^{-1}}{1-1.0691z^{-1}+0.3699z^{-2}} + \frac{1.8558-0.6304z^{-1}}{1-0.9972z^{-1}+0.2570z^{-2}} \qquad (2.31)$$

(ii) Bilinear transformation method :

In this procedure, we must utilize firstly the bilinear transformation as

$$s = j\Omega = \frac{2}{T}\frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \frac{2}{T} j \frac{\sin(\omega/2)}{\cos(\omega/2)} = \frac{2}{T} j\tan(\omega/2)$$

With $T = 1$, we write

$$\begin{cases} 20\log_{10}|H_a(j2\tan(0.2\pi/2)| \geq -1 \\ 20\log_{10}|H_a(j2\tan(0.3\pi/2)| \leq -15 \end{cases} \qquad (2.32)$$

$\Omega_p = j2\tan(0.2\pi/2)$ and $\Omega_s = j2\tan(0.3\pi/2)$

Reformulating (2.32), we find

$$\begin{cases} 1+\left(\dfrac{2\tan(0.1\pi)}{\Omega_c}\right)^{2N} = 10^{0.1} \\ 1+\left(\dfrac{2\tan(0.15\pi)}{\Omega_c}\right)^{2N} = 10^{1.5} \end{cases} \tag{2.33}$$

Solving (2.33), we get

$$N = \frac{1}{2}\frac{\log\left[(10^{1.5}-1)/(10^{0.1}-1)\right]}{\log\left[\tan(0.15\pi)/\tan(0.1\pi)\right]} = 5.3046 \tag{2.34}$$

We take N = 6. Applying the 2nd equation of (2.33) to find $\Omega_c = 0.7662$. This choice is justified by this approach in order to ensure the specifications of the digital filter. In this case, the bandwidth specifications will be exceeded while those of the stopping band are met. There are 12 poles of $|H_a(s)|^2$ which are distributed evenly in angle on a radius circle 0.7662 (Fig. 2.6). The TF $H_a(s)$ having poles with negative real parts is

$$H_a(s) = \frac{0.20238}{(s^2+0.396s+0.5871)(s^2+1.083s+0.5871)(s^2+1.4802s+0.5871)} \tag{2.35}$$

Taking $s = \dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}$ and with T = 1, the equivalent ZT that corresponds the discrete filter is:

$$H(z) = \frac{0.0007378(1+z^{-1})^6}{(1-1.2686z^{-1}+0.7051z^{-2})(1-1.0106z^{-1}+0.3583z^{-2})(1-0.9044z^{-1}+0.2155z^{-2})}$$

(2.36)

The recurrence function $y(n) = \sum_{k=0}^{M} a_k x(n-k) - \sum_{k=1}^{N} b_k y(n-k)$ is requested to execute the filter IIR by a computer. This recursive algorithm is obtained by the IZT of (2.36).

2. 4. 2 Chebyshev analog filter

In Butterworth filters, the characteristics in the pass-band and the stop-band have a monotone nature. Consequently, if the filter specifications are in terms of maximum pass-band approximation error, the specifications are exceeded toward the low-frequency end of the pass-band. A more efficient approach, which usually leads to a lower order filter, is to distribute the accuracy of the approximation uniformly over the pass-band or the stp-band or both. The Chebyshev filter type 1 class has the property that the magnitude of the frequency response is equiripple on the pass-band and monotonic in the stop-band [1, 2].

