

# Matrices

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## SECTION A Manipulation of matrices

By the end of this section you will be able to:

- understand the arithmetic of matrices
- perform transformations using matrices

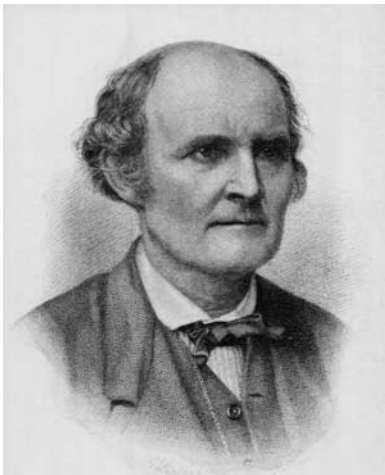
The next two chapters (Matrices and Vectors) belong to a field of mathematics called linear algebra.

Linear algebra is a fundamental area of mathematics and is arguably one of the most powerful mathematical tools ever developed. It is a core topic of study within fields as diverse as: business, economics, engineering, physics, computer science, ecology, sociology, demography and genetics.

For an example of linear algebra at work, one needs to look no further than Google's search engine, which relies upon linear algebra to rank the results of a search with respect to relevance.

### A1 Introduction

Matrices are used in control theory, electrical principles, vibrations, structural analysis, as well as many other fields. Also many engineering problems can be written in terms of simultaneous equations. If there are a large number of equations then in most cases it is easier to use matrices rather than the process of elimination discussed in **Chapter 2**.



**Arthur Cayley** (1821–95), an English lawyer who became a mathematician, was the first person to develop matrices as we know them today. He graduated in 1842 from Trinity College, Cambridge. In 1858 he published *Memoir on the Theory of Matrices* which contained the definition of a matrix, matrix addition, subtraction, etc. Cayley thought matrices were of no practical use, just a convenient notation. He could not have been more wrong. Matrices are used today in areas including economics, science, engineering, medicine and statistics.

Another application of matrices is in image processing. Many displays form images by lighting up tiny dots, called pixels, on their screens. Figure 1 (on the next page) shows a symbol represented by lighting up the appropriate pixels.

If we let 1 = 'pixel on' and 0 = 'pixel off' then we can represent this in matrix form as

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

This is an example of a  $4 \times 5$  (verbally stated as '4 by 5') matrix.

A matrix is an array of numbers enclosed in a bracket. It is used to store information. For example

$\begin{pmatrix} 12 & 6 \\ 3 & 7 \end{pmatrix}$  is a  $2 \times 2$  matrix (2 rows and 2 columns) and is called a square matrix.

$\begin{pmatrix} 3 & 2 \\ 7 & 6 \\ 5 & 9 \\ 2 & 1 \end{pmatrix}$  is a  $4 \times 2$  matrix (4 rows and 2 columns) and is

**not** a square matrix.

An example of a  $2 \times 4$  matrix is

$$\begin{pmatrix} 3 & 7 & 5 & 2 \\ 2 & 6 & 9 & 1 \end{pmatrix}$$

Note that when we say  $2 \times 4$  matrix we state the number of **rows first** and then the number of columns.  $2 \times 4$  ('2 by 4') refers to the **size** of the matrix.

Matrices is the plural of matrix. A common notation for matrices is:

$$\begin{array}{cc} \text{Column 1} & \text{Column 2} \\ \text{Row 1} & \begin{pmatrix} a_{11} & a_{12} \end{pmatrix} \\ \text{Row 2} & \begin{pmatrix} a_{21} & a_{22} \end{pmatrix} \end{array} = \mathbf{A}$$

where  $a_{12}$  is the element in the first row and second column. Generally in this chapter, bold capital letters will denote matrices, as seen above.

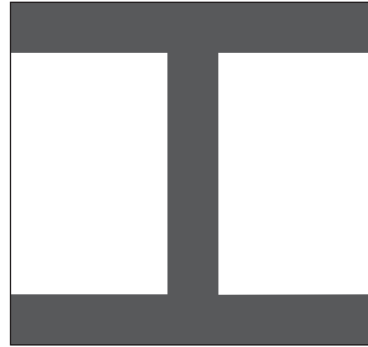


Fig. 1

## A2 Arithmetic of matrices

To add matrices, we add the corresponding locations in each matrix. For example:

$$\begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 9 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2 + 3 & 5 + 9 \\ 4 + 5 & 1 + 7 \end{pmatrix} = \begin{pmatrix} 5 & 14 \\ 9 & 8 \end{pmatrix}$$

Similarly for subtraction, we subtract the corresponding locations in each matrix:

$$\begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 9 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2 - 3 & 5 - 9 \\ 4 - 5 & 1 - 7 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -6 \end{pmatrix}$$

Matrices must be of the **same** size in order to add or subtract.

A scalar is a number which increases or decreases each entry of a matrix. The following are examples of scalar multiplication:

$$10 \begin{pmatrix} 3 & 1 \\ 7 & -9 \end{pmatrix} = \begin{pmatrix} 3 \times 10 & 1 \times 10 \\ 7 \times 10 & -9 \times 10 \end{pmatrix} = \begin{pmatrix} 30 & 10 \\ 70 & -90 \end{pmatrix}$$

and

$$\frac{1}{5} \begin{pmatrix} 20 & -10 \\ 5 & -35 \end{pmatrix} = \begin{pmatrix} 20/5 & -10/5 \\ 5/5 & -35/5 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & -7 \end{pmatrix}$$

Multiply each number in the matrix by the scalar. Also

$$7 \begin{pmatrix} -1 \\ 5 \\ 3 \\ 9 \end{pmatrix} = \begin{pmatrix} -7 \\ 35 \\ 21 \\ 63 \end{pmatrix}$$

Next we examine the multiplication of matrices.

### Row times column

Matrix multiplication is carried out by row times column. For example:

$$(1 \ 2) \begin{pmatrix} 3 \\ 4 \end{pmatrix} = (1 \times 3) + (2 \times 4) = 11$$

In general we have

$$(a \ b) \begin{pmatrix} c \\ d \end{pmatrix} = (a \times c) + (b \times d) = ac + bd$$

For  $2 \times 2$  matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

Applying this to the above matrices gives

$$\begin{pmatrix} 2 & 1 \\ 3 & 7 \end{pmatrix} \times \begin{pmatrix} 4 & 9 \\ 5 & 8 \end{pmatrix} = \begin{pmatrix} (2 \times 4) + (1 \times 5) & (2 \times 9) + (1 \times 8) \\ (3 \times 4) + (7 \times 5) & (3 \times 9) + (7 \times 8) \end{pmatrix} \\ = \begin{pmatrix} 13 & 26 \\ 47 & 83 \end{pmatrix} \quad \checkmark$$

The following is **not** how you multiply matrices:

$$\begin{pmatrix} 2 & 1 \\ 3 & 7 \end{pmatrix} \times \begin{pmatrix} 4 & 9 \\ 5 & 8 \end{pmatrix} \neq \begin{pmatrix} 2 \times 4 & 1 \times 9 \\ 3 \times 5 & 7 \times 8 \end{pmatrix} \quad \text{[Not equal]} \quad \times$$

We can **only** multiply matrices if the number of **columns** of the **first** matrix **equals** the number of **rows** of the **second** matrix.

$$\begin{array}{cc} \text{Col. 1} & \text{Col. 2} \\ \text{Row 1} & \begin{pmatrix} 2 & 1 \end{pmatrix} \\ \text{Row 2} & \begin{pmatrix} 3 & 7 \end{pmatrix} \end{array} \times \begin{pmatrix} 4 & 9 \\ 5 & 8 \end{pmatrix} = \begin{pmatrix} \text{Row 1} \times \text{Col. 1} & \text{Row 1} \times \text{Col. 2} \\ \text{Row 2} \times \text{Col. 1} & \text{Row 2} \times \text{Col. 2} \end{pmatrix}$$

### Example 1

Evaluate the following:

**a**  $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

**b**  $(3 \ 4) \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}$

**c**  $(1 \ 0) \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

**d**  $\begin{pmatrix} 2 & 3 & 6 \\ 1 & 5 & 7 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 4 & 2 \\ 1 & 3 \end{pmatrix}$

**e**  $(1 \ 2 \ 3) \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

**f**  $\begin{pmatrix} 2 & 3 & 6 \\ 1 & 5 & 7 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 4 & 2 \end{pmatrix}$

Example 1 *continued*

Solution

Matrix multiplication is row  $\times$  column.

$$\mathbf{a} \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} (2 \times 2) + (3 \times 1) \\ (-1 \times 2) + (5 \times 1) \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

$$\mathbf{b} \begin{pmatrix} 3 & 4 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} (3 \times 2) + (4 \times (-1)) & (3 \times 3) + (4 \times 5) \\ (2 \times 2) + (3 \times 1) & (2 \times 3) + (3 \times 1) \end{pmatrix} \\ = \begin{pmatrix} 2 & 29 \\ 7 & 17 \end{pmatrix}$$

$$\mathbf{c} \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} (1 \times 3) + (0 \times 4) \\ (3 \times 3) + (4 \times 4) \end{pmatrix} = \begin{pmatrix} 3 \\ 25 \end{pmatrix}$$

$$\mathbf{d} \begin{pmatrix} 2 & 3 & 6 \\ 1 & 5 & 7 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 4 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} (2 \times 3) + (3 \times 4) + (6 \times 1) & (2 \times 7) + (3 \times 2) + (6 \times 3) \\ (1 \times 3) + (5 \times 4) + (7 \times 1) & (1 \times 7) + (5 \times 2) + (7 \times 3) \end{pmatrix} \\ = \begin{pmatrix} 24 & 38 \\ 30 & 38 \end{pmatrix}$$

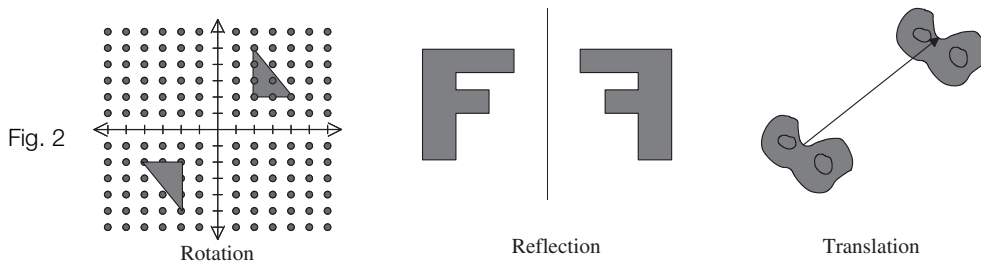
$$\mathbf{e} \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} (1 \times 4) + (2 \times 5) + (3 \times 6) \\ (5 \times 4) + (6 \times 5) + (7 \times 6) \end{pmatrix} = \begin{pmatrix} 32 \\ 89 \end{pmatrix}$$

$\mathbf{f} \begin{pmatrix} 2 & 3 & 6 \\ 1 & 5 & 7 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 4 & 2 \end{pmatrix}$ . Since the number of columns of the first matrix is 3 and number of rows of the second matrix is 2, we **cannot** multiply these matrices.

## A3 Applications to transformations

Computer graphics and games are based on transformations. For example, a computer animated sequence is based on modelling surfaces of connecting triangles. The computer stores the vertices of the triangle in its memory and then certain operations such as rotations, translations, reflections and enlargements are carried out. These operations are stored as matrices.

A transformation is a function like rotation, reflection and translation as shown in Fig. 2:



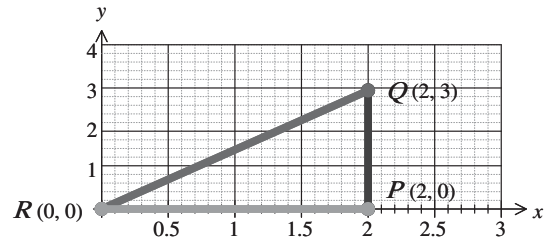
The figure shows various two-dimensional transformations. These transformations can be represented by a matrix.

For example, consider the vertices of a triangle  $P(2, 0)$ ,  $Q(2, 3)$  and  $R(0, 0)$  shown in Fig. 3 on the next page.

This can be represented in matrix form with the co-ordinates of the point  $P$  as the entries in the first column, the co-ordinates of the point  $Q$  as entries in the second column and the co-ordinates of the point  $R$  as entries in the last column. Let  $A$  denote this matrix then

$$A = \begin{pmatrix} P & Q & R \\ 2 & 2 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

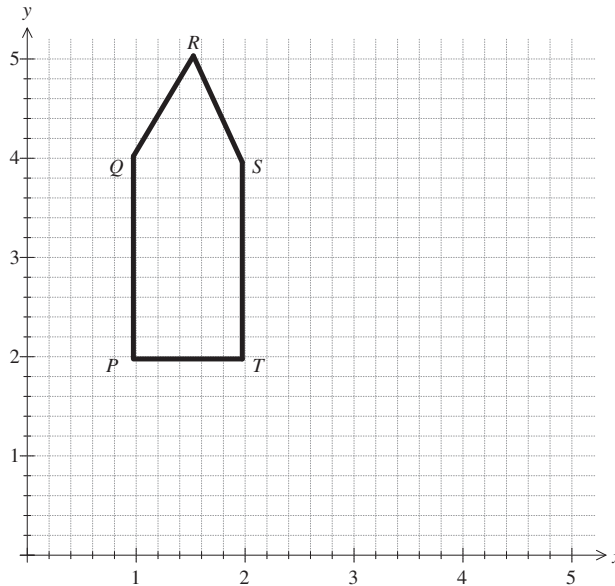
Fig. 3



The five corners of a house shown in Fig. 4 can be represented by a matrix  $H$  given by:

$$H = \begin{pmatrix} P & Q & R & S & T \\ 1 & 1 & 1.5 & 2 & 2 \\ 2 & 4 & 5 & 4 & 2 \end{pmatrix}$$

Fig. 4



### Example 2

Let  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $H = \begin{pmatrix} 1 & 1 & 1.5 & 2 & 2 \\ 2 & 4 & 5 & 4 & 2 \end{pmatrix}$  be the matrix representing  $PQRST$  shown in Fig. 4 above. Determine the image of the house under the transformation represented by the matrix multiplication  $AH$ . Illustrate and describe what effect this transformation has.

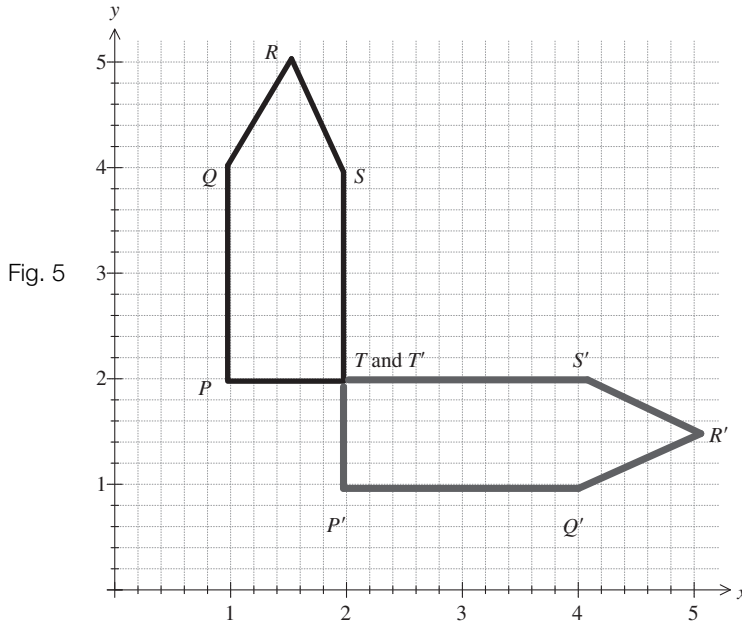
**Solution**

Let the image of the corners  $P, Q, R, S$  and  $T$  be denoted by  $P', Q', R', S'$  and  $T'$  respectively. Then the matrix multiplication  $AH$  yields:

$$AH = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P & Q & R & S & T \\ 1 & 1 & 1.5 & 2 & 2 \\ 2 & 4 & 5 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 5 & 4 & 2 \\ 1 & 1 & 1.5 & 2 & 2 \end{pmatrix}$$

### Example 2 *continued*

Plotting the given house  $PQRST$  and the transformed house  $P'Q'R'S'T'$  on the same plane (Fig. 5):



The transformation represented by matrix multiplication  $AH$  rotates the house  $90^\circ$  about the point  $T$  in a clockwise direction.

### ? Why does matrix $A$ in Example 2 produce this rotation of $90^\circ$ in a clockwise direction?

The matrix  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  changes the  $x$  and  $y$  co-ordinates of each vertex. For example, the vertex  $P(1, 2)$  under the matrix multiplication  $AH$  is transformed to  $P'(2, 1)$ , vertex  $Q(1, 4)$  is transformed to  $Q'(4, 1)$ , vertex  $R(1.5, 5)$  is transformed to  $R'(5, 1.5)$ , etc.

### Example 3

Let  $A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 3 & 0 \end{pmatrix}$  be the matrix representing the triangle  $PQR$  shown in Fig. 3 on page 365. Determine the image of the triangle under the transformation represented by  $2A$ .

Illustrate and describe what effect this transformation has.