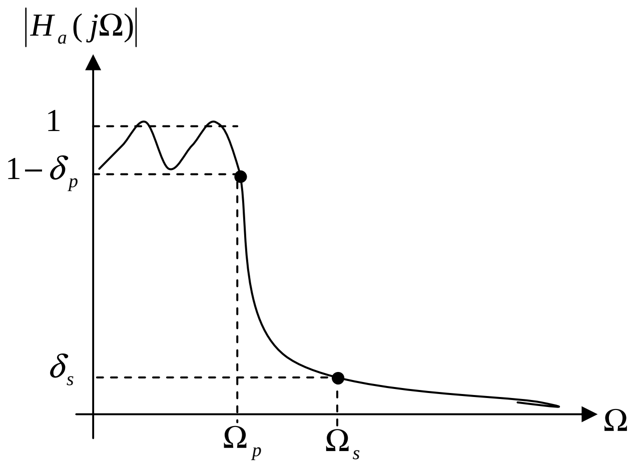


Fig. 2. 7 Approximation with a Chebyshev type 1 filter ($\sigma_p = \varepsilon$)

The analytical form of the squared magnitude is given by [1, 2]

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 V_N^2(\Omega/\Omega_c)} \tag{2.37}$$

where $V_N(x)$ is the Chebyshev polynomial of order N

$$V_N(x) = \cos(N\cos^{-1}(x)) \tag{2.38}$$

For example, if $N = 0$, $V_N(x) = 1$, if $N = 1$, $V_N(x) = \cos(\cos^{-1}(x)) = x$, if $N = 2$, $V_N(x) = \cos(2\cos^{-1}(x)) = 2x^2 - 1$. From equation (2.38) which defines the Chebyshev polynomials, it is clear that a recurrence formula can be obtained in which $V_{N+1}(x)$ can be obtained from $V_N(x)$ and $V_{N-1}(x)$. Hence

$$V_{N+1}(x) = 2xV_N(x) - V_{N-1}(x) \tag{2.39}$$

If $0 < x < 1$, equation (2.38) indicates that $V_N^2(x)$ varies between 0 and 1. if $x > 1$, $\cos^{-1}(x)$ is imaginary and $V_N(x)$ behaves like a hyperbolic cosine and therefore decreases monotonously. From (2.37), $|H_a(j\Omega)|^2$ fluctuates between 1 and $1/(1+\varepsilon^2)$ if $0 < \Omega/\Omega_c < 1$. To characterize the filter, three parameters must be determined; ε, Ω_c and N. In a typical design, ε is given by the ripple (fluctuation) of the bandwidth, Ω_c by the desired cut-off frequency, the N order is then selected in such a way that the bandwidth or stop specifications are met. Chebyshev type 1 low-pass filter properties are.

For $0 \le \Omega < \Omega_c$ and according to the properties of the Chebyshev polynomial, we have

$$\frac{1}{1+\varepsilon^2} \le |H_a(e^{j\omega})|^2 \le 1$$

2- From the above characteristics, we have

$$|H_a(0)|^2 = \begin{cases} 1/(1+\varepsilon^2), & \text{N even} \\ 1 & \text{N odd} \end{cases}$$

3- For $\Omega > \Omega_c$, the response has a monotonous and decreasing nature due to the monotonous behavior of $V_N(x)$ for $|x| > 1$.

It was shown in [1] that the poles of the Chebyshev filter extend over an elliptical of the S-plane. According to Fig. 2.8, the ellipse is defined by two circles corresponding to the major and minor axes. It was found that the rays of these circles are obtained according to ε, Ω_c and N.

$$\begin{cases} a = \frac{1}{2}\left(\alpha^{1/N} - \alpha^{-1/N}\right) \\ b = \frac{1}{2}\left(\alpha^{1/N} + \alpha^{-1/N}\right) \end{cases} \quad \text{with} \quad \alpha = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$$

To determine the poles of the Chebyshev filter on the ellipse (see Fig. 2. 8), one must first identify the points on the minor and major circles also spaced at an angle space, π / N so that the points are symmetrically located relative to the imaginary axis.