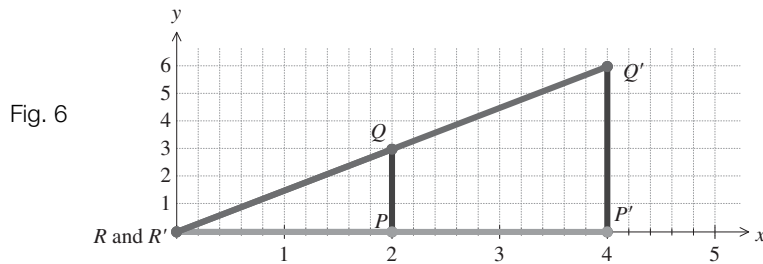
Solution

Carrying out the scalar multiplication we have:

$$2A = 2 \begin{pmatrix} P & Q & R \\ 2 & 2 & 0 \\ 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} P' & Q' & R' \\ 4 & 4 & 0 \\ 0 & 6 & 0 \end{pmatrix}$$

### Example 3 *continued*

Illustrating the given triangle  $PQR$  and the transformed triangle  $P'Q'R'$  (Fig. 6):



**What effect does scalar multiplication  $2\mathbf{A}$  have on the initial triangle  $PQR$ ?**

This scalar multiplication increases the length of sides of the triangle by a factor of 2. This means that  $2\mathbf{A}$  doubles the length of each side of the initial triangle.

**?** What would be the image of the initial triangle  $PQR$  under the scalar multiplication of  $\frac{1}{2}\mathbf{A}$ ?

It scales each side of the triangle to half its initial size.

### Example 4

Let  $\mathbf{A}$  be the matrix representing the triangle with the vertices  $P(2, 2)$ ,  $Q(2, 5)$  and  $R(4, 2)$ .

Write the matrix  $\mathbf{A}$  and determine the image of the triangle under the

transformation represented by matrix multiplication  $\mathbf{BA}$  where  $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

Illustrate and describe what effect this transformation has.

**Solution**

We are given the vertices of the triangle  $P(2, 2)$ ,  $Q(2, 5)$  and  $R(4, 2)$ , therefore

$$\mathbf{A} = \begin{matrix} & \begin{matrix} P & Q & R \end{matrix} \\ \begin{pmatrix} 2 & 2 & 4 \\ 2 & 5 & 2 \end{pmatrix} \end{matrix}$$

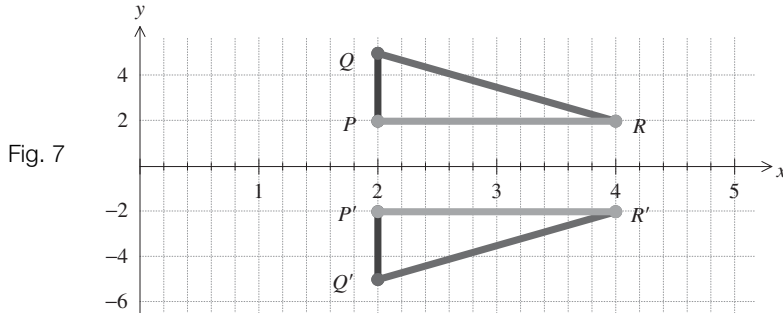
Let the images of points of the given triangle  $P$ ,  $Q$  and  $R$  under the transformation  $\mathbf{BA}$  be given by  $P'$ ,  $Q'$  and  $R'$ . We have

$$\mathbf{BA} = \begin{matrix} & \begin{matrix} P & Q & R \end{matrix} & & \begin{matrix} P' & Q' & R' \end{matrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 4 \\ 2 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 4 \\ -2 & -5 & -2 \end{pmatrix} \end{matrix}$$



### Example 4 *continued*

Plotting the given triangle  $P(2, 2)$ ,  $Q(2, 5)$  and  $R(4, 2)$  and the transformed triangle  $P'(2, -2)$ ,  $Q'(2, -5)$  and  $R'(4, -2)$  results in Fig. 7.



The effect of transformation **B** is to reflect the triangle  $PQR$  in the  $x$  (horizontal) axis.

The matrix **B** changes the  $+y$  co-ordinate of each vertex to  $-y$ .

### Example 5

Determine the image of the triangle  $PQR$  given in **Example 4** under the transformation represented by  $CA$  where  $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ .

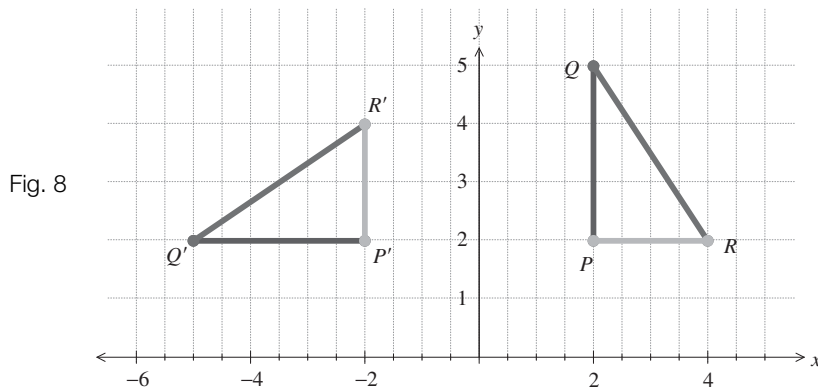
Illustrate and describe what effect this transformation has.

**Solution**

The transformation representing the matrix multiplication  $CA$  is given by

$$CA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 4 \\ 2 & 5 & 2 \end{pmatrix} = \begin{pmatrix} -2 & -5 & -2 \\ 2 & 2 & 4 \end{pmatrix}$$

We can illustrate this transformation as shown in Fig. 8 below:



The matrix multiplication  $CA$  produces a rotation of  $90^\circ$  anticlockwise about the origin.

We can also combine transformations which are normally called **composite** transformations. Composite transformations such as  $\mathbf{CB(A)}$  means that we **first** carry out the transformation **B** on **A** and **then** perform transformation **C** on the result of **BA**.

### Example 6

Determine the image of the triangle  $PQR$  (represented by matrix  $A$ ) given in **Example 4** above under the composite transformations represented by

- i**  $\mathbf{BC(A)}$       **ii**  $\mathbf{CB(A)}$

where the matrices **B** and **C** are given in **Examples 4** and **5** above, respectively.

Illustrate and describe what effect each composite transformation has. **What do you notice about your results?**



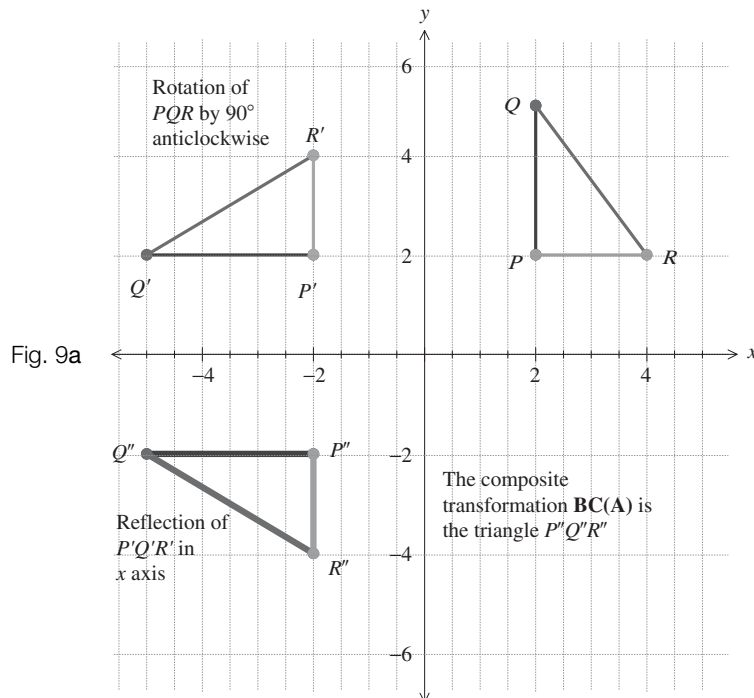
Solution

- i** The transformation representing the matrix multiplication  $\mathbf{BC(A)}$  is given by

$$\begin{aligned} \mathbf{BC(A)} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 4 \\ 2 & 5 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} P' & Q' & R' \\ 2 & 2 & 4 \end{pmatrix} = \begin{pmatrix} P'' & Q'' & R'' \\ -2 & -2 & -4 \end{pmatrix} \end{aligned}$$

Triangle rotated by 90° anticlockwise as seen in **Example 5**
Reflection in the x axis

We can illustrate this composite transformation as shown in Fig. 9a below:

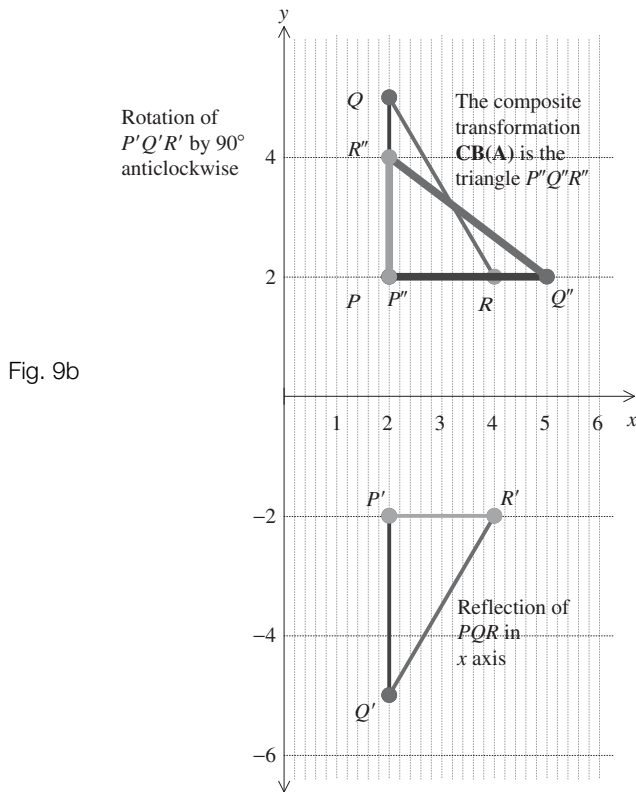


Example 6 *continued*

ii Similarly we have

$$\begin{aligned} \mathbf{CB(A)} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 4 \\ 2 & 5 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 2 & 2 & 4 \\ -2 & -5 & -2 \end{pmatrix}}_{\substack{\text{Reflection of triangle in } x \text{ axis} \\ \text{as seen in Example 4}}} = \underbrace{\begin{pmatrix} 2 & 5 & 2 \\ 2 & 2 & 4 \end{pmatrix}}_{\text{Rotation of } 90^\circ \text{ anticlockwise}} \end{aligned}$$

Figure 9b below illustrates the composite transformations:



$\mathbf{BC(A)}$  is the transformation of **first** rotating the given triangle  $PQR$  through  $90^\circ$  anticlockwise about the origin and then reflecting your result in the  $x$  axis.

$\mathbf{CB(A)}$  is the transformation of **first** reflecting the initial triangle  $PQR$  in the  $x$  axis and then rotating through  $90^\circ$  anticlockwise about the origin.

Note that the composite transformations  $\mathbf{BC}$  and  $\mathbf{CB}$  are **not** the same. Matrix multiplication does **not** give the same result when we multiply the matrices in the other order, that is,  $\mathbf{BC} \neq \mathbf{CB}$  [not equal]. The **order** of the transformation is important.

$\mathbf{BC} \neq \mathbf{CB}$ :

$$\mathbf{BC} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{CB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note also that the matrix representing the transformation that we apply first is the second matrix in the product and vice versa. This is because we apply the transformation to an object whose vertices are the columns of a matrix  $\mathbf{A}$ . So assume  $\mathbf{B}$  is the matrix representing the first transformation and  $\mathbf{C}$  is the matrix representing the second matrix. After applying the first transformation to  $\mathbf{A}$  we have  $\mathbf{BA}$  and then we apply the second transformation to this to get  $\mathbf{C(BA)}$ . Because matrix multiplication is associative, this is the same as  $(\mathbf{CB})\mathbf{A}$ . Consequently the composite transformation is represented by  $\mathbf{CB}$ .

**?** **Can we apply transformations to three dimensions?** Yes. It is difficult to visualize this in a book but we move on to this in **Section C**.

#### A4 Computation of matrices

**?** **Can we write the computation of matrices in compact form?** Yes, for example, let  $\mathbf{A}$  and  $\mathbf{B}$  be matrices, then  $\mathbf{A} + \mathbf{B}$  means add the matrices  $\mathbf{A}$  and  $\mathbf{B}$ . The computation  $2\mathbf{A}$  means multiply every entry of the matrix  $\mathbf{A}$  by 2.

**?** **How can the tedious nature of some of the calculations be avoided?** A software package such as MATLAB can be used for computing matrices. In fact, MATLAB is short for 'Matrix Laboratory'. It is a very useful tool and can be used to eliminate the tedium of lengthy calculations. Many mathematical software packages exist, such as MAPLE and MATHEMATICA, but MATLAB is particularly useful for matrices. In the present release of MATLAB the matrices are entered with square brackets rather than round ones.

For example, in MATLAB to write the matrix  $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$  we enter  $\mathbf{A} = [1 \ 2; 3 \ 4]$  after the command prompt  $>>$ . The semicolon indicates the end of the row. The output shows:

```
A =
     1     2
     3     4
```

#### Example 7

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 \\ 4 & 6 & 9 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & 5 & 6 \\ 1 & 7 & 2 \\ 9 & 6 & 1 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 5 & 4 & 9 \\ 7 & 4 & 0 \\ 6 & 9 & 8 \end{pmatrix}$$

Compute the following:

**a**  $\mathbf{B} + \mathbf{C}$    **b**  $3\mathbf{B} + 2\mathbf{C}$    **c**  $\mathbf{A} + \mathbf{B}$    **d**  $\mathbf{AB}$    **e**  $\mathbf{BA}$

Example 7 *continued*

**What do you notice about your results to parts d and e?**

**a** Adding the corresponding entries gives

$$\mathbf{B} + \mathbf{C} = \begin{pmatrix} 2 & 5 & 6 \\ 1 & 7 & 2 \\ 9 & 6 & 1 \end{pmatrix} + \begin{pmatrix} 5 & 4 & 9 \\ 7 & 4 & 0 \\ 6 & 9 & 8 \end{pmatrix} = \begin{pmatrix} 2+5 & 5+4 & 6+9 \\ 1+7 & 7+4 & 2+0 \\ 9+6 & 6+9 & 1+8 \end{pmatrix} = \begin{pmatrix} 7 & 9 & 15 \\ 8 & 11 & 2 \\ 15 & 15 & 9 \end{pmatrix}$$

To use MATLAB enter the matrices **B** and **C** after the command prompt `>>`. Separate each matrix by a comma. Then enter `B + C` and the output should give the same result as above.

**b** We have

$$\begin{aligned} 3\mathbf{B} + 2\mathbf{C} &= 3 \begin{pmatrix} 2 & 5 & 6 \\ 1 & 7 & 2 \\ 9 & 6 & 1 \end{pmatrix} + 2 \begin{pmatrix} 5 & 4 & 9 \\ 7 & 4 & 0 \\ 6 & 9 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 3 \times 2 & 3 \times 5 & 3 \times 6 \\ 3 \times 1 & 3 \times 7 & 3 \times 2 \\ 3 \times 9 & 3 \times 6 & 3 \times 1 \end{pmatrix} + \begin{pmatrix} 2 \times 5 & 2 \times 4 & 2 \times 9 \\ 2 \times 7 & 2 \times 4 & 2 \times 0 \\ 2 \times 6 & 2 \times 9 & 2 \times 8 \end{pmatrix} \quad \left[ \begin{array}{l} \text{Multiplying each} \\ \text{entry of matrix } \mathbf{B} \text{ by} \\ \text{3 and of matrix } \mathbf{C} \text{ by 2} \end{array} \right] \\ &= \begin{pmatrix} 6 & 15 & 18 \\ 3 & 21 & 6 \\ 27 & 18 & 3 \end{pmatrix} + \begin{pmatrix} 10 & 8 & 18 \\ 14 & 8 & 0 \\ 12 & 18 & 16 \end{pmatrix} \quad \left[ \begin{array}{l} \text{Simplifying each} \\ \text{entry} \end{array} \right] \\ &= \begin{pmatrix} 6+10 & 15+8 & 18+18 \\ 3+14 & 21+8 & 6+0 \\ 27+12 & 18+18 & 3+16 \end{pmatrix} = \begin{pmatrix} 16 & 23 & 36 \\ 17 & 29 & 6 \\ 39 & 36 & 19 \end{pmatrix} \quad \left[ \begin{array}{l} \text{Adding the} \\ \text{corresponding} \\ \text{entries} \end{array} \right] \end{aligned}$$

In MATLAB, multiplication is carried out by using the command `*`. Once the matrices **B** and **C** have been entered you do not need to enter them again. Enter `3 * B + 2 * C` to check the above evaluation.



**c** **A + B** is impossible because the matrices are of different sizes. **What sizes are matrix A and matrix B?**

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 \\ 4 & 6 & 9 \end{pmatrix} \text{ is a } 2 \times 3 \text{ matrix and } \mathbf{B} = \begin{pmatrix} 2 & 5 & 6 \\ 1 & 7 & 2 \\ 9 & 6 & 1 \end{pmatrix} \text{ is a } 3 \times 3 \text{ matrix, therefore we}$$

**cannot** add (or subtract) these matrices.

If we try this in MATLAB we receive a message saying 'Matrix dimensions must agree'.

**d** We have

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 1 & 2 & 5 \\ 4 & 6 & 9 \end{pmatrix} \times \begin{pmatrix} 2 & 5 & 6 \\ 1 & 7 & 2 \\ 9 & 6 & 1 \end{pmatrix} \\ &= \begin{pmatrix} (1 \times 2) + (2 \times 1) + (5 \times 9) & (1 \times 5) + (2 \times 7) + (5 \times 6) & (1 \times 6) + (2 \times 2) + (5 \times 1) \\ (4 \times 2) + (6 \times 1) + (9 \times 9) & (4 \times 5) + (6 \times 7) + (9 \times 6) & (4 \times 6) + (6 \times 2) + (9 \times 1) \end{pmatrix} \end{aligned}$$

Example 7 *continued*

$$\begin{aligned}
 &= \begin{pmatrix} 2 + 2 + 45 & 5 + 14 + 30 & 6 + 4 + 5 \\ 8 + 6 + 81 & 20 + 42 + 54 & 24 + 12 + 9 \end{pmatrix} \quad \text{[Simplifying]} \\
 &= \begin{pmatrix} 49 & 49 & 15 \\ 95 & 116 & 45 \end{pmatrix}
 \end{aligned}$$

**e** **BA** is impossible because the number of columns in the left-hand matrix, **B**, is **not** equal to the number of rows in the right-hand matrix, **A**. **How many columns does matrix B have?**

**B** =  $\begin{pmatrix} 2 & 5 & 6 \\ 1 & 7 & 2 \\ 9 & 6 & 1 \end{pmatrix}$  has 3 columns. **How many rows does matrix A have?**

Matrix **A** =  $\begin{pmatrix} 1 & 2 & 5 \\ 4 & 6 & 9 \end{pmatrix}$  has 2 rows. Since the number of columns of matrix **B**, 3, does **not** match the number of rows of matrix **A**, 2, we **cannot** evaluate **BA**. As obvious as this may seem, many students get this wrong. Watch out for this!

If we were to try this in MATLAB we would receive an error message saying 'Inner matrix dimensions must agree'.

Notice again that the matrix multiplication **AB** does **not** equal the matrix multiplication **BA**. In fact, as we have just established, in this case **BA** cannot even be evaluated, while **AB** is computed in part **d** above. So matrix multiplication is **not** the same as multiplying two real numbers. In matrix multiplication the **order** of the multiplication **does** matter.

**SUMMARY**

When multiplying two matrices it is row  $\times$  column. If **A** and **B** represent matrices then in general

$$\mathbf{AB} \neq \mathbf{BA}$$

**Exercise 11(a)**

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

You may like to check your numerical solutions by using MATLAB or any other appropriate software. Some calculators can also perform matrix calculations.

**1** For  $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ ,  $\mathbf{B} = \begin{pmatrix} 6 & -1 \\ 5 & 3 \end{pmatrix}$  and

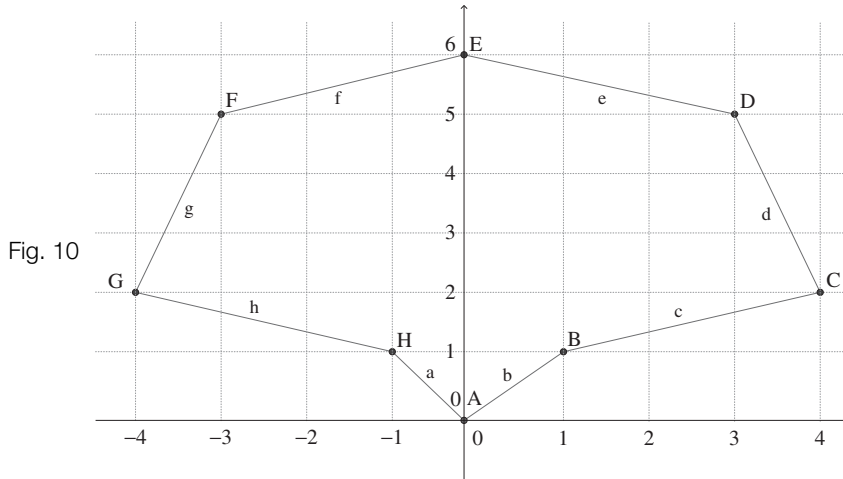
$\mathbf{C} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ , evaluate the following:

- |  |                                    |   |
|--|------------------------------------|---|
| <b>a</b> $\mathbf{A} + \mathbf{B}$     | <b>b</b> $\mathbf{B} + \mathbf{A}$ | <b>c</b> $\mathbf{A} + \mathbf{A} + \mathbf{A}$ |
| <b>d</b> $3\mathbf{A} + 2\mathbf{B}$   | <b>e</b> $\mathbf{A} + \mathbf{C}$ | <b>f</b> $\mathbf{B} + \mathbf{C}$              |
| <b>g</b> $\mathbf{AC}$                 | <b>h</b> $\mathbf{BC}$             | <b>i</b> $5\mathbf{A} - 7\mathbf{BC}$           |
| <b>j</b> $3\mathbf{AC} - 2\mathbf{BC}$ |                                    |   |

## Exercise 11(a) continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

2 Write down the matrix representing the ABCDEFGH in Fig. 10 below:



3 Let the top of a table be given by the co-ordinates  $P(1, 2)$ ,  $Q(2, 2)$ ,  $R(2, 4)$  and  $S(1, 4)$ . Write down the matrix  $A$  which represents this table top  $PQRS$ .

Let

$$B = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and}$$

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Determine the image of the table top under the following transformation and illustrate the effect that each transformation has:

- a**  $A - B$    **b**  $3A$    **c**  $CA$    **d**  $DA$   
**e**  $CD(A)$    **f**  $DC(A)$

4 Let  $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$  and find  $A^2 = A \times A$ .

[ $A^2$  in MATLAB is evaluated by the command  $A^2$ .] Comment upon your result. Can you get the same result when multiplying two non-zero real numbers?

5 Evaluate  $\begin{pmatrix} 5 & -1 & -2 \\ 10 & -2 & -4 \\ 15 & -3 & -6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ 2 & 3 & 8 \end{pmatrix}$ .

Comment upon your result. Can you get the same result when multiplying two non-zero real numbers?

## SECTION B Applications

By the end of this section you will be able to:

- apply matrices to solve equations
- evaluate the determinant of a  $2 \times 2$  matrix
- obtain the inverse matrix

## B1 Determinant of a matrix

### Example 8

Consider a triangle given by the co-ordinates  $P(0, 0)$ ,  $Q(2, 0)$  and  $R(0, 3)$  (the shaded triangle shown in Fig. 11 below). Let the matrix  $\mathbf{A}$  represent this triangle  $PQR$  and determine the image of this triangle under the transformation given by  $\mathbf{BA}$  where

$$\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}.$$

By illustrating this transformation, determine the areas of the triangle  $PQR$  and the transformed triangle  $P'Q'R'$ . **How does this transformation  $\mathbf{B}$  change the size of the area?**

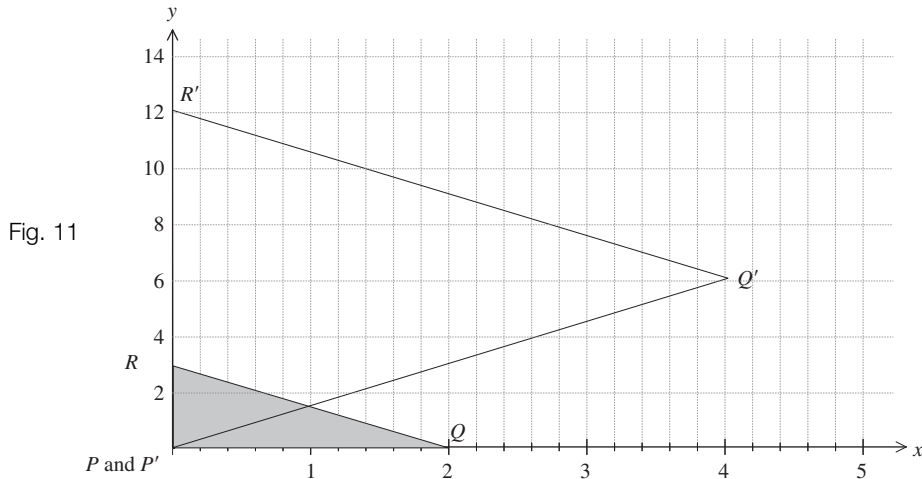
Solution

We are given co-ordinates  $P(0, 0)$ ,  $Q(2, 0)$  and  $R(0, 3)$ , therefore  $\mathbf{A} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ .

Evaluating the matrix multiplication  $\mathbf{BA}$ :

$$\mathbf{BA} = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 6 & 12 \end{pmatrix}$$

These triangles are plotted in Fig. 11 below:



The area of shaded triangle  $PQR = \frac{2 \times 3}{2} = 3$  and the area of the large triangle

$$P'Q'R' = \frac{12 \times 4}{2} = 24.$$

The transformation  $\mathbf{B}$  increases the area by a factor of 8 (because, of course,  $24/3 = 8$ ).



The factor 8 is called the **determinant** of the matrix **B** and describes the increase in size that **B** imposes on **A**. The determinant of the matrix **B** is calculated by:

$$\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix} = (2 \times 4) - (3 \times 0) = 8$$

In general, the determinant of a  $2 \times 2$  matrix,  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , is given by

$$\text{11.1} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is used to represent the determinant. Another notation is  $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ . Note that the

**?** determinant is a number, **not** a matrix. **Why?** Because the transformation  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  changes area by a **factor** ( $ad - bc$ ). For example

$$\det \begin{pmatrix} 5 & 4 \\ -2 & 1 \end{pmatrix} = (5 \times 1) - (-2 \times 4) = 13$$

The determinant is also used to find the inverse of a matrix which will be defined in Section **B3**.

## B2 Identity matrix

### Example 9

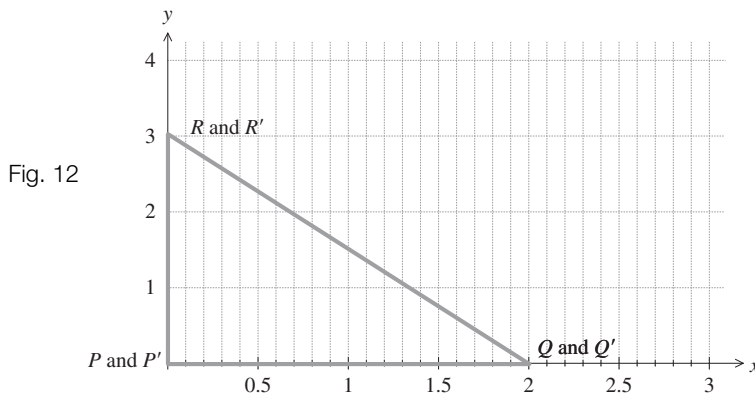
Determine the image of the triangle  $PQR$  (represented by **A**) in **Example 8** on the previous page under the transformations represented by the matrix multiplication  $\mathbf{IA}$  where  $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Illustrate and describe what effect this transformation has.

Solution

Carrying out the matrix multiplication  $\mathbf{IA}$  gives

$$\mathbf{IA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

We can plot this as shown in the Fig. 12 below:



The transformation given by the matrix multiplication  $\mathbf{IA}$  does **not** change the given triangle  $PQR$ . This means that the triangle  $PQR$  remains fixed under this transformation.

Now evaluate

$$\begin{pmatrix} 5 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 6 & 7 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

**?** What do you notice about your results?

$$\begin{pmatrix} 5 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 1 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 6 & 7 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 2 & 3 \end{pmatrix}$$

The resultant matrix in each case is unchanged by this operation. In general, we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  is called the identity matrix and is denoted by  $\mathbf{I}$ ,  $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

The identity matrix,  $\mathbf{I}$ , for matrices is analogous to the number 1 for numbers.

### B3 Inverse of a matrix

Let  $\mathbf{A}$  be the general  $2 \times 2$  (square) matrix:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The inverse matrix of  $\mathbf{A}$  is denoted by  $\mathbf{A}^{-1}$  and has the property

$$\mathbf{11.2} \quad \mathbf{A}^{-1} \times \mathbf{A} = \mathbf{I}$$

The notation  $\mathbf{A}^{-1}$  is the inverse matrix and not  $\mathbf{A}$  to the index  $-1$ .  $\left[ \mathbf{A}^{-1} \neq \frac{1}{\mathbf{A}}, \text{ not equal} \right]$

In transformations, this means that if the transformation  $\mathbf{A}$  is applied to a particular figure then the transformation  $\mathbf{A}^{-1}$  **undoes**  $\mathbf{A}$  so the net result of the combined transformation  $\mathbf{A}^{-1} \mathbf{A}$  is to leave the figure untouched, and this is why  $\mathbf{A}^{-1} \mathbf{A} = \mathbf{I}$ .

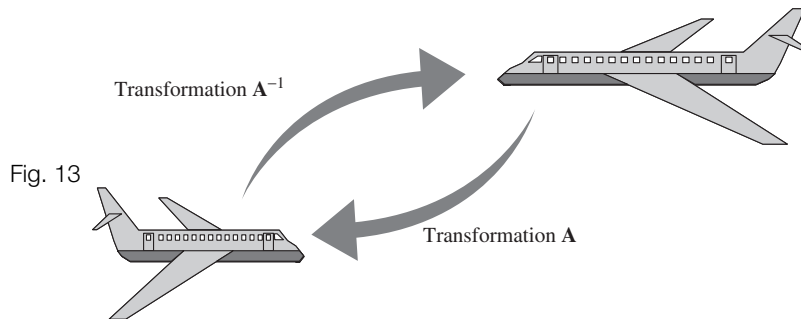


Fig. 13

In a computer game if a transformation  $\mathbf{A}$  is applied to an object as shown in Fig. 13, then the transformation  $\mathbf{A}^{-1}$  undoes  $\mathbf{A}$  and so the net result of  $\mathbf{A}^{-1} \mathbf{A} = \mathbf{I}$  is to leave the object unchanged.

If the determinant of  $A$  is equal to 1 then  $A^{-1}$  is given by

$$\mathbf{11.3} \quad A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad [\text{provided that } \det A = 1]$$

Remember this only works if  $\det A = 1$ .

### Example 10

For  $A = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$ , find  $A^{-1}$ .

Solution

First we obtain the determinant by using

$$\mathbf{11.1} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

$$\det \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix} = (3 \times 5) - (2 \times 7) = 1$$

Since the determinant of  $A$  is 1 we can find  $A^{-1}$  by using **11.3**, that is, interchanging numbers 3 and 5 and putting a negative sign in front of the other numbers:

$$A^{-1} = \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$$

We can confirm the result by checking the matrix multiplication  $A^{-1}A = I$ :

$$A^{-1}A = \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

### ? What are the limitations of this method?

The determinant of  $A$  has to equal 1. Of course there are going to be many matrices where the determinant does **not** equal 1. Consider

$$A = \begin{pmatrix} 11 & 4 \\ 7 & 3 \end{pmatrix}$$

The determinant is

$$\det A = (11 \times 3) - (7 \times 4) = 5$$

Therefore we **cannot** use  $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$  to find  $A^{-1}$ .

**?** However if we try using this, what matrix do we obtain?

Call this matrix **B**. Interchanging the numbers 3 and 11 and placing a negative sign in front of the other numbers:

$$\mathbf{B} = \begin{pmatrix} 3 & -4 \\ -7 & 11 \end{pmatrix}$$

Multiplying the two matrices gives

$$\mathbf{BA} = \begin{pmatrix} 3 & -4 \\ -7 & 11 \end{pmatrix} \begin{pmatrix} 11 & 4 \\ 7 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

Of course  $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$  but it can be made into the identity matrix.

**?** How?

By multiplying with a scalar of 1/5:

$$\frac{1}{5} \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

**?** Where did the number 5 appear previously in this discussion?

It is the determinant of the matrix **A**. Since  $\frac{1}{5} \mathbf{B}$  multiplied by **A** gives the identity matrix, so  $\frac{1}{5} \mathbf{B}$  must be the inverse of **A**. Thus

$$\mathbf{A}^{-1} = \frac{1}{5} \mathbf{B} = \frac{1}{5} \begin{pmatrix} 3 & -4 \\ -7 & 11 \end{pmatrix}$$

This means that when we apply the transformation **B** on **A** the area is increased by a factor of 5 (because  $\det \mathbf{B} = 5$ ) and so to find the inverse we need to divide by 5.

In general, if  $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  then the inverse matrix,  $\mathbf{A}^{-1}$ , is defined as

$$\mathbf{11.4} \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ provided that } \det \mathbf{A} \neq 0$$

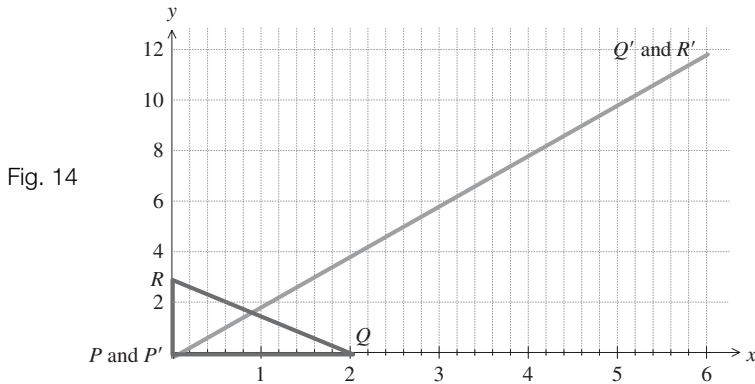
**?** If the determinant of **A** is zero then matrix **A** does not have an inverse. **Why not?**

Because the det of inverse matrix  $\mathbf{A}^{-1}$  has  $1/\det \mathbf{A}$  and if  $\det \mathbf{A} = 0$  then we cannot have  $\mathbf{A}^{-1}$ . We can demonstrate this by using transformations.