$$s_{1,2} = -a\Omega_c \sin(\pi/6) \pm jb\Omega_c \cos(\pi/6)$$

$$s_3 = -a\Omega_c$$

for $N > 3$

$$s_{1,2} = -a\Omega_c \sin(\pi/2N) \pm jb\Omega_c \cos(\pi/2N)$$

$$s_{3,4} = -a\Omega_c \sin(3\pi/2N) \pm jb\Omega_c \cos(3\pi/2N)$$

…etc

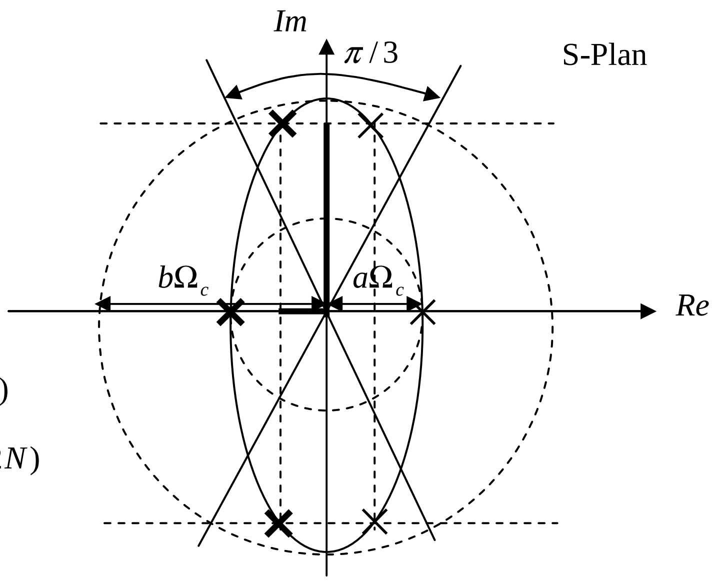


Fig. 2. 8 Location of Chebyshev filter poles of order $N = 3$

A point never falls on the imaginary axis and falls on the real axis if N is odd and never if N is even. After this calculation, it is well confirmed that the poles of the N-order filter can be obtained by the following general expression [2]:

$$s_k = -\Omega_c \sinh\left(\frac{1}{N}\sinh^{-1}\frac{1}{\varepsilon}\right)\sin\left(\frac{(2k+1)\pi}{2N}\right) + j\Omega_c \cosh\left(\frac{1}{N}\sinh^{-1}\frac{1}{\varepsilon}\right)\cos\left(\frac{(2k+1)\pi}{2N}\right) \quad (2.40)$$

$0 \le k \le N-1$.

The following identities are useful for the calculation of the inverse functions $\cosh^{-1}$ and $\sinh^{-1}$. Hence $\sinh^{-1}(x) = \ln\left(x+\sqrt{x^2+1}\right)$ and $\cosh^{-1}(x) = \ln\left(x+\sqrt{x^2-1}\right)$. The Chebyshev analog filter is given in the following form [2]

$$H_a(s) = \begin{cases} \frac{1}{\sqrt{1+\varepsilon^2}} \prod_{k=0}^{N-1} \frac{-s_k}{s-s_k}, & \text{N even} \\ \prod_{k=0}^{N-1} \frac{-s_k}{s-s_k} & \text{N odd} \end{cases} \quad (2.41)$$

- To satisfy the condition, $\frac{1}{1+\varepsilon^2 V_N^2(\Omega_p/\Omega_c)} \ge (1+\delta_p)^2$, we choose $\Omega_c = \Omega_p$

- To satisfy the condition, $\frac{1}{1+\varepsilon^2 V_N^2(\Omega_s/\Omega_c)} \le \delta_s^2$, we choose

$V_N(1/k) = \cosh(N\cosh^{-1}(1/k) \ge 1/d$, with $d = \sqrt{\frac{(1-\delta_p)^{-2}-1}{\delta_s^{-2}-1}}$ and $k = \frac{\Omega_p}{\Omega_s}$

The previous equality gives, $N \ge \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/k)}$

Exemple. 2. 3 :

To illustrate the design by the Chebyshev template, we take the same example above [1].