Consider the image of the triangle *PQR* (represented by matrix **A**) given in **Example 8** on page 575, under the transformation **BA** where  $\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$ . Multiplying the matrices we have:

$$\mathbf{BA} = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 6 \\ 0 & 12 & 12 \end{pmatrix}$$

The transformation **B** on **A** is illustrated in Fig. 14:



Under the transformation **B** the triangle represented by matrix **A** becomes a line  $P'Q'R'$  which means it has **no** area (Fig. 14). The transformation **B** collapses the area of the triangle to zero so that matrix **B** **cannot** have an inverse because there **no** way back to the original object. The area is increased by a factor of **zero**, therefore the determinant of **B** is zero ( $\det(\mathbf{B}) = 0$ ) and we can confirm this by evaluating using the formula:

$$\det \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix} = (3 \times 4) - (6 \times 2) = 0$$

### Example 11

Let

$$\mathbf{A} = \begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix}$$

Find  $\mathbf{A}^{-1}$ .

Solution

First we have to find  $\det \begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix}$ . By

$$\mathbf{11.1} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

$$\det \begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix} = (7 \times 7) - (5 \times 9) = 4$$

With  $\det \mathbf{A} = 4$ :

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{\det \mathbf{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 7 & -9 \\ -5 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 7/4 & -9/4 \\ -5/4 & 7/4 \end{pmatrix} \end{aligned}$$

Example 11 *continued*

To check that  $A^{-1}$  is indeed the inverse matrix of  $A$ , we carry out the following matrix multiplication:

$$\begin{aligned} A^{-1} \times A &= \frac{1}{4} \begin{pmatrix} 7 & -9 \\ -5 & 7 \end{pmatrix} \begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \end{aligned}$$

Thus  $A^{-1}$  is the inverse matrix of  $A$ .

It's good practice to check your inverse matrix.

## B4 Solving linear equations

Many problems in engineering and science are solved by using systems of simultaneous linear equations. Linear equations are equations in which the variables have an index of 1 or 0. For example, if you apply Hooke's law to a spring then this leads to a system of linear equations. If you apply Kirchhoff's law to a circuit which only contains resistors then the result is a set of linear equations. Linear equations arise in many different fields such as engineering, physics, computer science, technology, business and economics.

Generally a complex scientific or engineering problem can be reduced to a linear system of simultaneous equations.

We can use matrices to solve simultaneous equations, as the following example shows.

Example 12 *electrical principles*

By applying Kirchhoff's law to a circuit we obtain the following equations:

$$\begin{aligned} 7i_1 + 9i_2 &= 3 \\ 5i_1 + 7i_2 &= 1 \end{aligned}$$

where  $i_1$  and  $i_2$  represent currents. Find the values of  $i_1$  and  $i_2$ .

Solution

The product of matrices is row  $\times$  column so the above equations can be written in matrix form as

$$\begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Note that if we multiply the contents of the first two sets of brackets, using the row by column method of matrix multiplication, we get back



### Example 12 *continued*

$$\begin{aligned}7i_1 + 9i_2 &= 3 \\5i_1 + 7i_2 &= 1\end{aligned}$$

Writing equations in this format is why the multiplication rule of matrices is useful.

Let  $\mathbf{A} = \begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix}$ . Therefore we have

$$\mathbf{A} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Since we want to find  $i_1$  and  $i_2$  we need to obtain

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix}$$



**We need to remove the  $\mathbf{A}$  from the Left-Hand Side. How?**

Multiply both sides of  $\mathbf{A} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$  by the inverse matrix,  $\mathbf{A}^{-1}$ , which gives

$$\begin{aligned}\mathbf{A}^{-1}\mathbf{A} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} &= \mathbf{A}^{-1} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ \mathbf{I} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} &= \mathbf{A}^{-1} \begin{pmatrix} 3 \\ 1 \end{pmatrix}\end{aligned}$$

Remember that  $\mathbf{I}$  is the identity matrix, so on the Left-Hand Side we only have

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix}. \text{ Thus } \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

We need to find the inverse matrix,  $\mathbf{A}^{-1}$ , using

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

We have

$$\mathbf{A}^{-1} = \begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix}^{-1} = \frac{1}{(7 \times 7) - (5 \times 9)} \begin{pmatrix} 7 & -9 \\ -5 & 7 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 7 & -9 \\ -5 & 7 \end{pmatrix}$$

Substituting this value for  $\mathbf{A}^{-1}$  into the above  $\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$  gives

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 7 & -9 \\ -5 & 7 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} (7 \times 3) + (-9 \times 1) \\ (-5 \times 3) + (7 \times 1) \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 12 \\ -8 \end{pmatrix}$$

Multiplying each element of the last matrix by 1/4:

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 12/4 \\ -8/4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Hence  $i_1 = 3$  A (amp) and  $i_2 = -2$  A. To check this solution is correct we substitute these into our given equations:

$$\begin{aligned}7i_1 + 9i_2 &= 3 & 7(3) + 9(-2) &= 3 \\5i_1 + 7i_2 &= 1 & 5(3) + 7(-2) &= 1\end{aligned}$$

Normally an  $n \times 1$  matrix is called a **column vector** and is denoted by a bold small case letter ( $n$  is any positive whole number).

The above equations are examples of linear equations

$$\begin{aligned}7i_1 + 9i_2 &= 3 \\5i_1 + 7i_2 &= 1\end{aligned}$$

and can be written in matrix form as

$$\mathbf{A}\mathbf{u} = \mathbf{b}$$

where  $\mathbf{A} = \begin{pmatrix} 7 & 9 \\ 5 & 7 \end{pmatrix}$ ,  $\mathbf{u} = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ . (In this case  $\mathbf{u}$  and  $\mathbf{b}$  are column vectors.)

For the general case

$$\mathbf{A}\mathbf{u} = \mathbf{b}$$

where  $\mathbf{A}$  is a  $2 \times 2$  matrix with inverse  $\mathbf{A}^{-1}$ ,  $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$  ( $x$  and  $y$  are the variables whose values we are trying to find) and  $\mathbf{b}$  is the value of the equations. To solve these equations we need

to find the values of  $x$  and  $y$  which satisfy these equations, hence we need to find  $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ .

This can be determined by

$$\mathbf{11.5} \quad \mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$$

To solve simultaneous equations we can also use the following algorithm:

- 1** Write the given equations in matrix form:  $\mathbf{A}\mathbf{u} = \mathbf{b}$ .
- 2** Find  $\mathbf{A}^{-1}$ .
- 3** The solution  $\mathbf{u}$  is given by  $\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$ .

### Example 13

Solve

$$\begin{aligned}2x + 3y &= 4 \\10x + 4y &= 9\end{aligned}$$

Solution

Following the above algorithm.

- 1** We write the given simultaneous equations in matrix form:

$$\begin{pmatrix} 2 & 3 \\ 10 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$$

- 2** Let  $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 10 & 4 \end{pmatrix}$  so we need to find  $\mathbf{A}^{-1}$  by using

$$\mathbf{11.4} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 10 & 4 \end{pmatrix}^{-1} = \frac{1}{8 - 30} \begin{pmatrix} 4 & -3 \\ -10 & 2 \end{pmatrix} = -\frac{1}{22} \begin{pmatrix} 4 & -3 \\ -10 & 2 \end{pmatrix}$$



Example 13 *continued*

3 Hence

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \mathbf{A}^{-1} \begin{pmatrix} 4 \\ 9 \end{pmatrix} \\ &= -\frac{1}{22} \begin{pmatrix} 4 & -3 \\ -10 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 9 \end{pmatrix} = -\frac{1}{22} \begin{pmatrix} -11 \\ -22 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \end{aligned}$$

Therefore  $x = 1/2$ ,  $y = 1$ . Check for yourself that  $x = 1/2$ ,  $y = 1$  do indeed satisfy the given simultaneous equations.

**SUMMARY**

The determinant of a matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is defined as

$$\mathbf{11.1} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

The inverse matrix,  $\mathbf{A}^{-1}$ , has the property

$$\mathbf{11.2} \quad \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

where  $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  is the identity matrix.

The inverse matrix,  $\mathbf{A}^{-1}$ , of  $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is defined as

$$\mathbf{11.4} \quad \mathbf{A}^{-1} = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (\det \mathbf{A} \neq 0)$$

Solving simultaneous equations of the form

$$\mathbf{A}\mathbf{u} = \mathbf{b}$$

is given by

$$\mathbf{11.5} \quad \mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$$

**Exercise 11(b)**

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

**1** Compute  $\det \mathbf{A}$  and  $\det \mathbf{B}$  for the following:

$$\mathbf{a} \quad \mathbf{A} = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix}$$

$$\mathbf{b} \quad \mathbf{A} = \begin{pmatrix} -1 & 2 \\ 5 & 3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -1 & 5 \\ 2 & 3 \end{pmatrix}$$

$$\mathbf{c} \quad \mathbf{A} = \begin{pmatrix} \cos(-\pi) & \cos(\pi) \\ \sin(-\pi) & \sin(\pi) \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} \cos(-\pi) & \sin(-\pi) \\ \cos(\pi) & \sin(\pi) \end{pmatrix}$$

What do you notice about matrices  $\mathbf{A}$  and  $\mathbf{B}$  and your results for  $\det \mathbf{A}$  and  $\det \mathbf{B}$ ?

## Exercise 11(b) continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

2 For  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $B = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ , show that  $\det A = \det B$ .

3 Evaluate **i**  $\det A$  **ii**  $\det B$  **iii**  $\det A \times \det B$  **iv**  $\det(AB)$  for the following:

**a**  $A = \begin{pmatrix} 1 & 3 \\ 5 & 6 \end{pmatrix}$ ,  $B = \begin{pmatrix} 3 & 7 \\ -1 & 5 \end{pmatrix}$

**b**  $A = \begin{pmatrix} -1 & 10 \\ 170 & 1.5 \end{pmatrix}$ ,  $B = \begin{pmatrix} -30 & -9.3 \\ 61 & -1.9 \end{pmatrix}$

**c**  $A = \begin{pmatrix} 5 & -3 \\ 2.2 & -5.6 \end{pmatrix}$ ,  $B = \begin{pmatrix} -7.1 & -2.1 \\ -3.5 & -12.2 \end{pmatrix}$

What do you notice about your results to parts **iii** and **iv**.

4 Find the inverse of the following matrices:


**a**  $A = \begin{pmatrix} 9 & 2 \\ 13 & 3 \end{pmatrix}$  **b**  $A = \begin{pmatrix} 17 & 7 \\ 12 & 5 \end{pmatrix}$

**c**  $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

5 Find the inverse matrix of the following:

**a**  $A = \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}$  **b**  $A = \begin{pmatrix} 3 & 6 \\ 7 & 8 \end{pmatrix}$

**c**  $A = \begin{pmatrix} 7 & 1 \\ 14 & 2 \end{pmatrix}$

 The next two questions are in the field of [**electrical principles**].

6 We obtain the following equations from an electrical circuit:

$$\begin{aligned} 2i_1 + 3i_2 &= 5 \\ 6i_1 + 7i_2 &= 10 \end{aligned}$$

where  $i_1$  and  $i_2$  represent currents. Write these equations in matrix form and solve for  $i_1$  and  $i_2$ .

\*7 By applying Kirchoff's law to a circuit we obtain the following equations:

$$\begin{aligned} 30i_1 - 10i_2 &= 12 \\ -10i_1 + 35i_2 &= 5 \end{aligned}$$

where  $i_1$  and  $i_2$  represent currents. Find the exact values of  $i_1$  and  $i_2$ .

8 Solve the following equations:

**a**  $x + 3y = 14$

$2x - 3y = -8$

**b**  $-2x - 5y = -3$

$-5x - 2y = 3$

**c**  $4x + 6y = -1$

$7x - \frac{3}{4}y = 2$

SECTION C  $3 \times 3$  matrices

By the end of this section you will be able to:

- evaluate the determinant of a  $3 \times 3$  matrix
- find the inverse of a  $3 \times 3$  matrix
- solve equations using the inverse matrix method

This section may seem rather abstract compared to the previous two because it is much harder to visualize three-dimensional transformations.

C1 **Determinants**

If

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is a general  $3 \times 3$  matrix then

$$\mathbf{11.6} \quad \det \mathbf{A} = a \left[ \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} \right] - b \left[ \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} \right] + c \left[ \det \begin{pmatrix} d & e \\ g & h \end{pmatrix} \right]$$

The determinant of a  $3 \times 3$  matrix is the increase in **volume** associated with three-dimensional transformation.

**11.6** gives the determinant of a  $3 \times 3$  matrix. The simplest way of evaluating the above determinant is **not** to remember all the details of **11.6** but to use the following procedure:

**1** Delete the row and column containing the entry  $a$ :

$$\begin{pmatrix} (a) & & \\ & e & f \\ & h & i \end{pmatrix}$$

then evaluate the determinant of the remaining matrix  $\begin{pmatrix} e & f \\ h & i \end{pmatrix}$ .

**2** Delete the row and column containing the entry  $b$ :

$$\begin{pmatrix} & (b) & \\ d & & f \\ g & & i \end{pmatrix}$$

then evaluate the determinant of the remaining matrix  $\begin{pmatrix} d & f \\ g & i \end{pmatrix}$ .

**3** Delete the row and column containing the entry  $c$ :

$$\begin{pmatrix} & & (c) \\ d & e & \\ g & h & \end{pmatrix}$$

then evaluate the determinant of the remaining matrix  $\begin{pmatrix} d & e \\ g & h \end{pmatrix}$ .

**4** Combine these steps as in **11.6** to obtain the determinant of the above matrix.

A general aid in finding the determinant is to place your thumb over the number and ignore the horizontal row and vertical column containing that number.

## Example 14

Evaluate the determinant of

$$\mathbf{A} = \begin{pmatrix} 1 & 7 & 5 \\ 3 & 2 & -7 \\ 6 & -8 & 9 \end{pmatrix}$$

Solution

By using steps 1–4 and 11.6 we have

$$\begin{aligned} \det \mathbf{A} &= 1 \det \begin{pmatrix} 2 & -7 \\ -8 & 9 \end{pmatrix} - 7 \det \begin{pmatrix} 3 & -7 \\ 6 & 9 \end{pmatrix} + 5 \det \begin{pmatrix} 3 & 2 \\ 6 & -8 \end{pmatrix} \\ &\equiv [(2 \times 9) - (-8 \times (-7))] - 7[(3 \times 9) - (6 \times (-7))] + 5[(3 \times (-8)) - (6 \times 2)] \\ &\quad \text{by 11.1} \\ &= -701 \end{aligned}$$

## C2 Cofactors

A general  $3 \times 3$  matrix is  $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ . The determinant of the remaining matrix after

deleting the rows and columns of an entry is called the **minor** of that entry. For example, in the above case

$$\det \begin{pmatrix} e & f \\ h & i \end{pmatrix} \text{ is the minor of } a$$

$$\det \begin{pmatrix} d & f \\ g & i \end{pmatrix} \text{ is the minor of } b$$

? What is the minor of  $c$ ?

$$\det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

? What is the minor of  $e$ ?

Deleting the rows and columns containing the entry  $e$  we have

$$\begin{pmatrix} a & c \\ g & i \end{pmatrix}$$

Hence  $\det \begin{pmatrix} a & c \\ g & i \end{pmatrix}$  is the minor of  $e$ .

---


$$\text{11.1 } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

## Example 15

Evaluate the minor of  $-1$  in

$$\begin{pmatrix} 3 & 5 & 7 \\ -1 & 2 & 3 \\ -4 & 4 & -9 \end{pmatrix}$$

Solution

After deleting the row and column containing  $-1$

$$\begin{pmatrix} & 5 & 7 \\ (-1) & & \\ & 4 & -9 \end{pmatrix}$$

we obtain the matrix  $\begin{pmatrix} 5 & 7 \\ 4 & -9 \end{pmatrix}$ . The minor of  $-1$  is

$$\det \begin{pmatrix} 5 & 7 \\ 4 & -9 \end{pmatrix} \stackrel{\text{by 11.1}}{=} [5 \times (-9)] - [4 \times 7] = -73$$

The **cofactor** of  $-1$  in **Example 15** is  $-(-73) = 73$ . The cofactor of an entry is the minor with a **place sign**. The place sign is found from the rule:

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

The first entry of a matrix has a positive place sign and then the place signs alternate.

## Example 16

Find the cofactor of 5 of the matrix in **Example 15**.

Solution

The minor of 5 is

$$\det \begin{pmatrix} -1 & 3 \\ -4 & -9 \end{pmatrix} \stackrel{\text{by 11.1}}{=} [-1 \times (-9)] - [(-4) \times 3] = 21$$

According to the rule, the place sign is negative, so the cofactor of 5 is  $-21$ .

Note that the **minor** of 5 is 21 but the **cofactor** is  $-21$  because the position of 5 in the matrix gives it a negative place sign.

Hence the cofactor is a minor with a place sign.

In **Section C1** we calculated the determinant of a  $3 \times 3$  matrix using formula **11.6**.

---


$$\mathbf{11.1} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

We can rewrite a  $3 \times 3$  matrix in terms of its cofactors, that is, if  $A$  is a general  $3 \times 3$  matrix:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

We can also use a  $3 \times 3$  cofactor matrix to calculate its determinant using:

$$\mathbf{11.7} \quad \det A = a(\text{cofactor of } a) + b(\text{cofactor of } b) + c(\text{cofactor of } c)$$

We can find the determinant of a matrix by expanding along **any** of the rows or **any** of the columns. For example, the formula for expanding along the second row is

$$\det A = d(\text{cofactor of } d) + e(\text{cofactor of } e) + f(\text{cofactor of } f)$$

**?** What is the formula for expanding along the third row?

$$\det A = g(\text{cofactor of } g) + h(\text{cofactor of } h) + i(\text{cofactor of } i)$$

We can also expand along any column. Expansion along the first column is

$$\det A = a(\text{cofactor of } a) + d(\text{cofactor of } d) + g(\text{cofactor of } g)$$

and so on.

### Example 17

Find the determinant of

$$A = \begin{pmatrix} -1 & -6 & 3 \\ 5 & 6 & -7 \\ -2 & 0 & 1 \end{pmatrix}$$

Solution

Since there is a zero in the last row it is easier to expand along this row:

$$\begin{aligned} \det \begin{pmatrix} -1 & -6 & 3 \\ 5 & 6 & -7 \\ -2 & 0 & 1 \end{pmatrix} &= -2 \det \begin{pmatrix} -6 & 3 \\ 6 & -7 \end{pmatrix} - 0 \det \begin{pmatrix} -1 & 3 \\ 5 & -7 \end{pmatrix} + 1 \det \begin{pmatrix} -1 & -6 \\ 5 & 6 \end{pmatrix} \\ &\equiv -2(42 - 18) - 0 + (-6 + 30) \\ &\quad \text{by } \mathbf{11.1} \\ &= -24 \end{aligned}$$

Note that the middle term on the Right-Hand Side is zero and so we just need to evaluate the other two terms. We could also have found the determinant of  $A$  by expanding along the second column because this also contains (the same) zero.

---


$$\mathbf{11.1} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

Example 17 *continued*

If a row or column contains zero(s) then it is easier to expand along that row or column.

We can also obtain the determinant of a  $4 \times 4$ ,  $5 \times 5$ ,  $6 \times 6$ , etc. matrix but it becomes very laborious to do this by just using pen and paper. In these cases it is more convenient to use a graphical calculator or a computer algebra system.

**C3 Cofactor matrix**

Let  $C$  be the new matrix consisting of the cofactors of the general matrix  $A$ . If

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{then} \quad C = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}$$

where  $A$  is the cofactor of  $a$ ,  $B$  is the cofactor of  $b$ , etc.  $C$  is called the cofactor matrix and it is used in finding the inverse matrix.

## Example 18

Find the cofactor matrix  $C$  of

$$A = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 9 & 7 \\ -2 & 1 & 0 \end{pmatrix}$$

**Solution**

Cofactor of the first entry, 1, is

$$\det \begin{pmatrix} 9 & 7 \\ 1 & 0 \end{pmatrix} \underset{\text{by 11.1}}{\equiv} [(9 \times 0) - (1 \times 7)] = -7$$

Cofactor of  $-1$  is

$$\underset{\substack{\text{minus} \\ \text{place sign}}}{=} \det \begin{pmatrix} 3 & 7 \\ -2 & 0 \end{pmatrix} \underset{\text{by 11.1}}{\equiv} -[(3 \times 0) - (-2 \times 7)] = -14$$

Cofactor of 5 is

$$\det \begin{pmatrix} 3 & 9 \\ -2 & 1 \end{pmatrix} \underset{\text{by 11.1}}{\equiv} [(3 \times 1) - (-2 \times 9)] = 21$$

Cofactor of 3 is

$$\underset{\substack{\text{minus} \\ \text{place sign}}}{=} \det \begin{pmatrix} -1 & 5 \\ 1 & 0 \end{pmatrix} \underset{\text{by 11.1}}{\equiv} -[(-1 \times 0) - (1 \times 5)] = 5$$

---


$$\text{11.1} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

Example 18 *continued*

Cofactor of 9 is

$$\det \begin{pmatrix} 1 & 5 \\ -2 & 0 \end{pmatrix} \underset{\text{by 11.1}}{\equiv} [(1 \times 0) - (-2 \times 5)] = 10$$

Cofactor of 7 is

$$\underset{\substack{\text{minus} \\ \text{place sign}}}{\det} \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} \underset{\text{by 11.1}}{\equiv} -[(1 \times 1) - (-2 \times (-1))] = 1$$

Cofactor of -2 is

$$\det \begin{pmatrix} -1 & 5 \\ 9 & 7 \end{pmatrix} \underset{\text{by 11.1}}{\equiv} [(-1 \times 7) - (9 \times 5)] = -52$$

Cofactor of 1 (the 1 on the bottom row of the given matrix) is

$$\underset{\substack{\text{minus} \\ \text{place sign}}}{\det} \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix} \underset{\text{by 11.1}}{\equiv} -[(1 \times 7) - (3 \times 5)] = 8$$

Cofactor of 0 is

$$\det \begin{pmatrix} 1 & -1 \\ 3 & 9 \end{pmatrix} \underset{\text{by 11.1}}{\equiv} [(1 \times 9) - (3 \times (-1))] = 12$$

Hence by collecting these together and placing them in the corresponding position gives the cofactor matrix:

$$\mathbf{C} = \begin{pmatrix} -7 & -14 & 21 \\ 5 & 10 & 1 \\ -52 & 8 & 12 \end{pmatrix}$$

## C4 Inverse of a matrix

In order to find the inverse of a  $3 \times 3$  matrix, we must define some terms. The **transpose** of a matrix is evaluated by interchanging the rows and columns. Let  $\mathbf{A}$  be a matrix then the transpose is denoted by  $\mathbf{A}^T$ . The first row of  $\mathbf{A}$  becomes the first column of  $\mathbf{A}^T$ , the second row of  $\mathbf{A}$  becomes the second column of  $\mathbf{A}^T$  and so on. If

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ then } \mathbf{A}^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

---


$$\text{11.1 } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$



### Example 19

Find  $A^T$  and  $C^T$  of **Example 18**.

Solution

We have

$$A = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 9 & 7 \\ -2 & 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} -7 & -14 & 21 \\ 5 & 10 & 1 \\ -52 & 8 & 12 \end{pmatrix}$$

Interchanging rows and columns gives

$$A^T = \begin{pmatrix} 1 & 3 & -2 \\ -1 & 9 & 1 \\ 5 & 7 & 0 \end{pmatrix} \quad C^T = \begin{pmatrix} -7 & 5 & -52 \\ -14 & 10 & 8 \\ 21 & 1 & 12 \end{pmatrix}$$

We can find the transpose of **any** matrix, it doesn't need to be a square matrix. For example, if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ then } A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

The transpose of a matrix is an important concept in matrix theory.

The **adjoint** of a square matrix  $A$ , denoted  $\text{adj}A$ , is defined as

$$\text{11.8} \quad \text{adj}A = C^T$$

where  $C$  is the matrix of cofactors of  $A$ . For **Example 19**:

$$\text{adj}A = C^T = \begin{pmatrix} -7 & 5 & -52 \\ -14 & 10 & 8 \\ 21 & 1 & 12 \end{pmatrix}$$

The **inverse** of any square matrix  $A$ , denoted by  $A^{-1}$ , is given by

$$\text{11.9} \quad A^{-1} = \frac{1}{\det A} (\text{adj}A) \quad [\text{Provided that } \det A \neq 0]$$

and it is defined as the matrix which satisfies

$$A^{-1} \times A = I$$

where  $I$  is the identity matrix.

## Example 20

Determine  $A^{-1}$  of

$$A = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 9 & 7 \\ -2 & 1 & 0 \end{pmatrix}$$

Solution

Since  $A^{-1} = \frac{1}{\det A}(\text{adj}A)$  we need to find  $\text{adj}A$  and  $\det A$ .



**What is  $\text{adj}A$  equal to?**

Because the matrix  $A$  is the same matrix  $A$  as in **Examples 18** and **19**, we have already evaluated  $\text{adj}A (= C^T)$ , that is, the cofactor matrix transposed). Hence from above we have

$$\text{adj}A = \begin{pmatrix} -7 & 5 & -52 \\ -14 & 10 & 8 \\ 21 & 1 & 12 \end{pmatrix} \quad [= C^T]$$

We only need to find  $\det A$ . Expanding along the **bottom row** of the given  $A$  because it contains a 0:

$$\begin{aligned} \det A &= -2\det \begin{pmatrix} -1 & 5 \\ 9 & 7 \end{pmatrix} - 1\det \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix} + 0\det \begin{pmatrix} 1 & -1 \\ 3 & 9 \end{pmatrix} \\ &\stackrel{\text{by 11.1}}{=} -2[-7 - 45] - 1[7 - 15] + 0 \\ &= 112 \end{aligned}$$

Substituting these into **11.9** gives

$$A^{-1} = \frac{1}{\det A}(\text{adj}A) = \frac{1}{112} \begin{pmatrix} -7 & 5 & -52 \\ -14 & 10 & 8 \\ 21 & 1 & 12 \end{pmatrix}$$

## Example 21

Check that  $A^{-1}$  is indeed the inverse matrix of  $A$  in **Example 20**.

Solution

We need to check that

$$A^{-1} \times A = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

---

**11.1**  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$

### Example 21 *continued*

$\mathbf{I}$  is the identity matrix of a  $3 \times 3$  matrix, with 1's along the main diagonal and zeros elsewhere. Also remember that multiplication of matrices is row  $\times$  column.

$$\begin{aligned} \mathbf{A}^{-1} \times \mathbf{A} &= \frac{1}{112} \begin{pmatrix} -7 & 5 & -52 \\ -14 & 10 & 8 \\ 21 & 1 & 12 \end{pmatrix} \begin{pmatrix} 1 & -1 & 5 \\ 3 & 9 & 7 \\ -2 & 1 & 0 \end{pmatrix} \\ &= \frac{1}{112} \begin{pmatrix} 112 & 0 & 0 \\ 0 & 112 & 0 \\ 0 & 0 & 112 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \end{aligned}$$

The identity matrix is a square matrix with 1's along the diagonal and zeros elsewhere.



### Example 22 *electrical principles*

By applying Kirchhoff's law to a circuit we obtain the following equations:

$$i_1 + 2i_2 + i_3 = 4$$

$$3i_1 - 4i_2 - 2i_3 = 2$$

$$5i_1 + 3i_2 + 5i_3 = -1$$

where  $i_1$ ,  $i_2$  and  $i_3$  represent currents. By using the inverse matrix method, find the values of  $i_1$ ,  $i_2$  and  $i_3$ .

Solution

Writing the above equations in matrix form we obtain

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

Let  $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{pmatrix}$ , then we have

$$\mathbf{A} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

Multiplying both sides by the inverse matrix,  $\mathbf{A}^{-1}$ , gives

**\*** 
$$\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$



### Example 22 *continued*



We need to find  $A^{-1}$ . **How do we obtain  $A^{-1}$ ?**

We need  $\det A$  because  $A^{-1} = \frac{1}{\det A}(\text{adj}A)$ . Thus

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{pmatrix} &= \det \begin{pmatrix} -4 & -2 \\ 3 & 5 \end{pmatrix} - 2\det \begin{pmatrix} 3 & -2 \\ 5 & 5 \end{pmatrix} + \det \begin{pmatrix} 3 & -4 \\ 5 & 3 \end{pmatrix} \\ &\stackrel{\text{by 11.1}}{=} -14 - 50 + 29 \\ \det A &= -35 \end{aligned}$$



**What else do we need to find?**

$\text{adj}A$

$\text{adj}A$  is the cofactor matrix  $C$  transposed. Evaluating the cofactors of each entry horizontally we obtain the following:

Cofactor of 1 (first entry) is

$$\det \begin{pmatrix} -4 & -2 \\ 3 & 5 \end{pmatrix} \stackrel{\text{by 11.1}}{=} (-4 \times 5) - (3 \times (-2)) = -14$$

Cofactor of 2 is

$$-\det \begin{pmatrix} 3 & -2 \\ 5 & 5 \end{pmatrix} \stackrel{\text{by 11.1}}{=} -[(3 \times 5) - (5 \times (-2))] = -25$$

Cofactor of 1 is

$$\det \begin{pmatrix} 3 & -4 \\ 5 & 3 \end{pmatrix} = 29$$

Cofactor of 3 (in the second row) is

$$-\det \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} = -7$$

Cofactor of  $-4$  is

$$\det \begin{pmatrix} 1 & 1 \\ 5 & 5 \end{pmatrix} = 0$$

Cofactor of  $-2$  is

$$-\det \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix} = 7$$

Cofactor of 5 (the left 5 in the bottom row) is

$$\det \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} = 0$$

---

**11.1**  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$



### Example 22 *continued*

Cofactor of 3 (in the last row) is

$$-\det\begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} = 5$$

Cofactor of 5 (the right 5 in the bottom row) is

$$\det\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} = -10$$

Thus the cofactor matrix  $C = \begin{pmatrix} -14 & -25 & 29 \\ -7 & 0 & 7 \\ 0 & 5 & -10 \end{pmatrix}$

Transposing (rows  $\leftrightarrow$  columns) this matrix gives the adjoint:

$$\text{adj}A = C^T = \begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix}$$

Substituting this and  $\det A = -35$  into **11.9**

$$A^{-1} = \frac{1}{\det A} (\text{adj}A) = \frac{1}{-35} \begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix}$$

Substituting this into **\*** yields

$$\begin{aligned} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} &= A^{-1} \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} \\ &= \frac{1}{-35} \begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} \\ &= -\frac{1}{35} \begin{pmatrix} [-14 \times 4] + [-7 \times 2] + [0 \times (-1)] \\ [-25 \times 4] + [0 \times 2] + [5 \times (-1)] \\ [29 \times 4] + [7 \times 2] + [-10 \times (-1)] \end{pmatrix} = -\frac{1}{35} \begin{pmatrix} -70 \\ -105 \\ 140 \end{pmatrix} \\ \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} &= -\frac{1}{35} \begin{pmatrix} -70 \\ -105 \\ 140 \end{pmatrix} \\ &= \begin{pmatrix} -70/(-35) \\ -105/(-35) \\ 140/(-35) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \end{aligned}$$

Hence  $i_1 = 2$  A,  $i_2 = 3$  A and  $i_3 = -4$  A.

You can apply the inverse matrix method as described in **Example 22** to solve 3, 4, 5, . . . simultaneous equations. It might seem long and tedious but generally solving three or more equations is more laborious unless you use a graphical calculator or a computer algebra system.

**SUMMARY**

The determinant of a  $3 \times 3$  matrix is

$$\mathbf{11.6} \quad \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \left[ \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} \right] - b \left[ \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} \right] + c \left[ \det \begin{pmatrix} d & e \\ g & h \end{pmatrix} \right]$$

The cofactor of  $a$  in **11.6** is the determinant of the matrix after deleting the row and column containing  $a$  with a place sign.

The cofactor matrix is the matrix of cofactors and the adjoint is defined as the cofactor matrix transposed.

The inverse of a square matrix  $\mathbf{A}$  is

$$\mathbf{11.9} \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} (\text{adj} \mathbf{A}) \quad [\text{Provided that } \det \mathbf{A} \neq 0]$$

**Exercise 11(c)**

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- 1** Calculate the determinants of the following matrices:

$$\mathbf{a} \quad \mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ -6 & 3 & 1 \end{pmatrix}$$

$$\mathbf{b} \quad \mathbf{B} = \begin{pmatrix} 2 & -10 & 11 \\ 5 & 3 & -4 \\ 7 & 9 & 12 \end{pmatrix}$$

$$\mathbf{c} \quad \mathbf{C} = \begin{pmatrix} -12 & 9 & -5 \\ 3 & 9 & 1 \\ -7 & 2 & -2 \end{pmatrix}$$

- 2** Find  $\mathbf{A}^T$  ( $\mathbf{A}$  transposed) for the following:

$$\mathbf{a} \quad \mathbf{A} = \begin{pmatrix} -2 & 3 \\ 1 & 5 \end{pmatrix} \quad \mathbf{b} \quad \mathbf{A} = \begin{pmatrix} 1/3 & -2/5 \\ -3/7 & \pi \end{pmatrix}$$

$$\mathbf{c} \quad \mathbf{A} = \begin{pmatrix} 7 & 3 & 4 \\ 2 & 6 & 1 \\ -3 & -3 & 1 \end{pmatrix}$$

$$\mathbf{d} \quad \mathbf{A} = \begin{pmatrix} 1.17 & 1.36 \\ 9.39 & -1.45 \\ 2.11 & 5.20 \end{pmatrix}$$

$$\mathbf{e} \quad \mathbf{A} = \begin{pmatrix} -7 & 2 & 1 & 5 \\ 3 & 6 & -4 & 7 \\ 8 & 3 & -3 & 5 \\ -4 & 6 & 7 & 0 \end{pmatrix}$$

$$\mathbf{f} \quad \mathbf{A} = \begin{pmatrix} 1 & -3 & 7 \\ -9 & 4 & 6 \\ -4 & 2 & 8 \\ 9 & 19 & 11 \end{pmatrix}$$

- 3** Show that


$$\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & -2 \\ 4 & 2 & 7 \end{pmatrix} = 25\mathbf{i} - 57\mathbf{j} + 2\mathbf{k}$$

- 4** Find the values of  $t$  so that

$$\det \begin{pmatrix} 1 & 0 & 3 \\ 5 & t & -7 \\ 3 & 9 & t-1 \end{pmatrix} = 0$$

- 5** Find the cofactor matrix  $\mathbf{C}$  and  $\mathbf{C}^T$  of

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 3 & 7 \\ 6 & -1 & 0 \end{pmatrix}$$

- 6**  [electrical principles] By applying Kirchhoff's law and Ohm's law to a circuit, we obtain the following equations:

$$3i_1 - 5i_2 + 3i_3 = 7.5$$

$$2i_1 + i_2 - 7i_3 = -17.5$$

$$-10i_1 + 4i_2 + 5i_3 = 16$$

where  $i_1$ ,  $i_2$  and  $i_3$  represent currents. By using the inverse matrix method, find the values of  $i_1$ ,  $i_2$  and  $i_3$ .