(i) Impulse invariance approach:

Based on the Chebyshev polynomy, we write: $V_N(\Omega) = \begin{cases} \cos(N\cos^{-1}(\Omega)), & |\Omega| \le 1 \\ \cosh(N\cosh^{-1}(\Omega)), & |\Omega| > 1 \end{cases}$

For A_p = 1dB and A_s = 15dB, we have

$$\begin{cases} 20\log_{10}\left(|H_a(j\Omega)|\right) \ge -A_p = -1 \\ 20\log_{10}\left(|H_a(j\Omega)|\right) \le -A_s = -15 \end{cases} \quad (2.42)$$

Thus

$$\begin{cases} \varepsilon^2\cos^2\left(N\cos^{-1}(1)\right) = 10^{0.1A_p} - 1 \\ \varepsilon^2\cosh^2\left(N\cosh^{-1}(\Omega_s/\Omega_p)\right) = 10^{0.1As} - 1 \end{cases} \quad (2.43)$$

For $\Omega = \Omega_c = \Omega_p$, $\cos^2(N\cos^{-1}(1)) = \cos^2(N.0) = 1 \Rightarrow \varepsilon^2 = 10^{0.1A_p} - 1 \Rightarrow \delta_p = \varepsilon = 0.508847$

$$\cosh^2\left(N\cosh^{-1}(\Omega_s/\Omega_p)\right)=\frac{10^{0.1A_s}-1}{10^{0.1A_p}-1}\Rightarrow\ \cosh\left(N\cosh^{-1}(\Omega_s/\Omega_p)\right)=\sqrt{\frac{10^{0.1A_s}-1}{10^{0.1A_p}-1}}$$

By taking the $\cosh^{-1}$ on both sides, we get

$$N\geq\frac{\cosh^{-1}\left(\sqrt{(10^{0.1A_s}-1)/(10^{0.1A_p}-1)}\right)}{\cosh^{-1}(\Omega_s/\Omega_p)}=3.19 \quad (2.44)$$

So we take N = 4. We assume that the bandwidth specifications are met.

$\Omega_c=0.2\pi=0.628318$

$\alpha=4.1702$

$a=0.3646$

$b=1.0644$

The Chebyshev filter parameters are : $\begin{cases}\varepsilon=0.508841\\ \Omega_c=0.2\pi=0.6283\\ N=4\end{cases}$

They are used for core business calculations, s_k , k = 0, …, N-1. Hence

$s_{0,1}$= -0.0877 $\pm$ j0.6179

$s_{2,3}$= -0.2117 $\pm$ j0.2559

The corresponding TF is

$$H_a(s)=\frac{0.038286}{\left(s^2+0.4233s+0.1103\right)\left(s^2+0.17535s+0.3894\right)} \quad (2.45)$$

The TF of the IIR discrete filter is finally obtained as

$$H(z)=\frac{0.08327+0.0239z^{-1}}{1-1.5658z^{-1}+0.6549z^{-2}}+\frac{0.08327+0.0246z^{-1}}{1-1.4934z^{-1}+0.8392z^{-2}} \quad (2.46)$$

(ii) Bilinear transformation approach:

In this case, the specifications on the analog filter are

$$\begin{cases}20\log_{10}\left|j2\tan\left(\frac{0.2\pi}{2}\right)\right|\geq-1\\ 20\log_{10}\left|j2\tan\left(\frac{0.3\pi}{2}\right)\right|\leq-15\end{cases} \quad (2.47)$$

Thus, $\Omega_c=2\tan(0.2\pi/2)=0.3142$, $\varepsilon=0.50885$ and N=4. The poles are

$s_{0,1=}$ -0.0188 $\pm$ 0.1280i

$s_{2,3=}$ -0.0453 $\pm$ 0.1280i

The TF becomes

$$H_a(s) = \frac{0.04381}{\left(s^2 + 0.1814s + 0.4166\right)\left(s^2 + 0.4378s + 0.1180\right)} \tag{2.48}$$

Replacing, $s = 2\frac{1-z^{-1}}{1+z^{-1}}$, the transfer function of the digital filter is finally obtained by:

$$H(z) = \frac{0.001836(1+z^{-1})^4}{(1-1.4996z^{-1} + 0.8482z^{-2})(1-1.5548z^{-1} + 0.6493z^{-2})} \tag{2.49}$$

The recurrence function $y(n) = \sum_{k=0}^{M} a_k x(n-k) - \sum_{k=1}^{N} b_k y(n-k)\, y(n)$ is required to operate the underling filter given by (2.49).

2. 5 Frequency transformations of low-pass IIR filters

The examples of the previous IIR numerical filters are made using methods based on impulse invariance and bilinear transformation. These filters are inspired from analog filters i.e, Butterworth and Chebyshev with low-pass frequency selection properties. The answers of the most widely used ideal filters are shown in Fig. 2. 9.

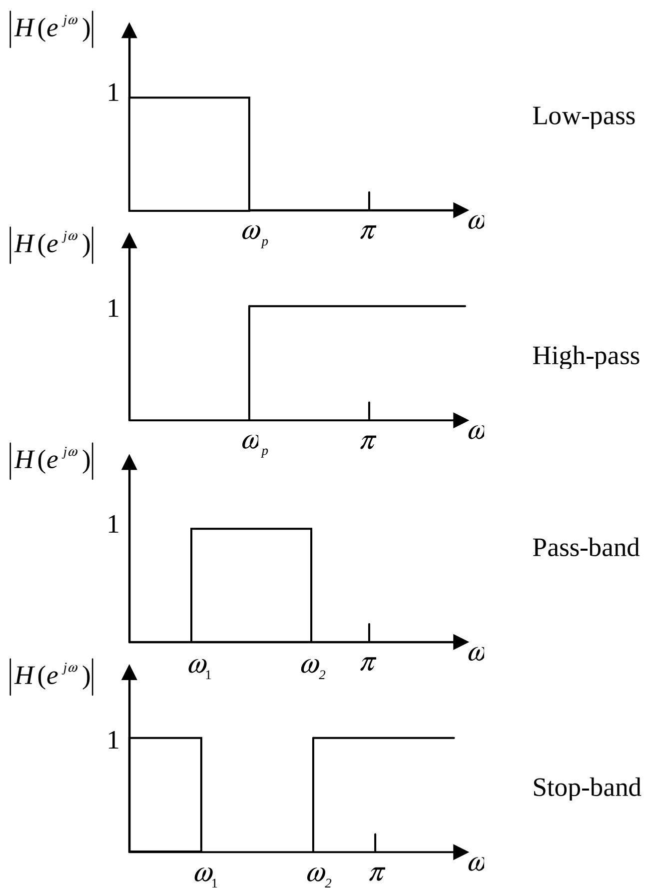


Fig. 2. 9 Possible frequency responses of ideal filters

Using rational transformation techniques, Filters shown in Fig. 2. 9 can be designed from available digital low-pass filter with a cut-off frequency θ_p. Table 2.1 shows the corresponding transformations [1]

Table. 2.1 Transformation from the prototype of IIR low-pass filter with a cut-off frequency θ_p, toward other filters with desired cut-off frequencies ω_p, ω_1 and ω_2.

Filtre type	Transformation	Associated design formulas
Low-pass	$z^{-1} = \frac{z^{-1}-\alpha}{1-\alpha z^{-1}}$	$\alpha = \sin\left((\theta_p - \omega_p)/2\right)/\sin\left((\theta_p + \omega_p)/2\right)$
High-pass	$z^{-1} = -\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}$	$\alpha = -\cos\left((\omega_p + \theta_p)/2\right)/\cos\left((\omega_p - \theta_p)/2\right)$
Pass-band	$z^{-1} = -\frac{z^{-2} - \frac{2\alpha k}{k+1}z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1}z^{-2} - \frac{2\alpha k}{k+1}z^{-1} + 1}$	$\alpha = \cos\left((\omega_2 + \omega_1)/2\right)/\cos\left((\omega_2 - \omega_1)/2\right)$ and $k = \cot\left((\omega_2 - \omega_1)/2\right)\tan\left(\theta_p/2\right)$
Stop-band	$z^{-1} = \frac{z^{-2} - \frac{2\alpha}{k+1}z^{-1} + \frac{1-k}{k+1}}{\frac{k-1}{k+1}z^{-2} - \frac{2\alpha k}{k+1}z^{-1} + 1}$	$\alpha = \cos\left((\omega_2 + \omega_1)/2\right)/\cos\left((\omega_2 - \omega_1)/2\right)$ and $k = \tan\left((\omega_2 - \omega_1)/2\right)\tan\left(\theta_p/2\right)$