## Exercise 11(c) continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

7 Find the determinant of the following:

$$\mathbf{a} \quad \mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 0 & 0 & 6 \\ 1 & 5 & 3 \end{pmatrix}$$

$$\mathbf{b} \quad \mathbf{B} = \begin{pmatrix} 6 & 7 & 1 \\ 1 & 3 & 2 \\ 0 & 1 & 5 \end{pmatrix}$$

$$\mathbf{c} \quad \mathbf{C} = \begin{pmatrix} 1 & 5 & 1 \\ 0 & 3 & 7 \\ 0 & 2 & 9 \end{pmatrix}$$

$$\mathbf{d} \quad \mathbf{D} = \begin{pmatrix} 9 & 5 & 1 \\ 13 & 0 & 2 \\ 11 & 0 & 3 \end{pmatrix}$$

For the following use a computer algebra system or a graphical calculator.

8 Evaluate the determinant of

$$\mathbf{a} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

$$\mathbf{b} \quad \mathbf{B} = \begin{pmatrix} -1.10 & 4.23 & 2.67 & 7.45 & 9.62 \\ 19.61 & 6.40 & 3.12 & 11.89 & 2.36 \\ -17.5 & -9.73 & 5.23 & 8.54 & 2.51 \\ 6.19 & 2.91 & 17.64 & 8.93 & 8.98 \\ 3.98 & 11.84 & 4.78 & 9.85 & 3.22 \end{pmatrix}$$

## SECTION D Gaussian elimination

By the end of this section you will be able to:

- understand the process of Gaussian elimination
- apply the Gaussian elimination procedure to a system

Solving three (or more) simultaneous equations by the inverse matrix method can be a lengthy task, as we observed in the previous section. The Gaussian elimination procedure is often a more straightforward method for solving simultaneous linear equations.

## D1 Gaussian elimination

The Gaussian elimination procedure (named after the great German mathematician **K.F. Gauss**) is often used in computer software to solve systems of linear equations.

For square matrices, MATLAB uses Gaussian elimination to reduce a linear system to an appropriate format.

## Example 23

Solve the following linear equations using the Gaussian elimination procedure:

$$\begin{aligned} x - 3y + 5z &= -9 \\ 2x - y - 3z &= 19 \\ 3x + y + 4z &= -13 \end{aligned}$$

### Example 23 *continued*

Solution

This can be written in matrix form as

$$\begin{pmatrix} 1 & -3 & 5 \\ 2 & -1 & -3 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -9 \\ 19 \\ -13 \end{pmatrix}$$



**What are we trying to find?**

The values of  $x$ ,  $y$  and  $z$  which satisfy the given equations.

If we can somehow transform the above matrix to

$$\begin{array}{l} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{array} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & A \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} * \\ * \\ B \end{pmatrix}$$



then we can find  $z$ . **How?**

Look at the last row of the  $3 \times 3$  matrix in **11.10** and multiply it out with the adjacent

column vector,  $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ . (An  $n \times 1$  matrix is generally called a column vector.) We obtain

$$(0 \times x) + (0 \times y) + (A \times z) = B$$

$$Az = B \text{ gives } z = \frac{B}{A}$$

Therefore from the last row we can find  $z$ .

Look at row 2. On the Left-Hand Side we get an equation in  $y$  and  $z$ . From above we know the value of  $z = B/A$ , so we can substitute  $z = B/A$  and also obtain  $y$ . Similarly from the first row we can find  $x$  by substituting the values of  $y$  and  $z$ .

The matrix in **11.10** shows that we need to achieve 0's in the position shown in order to create equations with fewer unknowns so that by substitution we can find the answers.

Let's try this method for the matrix in question:

$$\begin{pmatrix} 1 & -3 & 5 \\ 2 & -1 & -3 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -9 \\ 19 \\ -13 \end{pmatrix}$$

We need to enter the Right-Hand Side within the matrix:

$$\begin{array}{l} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \begin{pmatrix} 1 & -3 & 5 & -9 \\ 2 & -1 & -3 & 19 \\ 3 & 1 & 4 & -13 \end{pmatrix}$$

This is called an **augmented** matrix.  $R_1$ ,  $R_2$  and  $R_3$  just refer to row 1, row 2 and row 3 respectively. We need to obtain the matrix in **11.10** from the above matrix, therefore we



Example 23 *continued*

require zeros in the bottom left-hand corner of this matrix; that is, the 2 from the second row must become zero and the 3 and 1 from the bottom row must become zero. Hence  $2 \rightarrow 0$ ,  $3 \rightarrow 0$  and  $1 \rightarrow 0$ .

?

**How is this achieved?**

Recall from our work with simultaneous equations in **Chapter 1**, that we can multiply an entire equation by a non-zero constant, and take one equation away from another. In terms of matrices this means that we can multiply a whole row by a non-zero constant and take one row away from another.

?

**How do we get 0 in place of 2?**

To get 0 in place of 2, we multiple row 1,  $R_1$ , by 2 and subtract from row 2,  $R_2$ ; that is, we carry out the row operation  $R_2 - 2R_1$ :

$$\begin{array}{l} R_1 \\ R_2^* = R_2 - 2R_1 \\ R_3 \end{array} \left( \begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 2 - 2(1) & -1 - [2 \times (-3)] & -3 - (2 \times 5) & 19 - (2 \times (-9)) \\ 3 & 1 & 4 & -13 \end{array} \right)$$

We call the new middle row  $R_2^*$  say. Simplifying the middle row gives

$$\begin{array}{l} R_1 \\ R_2^* \\ R_3 \end{array} \left( \begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 3 & 1 & 4 & -13 \end{array} \right)$$

?

**Where else do we need a 0?**

?

We need to get a 0 in place of 3 in the bottom row. **How?**

We multiply the top row  $R_1$  by 3 and subtract from the bottom row  $R_3$ ; that is, we carry out the row operation,  $R_3 - 3R_1$ :

$$\begin{array}{l} R_1 \\ R_2^* \\ R_3^* = R_3 - 3R_1 \end{array} \left( \begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 3 - 3(1) & 1 - (3 \times (-3)) & 4 - (3 \times 5) & -13 - (3 \times (-9)) \end{array} \right)$$

We can call  $R_3^*$  the new bottom row of this matrix. (Of course you can use other symbols such as small  $r$  for the new rows, that is,  $r_2$  and  $r_3$ .)

Simplifying the arithmetic in the entries gives:

$$\begin{array}{l} R_1 \\ R_2^* \\ R_3^* \end{array} \left( \begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 0 & 10 & -11 & 14 \end{array} \right)$$

?

Note that we only need to convert the 10 into 0 in the bottom row. **How do we get a 0 in place of 10?**

### Example 23 *continued*

We can **only** use the bottom two rows,  $R_2^*$  and  $R_3^*$ , otherwise we get a non-zero number in place of zeros already established. We execute  $R_3^{**} - 2R_2^*$  because

$$10 - (2 \times 5) = 0 \text{ (gives a 0 in place of 10)}$$

Therefore we have:

$$\begin{array}{l} R_1 \\ R_2^* \\ R_3^{**} = R_3^* - 2R_2^* \end{array} \left( \begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 0 - (2 \times 0) & 10 - (2 \times 5) & -11 - [2 \times (-13)] & 14 - (2 \times 37) \end{array} \right)$$

which simplifies to

$$\begin{array}{l} R_1 \\ R_2^* \\ R_3^{**} \end{array} \begin{array}{ccc|c} x & y & z & \\ \hline (†) & \left( \begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 0 & 0 & 15 & -60 \end{array} \right) \end{array}$$

From the bottom row  $R_3^{**}$  we have:

$$15z = -60 \text{ which gives } z = -\frac{60}{15} = -4$$



**How do we find the other two unknowns  $x$  and  $y$ ?**

By expanding the middle row  $R_2^*$  of (†) we have:

$$5y - 13z = 37$$

We can find  $y$  by substituting  $z = -4$  into this:

$$5y - 13(-4) = 37 \quad [\text{Substituting } z = -4]$$

$$5y + 52 = 37 \text{ implies that } 5y = -15, \text{ therefore } y = -\frac{15}{5} = -3$$

**How can we find the last unknown  $x$ ?**

By expanding the first row  $R_1$  of (†) we have:

$$x - 3y + 5z = -9$$

Substituting  $y = -3$  and  $z = -4$  into this:

$$x - [3 \times (-3)] + [5 \times (-4)] = -9$$

$$x + 9 - 20 = -9 \text{ which gives } x = 2$$

Hence our solution to the linear system is  $x = 2$ ,  $y = -3$  and  $z = -4$ . Check that this solution is correct by substituting these values into the given equations:

$$\begin{array}{ll} x - 3y + 5z = -9 & 2 - 3(-3) + 5(-4) = -9 \\ 2x - y - 3z = 19 & 2(2) - (-3) - 3(-4) = 19 \\ 3x + y + 4z = -13 & 3(2) + (-3) + 4(-4) = -13 \end{array}$$

The above process is called **Gaussian elimination**. We can:

- 1 interchange any two rows or columns (apart from the augmented column)
- 2 multiply any row by a non-zero scalar
- 3 add or subtract rows from each other.

The aim of the Gaussian elimination process is to produce the 'triangular' matrix shown in **11.10**, that is, zeros in the lower triangle.

When interchanging two columns we need to be careful. Consider the matrix used in

**Example 23:**

$$\begin{array}{ccc} x & y & z \\ \left( \begin{array}{ccc} 1 & -3 & 5 \\ 2 & -1 & -3 \\ 3 & 1 & 4 \end{array} \right) \end{array}$$

Remember that column 1 represents the coefficients of  $x$ , column 2 represents the coefficients of  $y$  and column 3 represents the coefficients of  $z$ . So when interchanging any of these columns we need to remember the variable associated with that column.

Interchanging column 1 and column 3 in the above gives

$$\begin{array}{ccc} z & y & x \\ \left( \begin{array}{ccc} 5 & -3 & 1 \\ -3 & -1 & 2 \\ 4 & 1 & 3 \end{array} \right) \end{array}$$



### Example 24 *mechanics*

A pulley system gives the following equations

$$\begin{aligned} \ddot{x}_1 + \ddot{x}_2 &= 0 \\ 2\ddot{x}_1 &= 20 - T \\ 5\ddot{x}_2 &= 50 - T \end{aligned}$$

where  $\ddot{x}_1, \ddot{x}_2$  represent acceleration and  $T$  represents tension in the rope. Determine  $\ddot{x}_1, \ddot{x}_2$  and  $T$  by using Gaussian elimination.

Solution

Rearranging the above equations gives

$$\begin{aligned} \ddot{x}_1 + \ddot{x}_2 &= 0 \\ 2\ddot{x}_1 + T &= 20 \\ 5\ddot{x}_2 + T &= 50 \end{aligned}$$

Putting this in matrix form:

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ T \end{pmatrix} = \begin{pmatrix} 0 \\ 20 \\ 50 \end{pmatrix}$$

Example 24 *continued*

In augmented matrix form we have

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left( \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 20 \\ 0 & 5 & 1 & 50 \end{array} \right)$$

To get a 0 in place of 2 we execute  $R_2 - 2R_1$ :

$$\begin{array}{l} R_1 \\ R_2^* \\ R_3 \end{array} = \begin{array}{l} R_1 \\ R_2 - 2R_1 \\ R_3 \end{array} \left( \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 20 \\ 0 & 5 & 1 & 50 \end{array} \right)$$



We need to get 0 in place of 5 in the bottom row,  $R_3$ . How?

Carrying out the row operation  $R_3 + \frac{5}{2}R_2^*$ :

$$\begin{array}{l} R_1 \\ R_2^* \\ R_3^* \end{array} = \begin{array}{l} R_1 \\ R_2^* \\ R_3 + \frac{5}{2}R_2^* \end{array} \left( \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 20 \\ 0 & 0 & 7/2 & 100 \end{array} \right)$$

From the bottom row,  $R_3^*$ , we can now conclude

$$\frac{7}{2}T = 100 \text{ which gives } T = \frac{200}{7} \text{ N}$$

Substituting  $T = \frac{200}{7}$  into the middle row,  $R_2^*$ , we have

$$-2\ddot{x}_2 + \frac{200}{7} = 20$$

Solving

$$\ddot{x}_2 = \frac{30}{7} \text{ m/s}^2$$

Similarly from the first row,  $R_1$ , we obtain  $\ddot{x}_1 = -\frac{30}{7} \text{ m/s}^2$ .

Check for yourself that these values do indeed satisfy the original equations.

Example 25 *electrical principles*

The mesh equations of a circuit are given by:

$$\begin{aligned} i_1 + 3i_2 + 2i_3 &= 13 \\ 4i_1 + 4i_2 - 3i_3 &= 3 \\ 5i_1 + i_2 + 2i_3 &= 13 \end{aligned}$$

Determine the values of  $i_1$ ,  $i_2$  and  $i_3$ .



### Example 25 *continued*

Solution



**How can we find the unknowns  $i_1$ ,  $i_2$  and  $i_3$ ?**

We use **Gaussian elimination with back substitution**.

Let's try this method for the given equations. The augmented matrix is:

$$\begin{array}{l} \text{Row 1 } R_1 \\ \text{Row 2 } R_2 \\ \text{Row 3 } R_3 \end{array} \left( \begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 4 & 4 & -3 & 3 \\ 5 & 1 & 2 & 13 \end{array} \right)$$

We need to convert this matrix so that there are zeros in the bottom left-hand corner triangle of this matrix, that is, first 4 in the second row needs to become 0, and 5 and 1 from the bottom row also need to become 0. Hence  $4 \rightarrow 0$ ,  $5 \rightarrow 0$  and  $1 \rightarrow 0$ .

To get 0 in place of the first 4 in the middle row we multiply row 1,  $R_1$ , by 4 and take it away from row 2,  $R_2$ , that is,  $R_2 - 4R_1$ . To get 0 in place of 5 in the bottom row we multiply row 1,  $R_1$ , by 5 and take away from row 3,  $R_3$ , that is,  $R_3 - 5R_1$ . Combining the two row operations,  $R_2 - 4R_1$  and  $R_3 - 5R_1$ , we have

$$\begin{array}{l} R_1 \\ R_2^\dagger = R_2 - 4R_1 \\ R_3^\dagger = R_3 - 5R_1 \end{array} \left( \begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 4 - (4 \times 1) & 4 - (4 \times 3) & -3 - (4 \times 2) & 3 - (4 \times 13) \\ 5 - (5 \times 1) & 1 - (5 \times 3) & 2 - (5 \times 2) & 13 - (5 \times 13) \end{array} \right)$$

We call the new rows 2 and 3 by  $R_2^\dagger$  and  $R_3^\dagger$  respectively. This simplifies to

$$\begin{array}{l} R_1 \\ R_2^\dagger \\ R_3^\dagger \end{array} \left( \begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & -14 & -8 & -52 \end{array} \right)$$

The new row 3 is called  $R_3^\dagger$  say.

We have nearly obtained the required matrix with zeros in the bottom left-hand corner.



We need a 0 in place of  $-14$  in the new bottom row,  $R_3^\dagger$ . We can **only** use the second and third rows,  $R_2^\dagger$  and  $R_3^\dagger$ . **Why?**

Because if we use the first and second rows,  $R_1$  and  $R_2^\dagger$ , we will get a non-zero number in place of the zero already established in  $R_2^\dagger$ . Similarly this holds for  $R_1$  and  $R_3^\dagger$ .



**How can we obtain 0 in place of  $-14$ ?**

$$R_3^\dagger - \frac{14}{8}R_2^\dagger \text{ because } -14 - \left[ \frac{14}{8} \times (-8) \right] = 0 \quad \text{[gives 0 in place of } -14\text{]}$$

Therefore

$$\begin{array}{l} R_1 \\ R_2^\dagger \\ R_3^{\dagger\dagger} = R_3^\dagger - \frac{14}{8}R_2^\dagger \end{array} \left( \begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & -14 - \left[ \frac{14}{8} \times (-8) \right] & -8 - \left[ \frac{14}{8} \times (-11) \right] & -52 - \left[ \frac{14}{8} \times (-49) \right] \end{array} \right)$$



### Example 25 *continued*

which simplifies to

$$\begin{array}{l}
 R_1 \\
 * \\
 R_2^\dagger \\
 R_3^{\dagger\dagger}
 \end{array}
 \left( \begin{array}{ccc|c}
 i_1 & i_2 & i_3 & \\
 1 & 3 & 2 & 13 \\
 0 & -8 & -11 & -49 \\
 0 & 0 & 45/4 & 135/4
 \end{array} \right)$$

This time we have called the bottom row  $R_3^{\dagger\dagger}$ . From this row,  $R_3^{\dagger\dagger}$ , we have

$$\frac{45}{4} i_3 = \frac{135}{4} \text{ which gives } i_3 = \frac{135}{45} = 3$$



**How can we find the unknown  $i_2$ ?**

Examining the second row  $R_2^\dagger$  in  $*$  we have

$$-8i_2 - 11i_3 = -49$$

We have established that  $i_3 = 3$ , therefore substituting  $i_3 = 3$  gives

$$\begin{aligned}
 -8i_2 - 11(3) &= -49 \\
 -8i_2 - 33 &= -49 \\
 -8i_2 &= -49 + 33 = -16 \text{ which yields } i_2 = (-16)/(-8) = 2
 \end{aligned}$$



So far we have  $i_2 = 2$  and  $i_3 = 3$ . **How can we find the last unknown  $i_1$ ?**

By expanding the first row,  $R_1$ , of  $*$  we have

$$i_1 + 3i_2 + 2i_3 = 13$$

Substituting our values already found,  $i_2 = 2$  and  $i_3 = 3$ , we have

$$\begin{aligned}
 i_1 + (3 \times 2) + (2 \times 3) &= 13 \\
 i_1 + 6 + 6 &= 13 \text{ which gives } i_1 = 1
 \end{aligned}$$

Hence  $i_1 = 1$ ,  $i_2 = 2$  and  $i_3 = 3$  is our solution. You can check that this solution is correct by substituting these values,  $i_1 = 1$ ,  $i_2 = 2$  and  $i_3 = 3$ , into the given equations.