Example. 2. 4 :

We wish to obtain a high-pass filter from the Chebyshev filter from a low-pass filter having the following cut-off frequency, $\theta_p = 0.2\pi$ and discrete TF, $H_l(z)$:

$$H_l(z) = \frac{0.001836(1+z^{-1})^4}{(1-1.5548z^{-1}+0.6493z^{-2})(1-1.4996z^{-1}+0.8482z^{-2})} \tag{2.50}$$

If the desired cut frequency of the desired high-pass filter is, $\omega_p = 0.6\pi$, we get from Table. 2. 1

$\alpha = -\cos\left((0.2\pi + 0.6\pi)/2\right)/\cos\left((0.6\pi - 0.2\pi)/2\right)$=-0.38197

By replacing the latter into $H_l(z)$, we find

$$H_d(z) = H_l(z)\Big|_{z^{-1}=-\frac{z^{-1}-0.38197}{1-0.38197z^{-1}}} = \frac{0.02426(1+z^{-1})^4}{(1-1.0416z^{-1}+0.4019z^{-2})(1-0.5561z^{-1}+0.7647z^{-2})} \tag{2.51}$$

2. 6 Related Matlab codes [3] :

To recognize the Matlab commands for the design of IIR filters, we consider the following examples:

(i) Butterworth filter:

The Matlab instruction 'butter' is used as follows

```
clear all;clc;
N=6;
wc=0.7032;
[b a]=butter(N,wc/pi);
figure(1);
freqz(b,a)
axis([0 1 -30 0]);
```

(ii) Chebyshev filter:

The Matlab instruction 'cheby1' leads to compute the Chebyshev type1 filter.

```
Clear all;clc ;
N=6;
wc=0.6498;
[b2,a2] = cheby1(4,0.50885,wc/pi);
figure(3);
freqz(b2,a2);
axis([0 1 -30 0])
```

Tutorial #2

Ex#1 :

We want to design a digital low-pass IIR filter from a Butterworth analog filter using the bilinear transformation method. The analog filter specifications are as follows:

Sampling frequency

- Pass-band attenuation of 3dB at Ω_p where f_e = 25kHz and f_c = 2.25kHz .
- Stop-band attenuation of 10dB at $\Omega_s = 0.45\pi$.

NB: f_e is the sampling frequency and f_c is the cutoff frequency.

Ex#2 :

We wish to design a digital low-pass IIR filter from a Butterworth analog filter using the impulse invariance method. The analog filter specifications are given below:

- Pass-band attenuation of 1dB at $\Omega_p = 0.25\pi$.
- Stop-band attenuation of 10dB at $\Omega_s = 0.45\pi$.

Ex#3 :

We want to design a digital low-pass filter from an analog Chebyshev filter using the method based on impulse invariance. The analog filter specifications are as follows:

- Pass-band attenuation of 3dB at $\Omega_p = 0.2\pi$.
- Stop-band attenuation of 10dB at $\Omega_s = 0.4\pi$.

Calculate N, Ω_c and the poles where the actual part is negative as well as the transfer function of the analog and digital filter assuming that the bandwidth specifications are met.

Ex#4 :

We want to design a digital low-pass filter from an analog Chebyshev filter using the method based on bilinear transformation. The analog filter specifications are as follows:

- Pass-band attenuation of 1dB at $\Omega_p = 0.25\pi$.
- Stop-band attenuation of 10dB at $\Omega_s = 0.45\pi$.