## SUMMARY

The Gaussian elimination procedure obtains the following triangular matrix

$$\text{11.10} \quad \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

and then uses back substitution to solve the given linear equations.

## Exercise 11(d)


Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

1 By using Gaussian elimination, solve

$$\begin{aligned} \mathbf{a} \quad x + 2y - 3z &= 3 \\ 2x - y - z &= 11 \\ 3x + 2y + z &= -5 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 2x + y + 2z &= 10 \\ x + 4y - 3z &= 0 \\ 3x + 6y - z &= 12 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad x + y + 2z &= 1 \\ 2x + 3y + 2z &= 2 \\ 5x + 2y + 8z &= 4 \end{aligned}$$


2  [electrical principles] Mesh equations of a circuit are given by

$$\begin{aligned} (9 \times 10^3)i_1 - (3 \times 10^3)i_2 &= 10 \\ -(3 \times 10^3)i_1 + (13 \times 10^3)i_2 &= 0 \end{aligned}$$


By using Gaussian elimination, solve these equations for the currents  $i_1$  and  $i_2$ .[Hint: Work in kilo (lose the  $10^3$ ), then the answer comes out automatically in mA ( $10^{-3}$ ) for  $i_1$  and  $i_2$ .]

3 By using Gaussian elimination solve


$$\begin{aligned} 10x + y - 5z &= 18 \\ -20x + 3y + 20z &= 14 \\ 5x + 3y + 5z &= 9 \end{aligned}$$

4  [mechanics] A simple pulley system gives the equations

$$\begin{aligned} \ddot{x}_1 &= T - g \\ 2\ddot{x}_2 &= T - 2g \\ \ddot{x}_1 + \ddot{x}_2 &= 0 \end{aligned}$$

where  $\ddot{x}_1, \ddot{x}_2$  represent acceleration,  $T$  is the tension in the rope and  $g$  is the acceleration due to gravity. Determine  $\ddot{x}_1, \ddot{x}_2$  and  $T$ .5  [mechanics] A simple pulley system gives the equations

$$\begin{aligned} \ddot{x}_1 + \ddot{x}_2 &= 0 \\ -3\ddot{x}_1 + T &= 3g \\ -2\ddot{x}_2 + T &= 2g \end{aligned}$$

where  $\ddot{x}_1, \ddot{x}_2$  represent acceleration,  $T$  is the tension in the rope and  $g$  is the acceleration due to gravity. Determine  $\ddot{x}_1, \ddot{x}_2$  and  $T$ .6  [electrical principles] The mesh equations of a circuit are given by

$$\begin{aligned} (4 \times 10^3)I_1 - (3 \times 10^3)I_2 &= 10 \\ -(3 \times 10^3)I_1 + (18 \times 10^3)I_2 \\ - (10 \times 10^3)I_3 &= 0 \\ -(10 \times 10^3)I_2 + (23 \times 10^3)I_3 &= -15 \end{aligned}$$

By using Gaussian elimination, solve these equations for  $I_1, I_2$  and  $I_3$ .

[See Hint for question 2.]

## SECTION E Linear equations

By the end of this section you will be able to:

- investigate equations which can be written in the form  $\mathbf{A}\mathbf{u} = \mathbf{b}$
- understand that a solution to  $\mathbf{A}\mathbf{u} = \mathbf{b}$  might still exist even if  $\mathbf{A}^{-1}$  does not exist
- understand the concept of an infinite number of solutions

## E1 Properties of linear equations

Linear equations are equations whose variables have an index of 1 or 0. So far we have looked at linear equations of the form

$$\begin{aligned}2x + 3y + 5z &= 9 \\x + 2y + 7z &= 13 \\4x + 11y + 3z &= -1\end{aligned}$$

These are called a system of linear equations.

**?** How can we write these in matrix form?

$$\begin{pmatrix} 2 & 3 & 5 \\ 1 & 2 & 7 \\ 4 & 11 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 13 \\ -1 \end{pmatrix}$$

Also this can be written as

$$\mathbf{A}\mathbf{u} = \mathbf{b} \quad \text{where } \mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 2 & 7 \\ 4 & 11 & 3 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 9 \\ 13 \\ -1 \end{pmatrix}.$$

**?** How do we find the values of  $x$ ,  $y$  and  $z$ ?

We find the inverse matrix  $\mathbf{A}^{-1}$  and multiply both sides by  $\mathbf{A}^{-1}$  to obtain

$$\begin{aligned}\mathbf{A}^{-1}\mathbf{A}\mathbf{u} &= \mathbf{A}^{-1}\mathbf{b} \\ \mathbf{I}\mathbf{u} &= \mathbf{A}^{-1}\mathbf{b} \quad [\mathbf{I} \text{ is the identity matrix}] \\ \mathbf{u} &= \mathbf{A}^{-1}\mathbf{b}\end{aligned}$$

This solution,  $\mathbf{u}$ , is unique. In earlier sections we used inverse matrix,  $\mathbf{A}^{-1}$ , to solve linear equations. The solutions of simultaneous equations depend on the determinant of matrix  $\mathbf{A}$  which we examine next.

We need to find  $\mathbf{A}^{-1}$  for this method to work. We can only find  $\mathbf{A}^{-1}$  if  $\det \mathbf{A} \neq 0$  because

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}(\text{adj} \mathbf{A}) \text{ and of course we cannot divide by 0.}$$

In general if  $\mathbf{A}$  is an  $n \times n$  matrix,  $\mathbf{u}$  and  $\mathbf{b}$  are  $n \times 1$  matrices ( $n \times 1$  matrices are called vectors) such that

$$\mathbf{11.11} \quad \mathbf{A}\mathbf{u} = \mathbf{b} \text{ with } \det \mathbf{A} \neq 0$$

then we have a unique solution, that is, we have a unique  $\mathbf{u}$  which satisfies  $\mathbf{A}\mathbf{u} = \mathbf{b}$ .

**?** However if  $\det \mathbf{A} = 0$  then what is the inverse matrix,  $\mathbf{A}^{-1}$ , of  $\mathbf{A}$ ?

We **cannot** find  $\mathbf{A}^{-1}$  because  $\mathbf{A}^{-1}$  is defined with  $\det \mathbf{A}$  in the denominator.

Hence the inverse matrix,  $\mathbf{A}^{-1}$ , does **not** exist.

**?** How can we solve linear equations when  $\mathbf{A}^{-1}$  does not exist? Is it possible?

In the next example we examine these questions.



## Example 26

Solve for  $x$  and  $y$ :

$$\begin{array}{ll} \mathbf{a} & 2x + 3y = 7 \\ & 2x + 3y = 12 \end{array} \qquad \begin{array}{ll} \mathbf{b} & 2x + 10y = 10 \\ & 6x + 30y = 30 \end{array}$$

Solution

**a** Writing equations **a** in matrix form:

$$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \end{pmatrix}$$

Let  $A = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$ , then

$$\det A = (2 \times 3) - (2 \times 3) = 0 \quad [\text{By } 11.1]$$

?

**What is  $A^{-1}$ ?**Hence we do not have an inverse matrix,  $A^{-1}$ .

?

**So what are the values of  $x$  and  $y$ ?**

Look at these equations:

$$* \quad 2x + 3y = 7$$

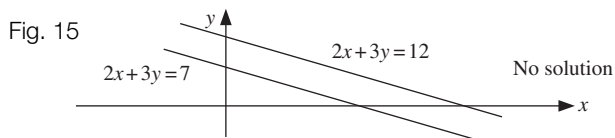
$$** \quad 2x + 3y = 12$$

?

**What do you notice?**

The coefficients of  $x$  and  $y$  are the same but the Right-Hand Side values are different, 7 and 12. Clearly they are **inconsistent**. Moreover if we subtract **\*\*** from **\*** then we obtain  $0 = -5$  which is clearly incorrect. Thus there is **no** solution to these simultaneous equations. The graphs of these equations are shown in Fig. 15.

We have parallel lines therefore the lines don't cross, which explains why there is no solution to these simultaneous equations.

**b** Writing the equations **b** in matrix form:

$$\begin{pmatrix} 2 & 10 \\ 6 & 30 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 30 \end{pmatrix}$$

Let  $A = \begin{pmatrix} 2 & 10 \\ 6 & 30 \end{pmatrix}$  then

$$\det A = (2 \times 30) - (6 \times 10) = 0 \quad [\text{By } 11.1]$$

$$11.1 \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

Example 26 *continued*

?

What is  $A^{-1}$ ?

?

Again  $A^{-1}$  does not exist. Are there any solutions to these equations?

?

Yes, try  $x = 0, y = 1$ . Do these equations have any other solutions?

Many:

$$\begin{aligned}x &= 2, & y &= 3/5 \\x &= 3, & y &= 2/5 \\x &= -1, & y &= 6/5 \\&\vdots & &\vdots\end{aligned}$$

These equations have an infinite number of solutions. They can be found as follows:

The equations are

$$\dagger \quad 2x + 10y = 10$$

$$\dagger\dagger \quad 6x + 30y = 30$$

From  $\dagger$  we have

$$\begin{aligned}10y &= 10 - 2x \\y &= \frac{10 - 2x}{10} = \frac{10}{10} - \frac{2}{10}x = 1 - \frac{x}{5} \quad \text{[Cancelling]}\end{aligned}$$

Thus we have  $y = 1 - \frac{x}{5}$ .

Let  $x = a$ , where  $a$  is any real number, then

$$y = 1 - \frac{a}{5}$$

Putting  $a = 1$  we obtain  $x = 1, y = 4/5$ .

Putting  $a = 2$  we obtain  $x = 2, y = 3/5$  and so on.

?

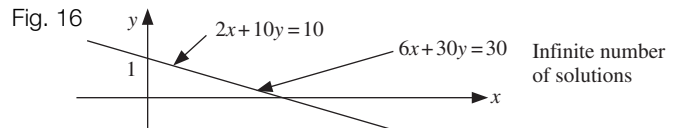
What do you notice about equations b?

$$2x + 10y = 10$$

$$6x + 30y = 30$$

The second equation is three times the first equation. The graphs of these are shown in Fig. 16.

Since the lines lie on top of each other, we have an infinite number of solutions.



If we consider the general  $n$  simultaneous equations written in matrix form as

$$\mathbf{11.12} \quad \mathbf{A}\mathbf{u} = \mathbf{b} \quad [\mathbf{A} \text{ is a square matrix, } \mathbf{b} \neq \mathbf{0}]$$

with  $\det A = 0$ , then the inverse matrix,  $A^{-1}$ , does not exist. Furthermore 11.12 satisfies one of the following:

- i there is no solution (there is no  $\mathbf{u}$  satisfying  $\mathbf{A}\mathbf{u} = \mathbf{b}$ ), or
- ii there are an infinite number of solutions (there are an infinite number of  $\mathbf{u}$ 's which satisfy  $\mathbf{A}\mathbf{u} = \mathbf{b}$ ).

In **Example 26a**:  $A = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$ ,  $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} 7 \\ 12 \end{pmatrix}$  and there were **no** solutions.

In **Example 26b**:  $A = \begin{pmatrix} 2 & 10 \\ 6 & 30 \end{pmatrix}$ ,  $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} 10 \\ 30 \end{pmatrix}$  and we had an **infinite** number

of solutions of the form:  $x = a$ ,  $y = 1 - \frac{a}{5}$  where  $a$  is any real number.

In both cases we had  $\det A = 0$ .

Now we look at equations of the form

$$\mathbf{A}\mathbf{u} = \mathbf{0} \text{ with } \det A = 0$$

Note that the Right-Hand Side is zero. Consider the equations

$$* \quad 2x + 10y = 0$$

$$** \quad 6x + 30y = 0$$

From **\*** we have  $x = -5y$ , that is,  $x$  is  $-5$  times the  $y$  value. Hence  $y = 1$ ,  $x = -5$  is a solution to both equations **\*** and **\*\***. The following are also solutions of the simultaneous equations:

$$\begin{array}{l} y = 2, \quad x = -10 \\ y = 3, \quad x = -15 \\ y = 4, \quad x = -20 \\ \vdots \quad \quad \quad \vdots \end{array}$$

There are an infinite number of solutions. The equations can be written in matrix form as

$$\begin{pmatrix} 2 & 10 \\ 6 & 30 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or  $\mathbf{A}\mathbf{u} = \mathbf{0}$

where  $A = \begin{pmatrix} 2 & 10 \\ 6 & 30 \end{pmatrix}$ ,  $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ ,  $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  and  $\det A = 0$ .

In general if

$$11.13 \quad \mathbf{A}\mathbf{u} = \mathbf{0} \text{ with } \det A = 0$$

then 11.13 has an infinite number of solutions. That is an infinite number of  $\mathbf{u}$ 's that satisfy  $\mathbf{A}\mathbf{u} = \mathbf{0}$ .

Now consider equations of the form

$$\mathbf{A}\mathbf{u} = \mathbf{0} \text{ with } \det A \neq 0$$

**?** Does  $A^{-1}$  exist?

Yes. Multiply both sides by  $A^{-1}$

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{A}\mathbf{u} &= \mathbf{A}^{-1}\mathbf{0} = \mathbf{0} \\ \mathbf{I} \mathbf{u} &= \mathbf{0} \\ \text{because} & \\ \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} & \\ \mathbf{u} &= \mathbf{0} \quad [\text{Because } \mathbf{I}\mathbf{u} = \mathbf{u}] \end{aligned}$$

The solution  $\mathbf{u} = \mathbf{0}$  means  $x = 0, y = 0, z = 0, \dots$ . This is known as the **trivial** solution.

In general if

$$\mathbf{11.14} \quad \mathbf{A}\mathbf{u} = \mathbf{0} \text{ with } \det\mathbf{A} \neq 0$$

then there is only the trivial solution,  $x = 0, y = 0, z = 0, \dots$ .

Summarizing the above, we have four cases of

$$\ast \quad \mathbf{A}\mathbf{u} = \mathbf{b}$$

$$\mathbf{11.11} \quad \text{If } \det\mathbf{A} \neq 0 \text{ then } \ast \text{ has a unique solution}$$

$$\mathbf{11.12} \quad \text{If } \mathbf{b} \neq \mathbf{0}, \det\mathbf{A} = 0 \text{ then either } \ast \text{ has no solution or there are an infinite number of solutions}$$

$$\mathbf{11.13} \quad \text{If } \mathbf{b} = \mathbf{0}, \det\mathbf{A} = 0 \text{ then } \ast \text{ has an infinite number of solutions}$$

$$\mathbf{11.14} \quad \text{If } \mathbf{b} = \mathbf{0}, \det\mathbf{A} \neq 0 \text{ then } \ast \text{ only has the trivial solution } \mathbf{u} = \mathbf{0}; x = 0, y = 0, z = 0, \dots$$

Note that for the case with  $\mathbf{b} = \mathbf{0}$  we have non-trivial solutions only if  $\det\mathbf{A} = 0$ .

The next three examples look into the various types of solution outlined in **11.11** to **11.14**.

### Example 27

Find all the solutions of

$$\begin{aligned} x + 2y + z &= 4 \\ 3x - 4y - 2z &= 2 \\ 5x + 3y + 5z &= -1 \end{aligned}$$

Solution

These equations can be written in matrix form as

$$\mathbf{A}\mathbf{u} = \mathbf{b}$$

where  $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{pmatrix}$ ,  $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$ . The equations were solved in

**Example 22** with the variables  $i_1, i_2$  and  $i_3$  instead of  $x, y$  and  $z$ , hence

$$\ast \quad x = 2, y = 3 \text{ and } z = -4$$

Since  $\det\mathbf{A} = -35 \neq 0$  (evaluated in **Example 22**), we can conclude from **11.11** that these solutions are unique, that is  $x = 2, y = 3$  and  $z = -4$  are the only values which satisfy the given equations.

## Example 28

Find all the values of  $x$ ,  $y$  and  $z$  that satisfy

$$2x + 3y + 4z = 0$$

$$5x + 7y + 3z = 0$$

$$7x + 3y + 11z = 0$$

Solution

Putting these in matrix form gives

$$\begin{pmatrix} 2 & 3 & 4 \\ 5 & 7 & 3 \\ 7 & 3 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This is of the form  $\mathbf{A}\mathbf{u} = \mathbf{b}$  where  $\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 7 & 3 \\ 7 & 3 & 11 \end{pmatrix}$ ,  $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ .

We first obtain  $\det \mathbf{A}$ :

$$\begin{aligned} \det \begin{pmatrix} 2 & 3 & 4 \\ 5 & 7 & 3 \\ 7 & 3 & 11 \end{pmatrix} &= 2\det \begin{pmatrix} 7 & 3 \\ 3 & 11 \end{pmatrix} - 3\det \begin{pmatrix} 5 & 3 \\ 7 & 11 \end{pmatrix} + 4\det \begin{pmatrix} 5 & 7 \\ 7 & 3 \end{pmatrix} \\ &= 2[(7 \times 11) - (3 \times 3)] - 3[(5 \times 11) - (7 \times 3)] + 4[(5 \times 3) - (7 \times 7)] \\ &= -102 \end{aligned}$$

Since  $\det \mathbf{A} = -102 \neq 0$  we can conclude by [11.14](#) that we only have the trivial solution:

$$x = 0, y = 0 \text{ and } z = 0$$

## Example 29

Find the values of  $\lambda$  that give a non-trivial solution of

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{u} = \mathbf{0}$$

where  $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 7 & 1 \end{pmatrix}$ ,  $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$  and  $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  is the identity matrix.

Solution

This is quite a difficult example to follow. It requires use of some algebra as well as matrix manipulation.

By [11.13](#) we have a non-trivial solution if

$$\star \quad \det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

Example 29 *continued*

Substituting the given information:

$$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) &= \det\left[\begin{pmatrix} 2 & 3 \\ 7 & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \\ &= \det\left[\begin{pmatrix} 2 & 3 \\ 7 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] \\ &= \det\begin{pmatrix} 2-\lambda & 3 \\ 7 & 1-\lambda \end{pmatrix} \\ &\equiv (2-\lambda)(1-\lambda) - (7 \times 3) \\ &\quad \text{by 11.1} \\ &= 2 - 3\lambda + \lambda^2 - 21 = \lambda^2 - 3\lambda - 19\end{aligned}$$

By \* we equate this to zero:

$$\lambda^2 - 3\lambda - 19 = 0 \quad \text{[Quadratic equation]}$$

?

**How do we find  $\lambda$ ?**

By using the quadratic formula

$$1.16 \quad \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with  $a = 1$ ,  $b = -3$  and  $c = -19$ :

$$\begin{aligned}\lambda &= \frac{3 \pm \sqrt{9 - (4 \times 1 \times (-19))}}{2} \\ &= \frac{3 \pm \sqrt{85}}{2} \\ \lambda_1 &= \frac{3 + \sqrt{85}}{2}, \quad \lambda_2 = \frac{3 - \sqrt{85}}{2}\end{aligned}$$

Different  $\lambda$ 's are normally denoted by  $\lambda_1$  and  $\lambda_2$ .

These  $\lambda$  are called the **eigenvalues** of the matrix  $\mathbf{A}$ . Eigenvalues are the subject of the next section.

These terms *eigenvalue* and *eigenvector* are derived from the German word *Eigenwerte* which means 'proper value'. The word *eigen* is pronounced 'i-gun' (the eigenvalues are usually represented by the Greek letter lambda  $\lambda$ ).

## SUMMARY

A system of linear equations can be written as

$$* \quad \mathbf{A}\mathbf{u} = \mathbf{b}$$

It has the following properties:

$$11.11 \quad \text{If } \det\mathbf{A} \neq 0 \text{ then } * \text{ has a unique solution}$$

$$11.1 \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

- 11.12** If  $\mathbf{b} \neq \mathbf{0}$ ,  $\det \mathbf{A} = 0$  then either \* has no solution or there are an infinite number of solutions
- 11.13** If  $\mathbf{b} = \mathbf{0}$ ,  $\det \mathbf{A} = 0$  then \* has an infinite number of solutions
- 11.14** If  $\mathbf{b} = \mathbf{0}$ ,  $\det \mathbf{A} \neq 0$  then \* only has the trivial solution  $x = 0, y = 0, z = 0, \dots$

**Exercise 11(e)**

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

**1** Solve:

$$\begin{aligned}x + 3y + 5z &= 0 \\7x - y + z &= 0 \\2x + 3y - 8z &= 0\end{aligned}$$

**2** Solve and find whether the solution is unique:

$$\begin{aligned}2x + 3y &= 5 \\4x + 5y &= 9\end{aligned}$$

**3** Find  $x$  and  $y$  which satisfy:

$$\begin{aligned}2x + y &= 0 \\4x + 2y &= 0\end{aligned}$$

Is the solution unique?

**4** Test whether the solutions of questions **1**, **4** and **5** of **Exercise 11(d)** are unique.**5** Find the values of  $\lambda$  that give a non-trivial solution of

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{u} = \mathbf{0}$$

for

$$\mathbf{a} \quad \mathbf{A} = \begin{pmatrix} -1 & 1 \\ 9 & -1 \end{pmatrix} \quad \mathbf{b} \quad \mathbf{A} = \begin{pmatrix} 1 & 3 \\ -6 & 5 \end{pmatrix}$$

$$\left[ \mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \mathbf{I} \text{ is the identity matrix} \right]$$

**SECTION F Eigenvalues and eigenvectors**

By the end of this section you will be able to:

- understand the terms eigenvalues and eigenvectors
- find the eigenvalues and eigenvectors of  $2 \times 2$  matrices
- find the eigenvalues and eigenvectors of  $3 \times 3$  matrices
- examine vibrations of buildings

This is a challenging section because it assumes a thorough understanding of matrices and it also requires a proficiency in algebra on topics such as expansion of brackets, solving equations and substitution.

**F1 Introduction****Example 30**

If  $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$  and  $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  then evaluate  $\mathbf{A}\mathbf{u}$ .

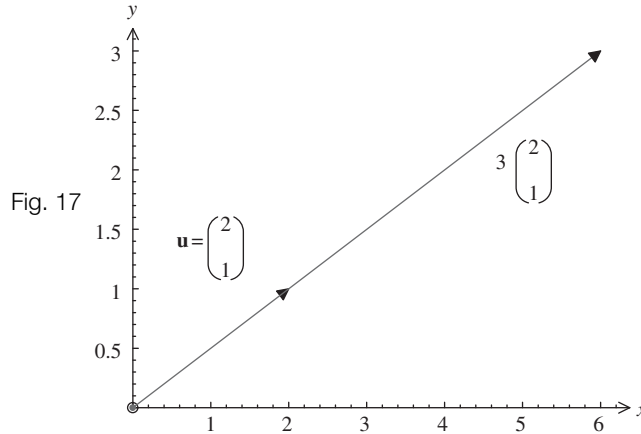
Example 30 *continued*

Solution

Multiplying the matrix and column vector:

$$\mathbf{A}\mathbf{u} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3\mathbf{u}$$

In geometric terms, this means that the matrix  $\mathbf{A}$  scalar multiples the vector  $\mathbf{u}$  by 3 (Fig. 17):



Note that we have  $\mathbf{A}\mathbf{u} = 3\mathbf{u}$ .

In general terms, this can be stated as

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$$

where  $\mathbf{A}$  is a square matrix,  $\mathbf{u}$  is a column vector and  $\lambda$  is a scalar (a number). Can you think of a vector,  $\mathbf{u}$ , which satisfies

$$\mathbf{u} = \mathbf{0}$$

This is called the trivial solution:  $x = 0, y = 0, z = 0 \dots$

$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$  is only interesting if  $\mathbf{u} \neq \mathbf{0}$ , otherwise we always have the trivial solution. For  $\mathbf{u} \neq \mathbf{0}$ ,  $\lambda$  is called an **eigenvalue** of the matrix  $\mathbf{A}$  and  $\mathbf{u}$  is called an **eigenvector** belonging to  $\lambda$ .

In this section the problem is that we need to find the eigenvalue,  $\lambda$ , and the corresponding eigenvector,  $\mathbf{u}$ , of a square matrix  $\mathbf{A}$ . How can we find  $\lambda$ ?

We have

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$$

$$\mathbf{A}\mathbf{u} - \lambda\mathbf{u} = \mathbf{0}$$

Remember the properties of the identity matrix,  $\mathbf{I}$ :

$$\mathbf{u} = \mathbf{I}\mathbf{u} \text{ and so } \lambda\mathbf{u} = \lambda\mathbf{I}\mathbf{u}$$

therefore becomes

$$\mathbf{A}\mathbf{u} - \lambda\mathbf{I}\mathbf{u} = \mathbf{0}$$



Factorizing we obtain

$$11.15 \quad (A - \lambda I)\mathbf{u} = \mathbf{0}$$

? Are there any solutions of 11.15?

Yes, the trivial solution  $\mathbf{u} = \mathbf{0}$ . However for eigenvalues and eigenvectors,  $\mathbf{u} \neq \mathbf{0}$ .

? What does matrix  $A$  need to satisfy in order to have a non-trivial solution to  $(A - \lambda I)\mathbf{u} = \mathbf{0}$ ?

As discussed in **Example 29** of the previous section, for non-trivial solution

$$11.16 \quad \det(A - \lambda I) = 0$$

Hence  $\lambda$  is an eigenvalue of  $A$  if and only if 11.16 holds (see **Example 29**).

Eigenvalues were initially developed in the field of differential equations by **Jean D'Alembert**.



Alembert, 1717–83, was a French mathematician and the illegitimate son of Madame Tencin and an army officer, Louis Destouches. His mother left him on the steps of a local church and he was sent to an orphanage. His father recognized his difficulties and placed him under the care of Madame Rousseau. He always thought of Madame Rousseau as his mother.

Sadly, Alembert's father died when he was only 9 years old and his father's family had to look after his financial situation so that he could be educated.

In 1735 Alembert graduated and thought a career in law would suit him, but his real thirst and enthusiasm were for mathematics which he studied in his spare time. Three years later he did qualify as an advocate but did not pursue a career in this field, choosing mathematics instead.

For most of his life he worked for the Paris Academy of Science and the French Academy.

He was well known to have a short fuse and often argued with his contemporaries.

## F2 Finding eigenvalues and eigenvectors

### Example 31

Find the eigenvalues of  $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ .

Example 31 *continued*

Solution

We need to find  $\lambda$  in  $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ . First we obtain  $\mathbf{A} - \lambda\mathbf{I}$ :

$$\begin{aligned}\mathbf{A} - \lambda\mathbf{I} &= \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && \left[ \mathbf{I} = \text{identity matrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \\ &= \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 2 - \lambda & 0 \\ 1 & 3 - \lambda \end{pmatrix}\end{aligned}$$

Substituting this into  $\det(\mathbf{A} - \lambda\mathbf{I})$  gives

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{pmatrix} 2 - \lambda & 0 \\ 1 & 3 - \lambda \end{pmatrix} \stackrel{\text{by 11.1}}{=} (2 - \lambda)(3 - \lambda) - 0$$

For eigenvalues we equate the determinant to zero:

$$(2 - \lambda)(3 - \lambda) = 0$$

$$\lambda = 2, \lambda = 3 \quad \text{[Solving]}$$

## Example 32

Find the eigenvector corresponding to  $\lambda = 2$  and  $\lambda = 3$  for  $\mathbf{A}$  in **Example 31**.

Solution

Let  $\mathbf{u}$  be the eigenvector corresponding to  $\lambda = 2$ . Substituting  $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$  and  $\lambda = 2$  into

$$\mathbf{11.15} \quad [\mathbf{A} - \lambda\mathbf{I}]\mathbf{u} = \mathbf{0}$$

gives

$$\begin{aligned}\left[ \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right] \mathbf{u} &= \mathbf{0} \\ \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \mathbf{u} &= \mathbf{0} \quad \text{[Simplifying]}\end{aligned}$$

Remember the zero column vector,  $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ . Let  $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$  and so we have

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiplying out gives

$$0 + 0 = 0$$

$$x + y = 0$$

---


$$\mathbf{11.1} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

Example 32 *continued*

Remember that the eigenvector cannot be zero;  $\mathbf{u} \neq \mathbf{0}$  therefore at least one of the values,  $x$  or  $y$ , must be non-zero. From the second equation we have

$$\begin{aligned}x + y &= 0 \\x &= -y\end{aligned}$$

Solutions are

$$\begin{aligned}x &= 1, y = -1 \\x &= 2, y = -2 \\x &= \pi, y = -\pi\end{aligned}$$

We have an infinite number of solutions, so for the time being we need to write the eigenvector  $\mathbf{u}$  in general terms. **How?**

Let  $x = a$  then  $y = -a$  where  $a \neq 0$  and is any real number.

The eigenvectors corresponding to  $\lambda = 2$  are

$$\mathbf{u} = \begin{pmatrix} a \\ -a \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ where } a \neq 0$$

Similarly we find the general eigenvector  $\mathbf{v}$  corresponding to  $\lambda = 3$ . Putting  $\lambda = 3$  into  $[\mathbf{A} - \lambda\mathbf{I}]\mathbf{v} = \mathbf{0}$  gives

$$\begin{aligned}\left[ \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \right] \mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{v} &= \mathbf{0}\end{aligned}$$

By writing  $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$  [different  $x$  and  $y$  from those above] and  $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  we obtain

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiplying out:

$$\begin{aligned}-x + 0 &= 0 \\x + 0 &= 0\end{aligned}$$

From these equations we must have  $x = 0$ . **What is  $y$  equal to?**

We can choose  $y$  to be any real number apart from zero. Thus

$$y = a \text{ where } a \neq 0$$

The general eigenvector of  $\lambda = 3$  is found by substituting  $x = 0$  and  $y = a$ :

$$\mathbf{v} = \begin{pmatrix} 0 \\ a \end{pmatrix} = a \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Note that for one eigenvalue there are infinitely many eigenvectors. We can choose any one of these eigenvectors for the corresponding eigenvalue. By putting  $a = 1$ , an eigenvector can be selected. In **Example 32**:

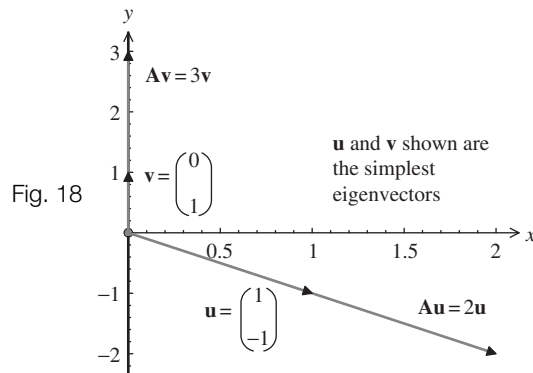
$$\begin{aligned}\mathbf{u} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ is an eigenvector corresponding to } \lambda = 2 \\ \mathbf{v} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is an eigenvector corresponding to } \lambda = 3\end{aligned}$$

### ? What does all this mean?

It means that we must have  $\mathbf{A}\mathbf{u} = \lambda_1\mathbf{u}$  and  $\mathbf{A}\mathbf{v} = \lambda_2\mathbf{v}$  which in this case is

$$\mathbf{A}\mathbf{u} = 2\mathbf{u} \text{ and } \mathbf{A}\mathbf{v} = 3\mathbf{v}$$

Plotting these eigenvectors and the effect of multiplying by the matrix  $\mathbf{A}$  gives:



The effect on the eigenvectors of multiplying by the matrix  $\mathbf{A}$  is to produce scalar multiples of itself, as you can see in Fig. 18.

Eigenvectors are non-zero vectors which are transformed to scalar multiples of themselves by a square matrix  $\mathbf{A}$ .

Generally the values of the elements of the eigenvectors are scaled to the smallest possible size of whole numbers.

The resulting equation from  $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$  is called the characteristic equation.

You need to be very careful with the algebra when expanding brackets. For example, expanding brackets such as  $(-1 - \lambda)(-2 - \lambda)$  can lead to confusion with the minus signs.

However if you factor out the two minus signs then it should be easier to expand this:

$$(-1 - \lambda)(-2 - \lambda) = \underset{=+}{\underset{=+}{(-1 - \lambda)(-2 - \lambda)}} = (1 + \lambda)(2 + \lambda) = (1 + \lambda)(2 + \lambda)$$

## F3 Applications to vibrations

### Example 33

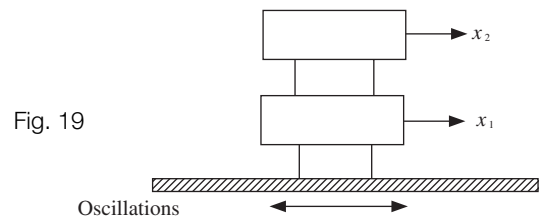
Consider a two-storey building subject to earthquake oscillations, as shown in Fig. 19.

The equations of motion are expressed as

$$\ddot{\mathbf{x}} = \mathbf{A}\mathbf{x} \text{ where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The period,  $T$ , of natural vibrations is given by

$$T = \frac{2\pi}{\sqrt{-\lambda}} \text{ where } \lambda \text{ is the eigenvalue of the matrix } \mathbf{A}. \text{ Find the period(s) if } \mathbf{A} = \begin{pmatrix} -20 & 10 \\ 10 & -10 \end{pmatrix}.$$



Example 33 *continued*

Solution

Substituting the given matrix  $A$  into  $A - \lambda I$  yields

$$\begin{aligned} A - \lambda I &= \begin{pmatrix} -20 & 10 \\ 10 & -10 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -20 & 10 \\ 10 & -10 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} -20 - \lambda & 10 \\ 10 & -10 - \lambda \end{pmatrix} \end{aligned}$$

Therefore  $\det(A - \lambda I)$  is given by

$$\begin{aligned} \det \begin{pmatrix} -20 - \lambda & 10 \\ 10 & -10 - \lambda \end{pmatrix} &= (-20 - \lambda)(-10 - \lambda) - (10 \times 10) \quad [\text{By } \mathbf{11.1}] \\ &= (20 + \lambda)(10 + \lambda) - 100 \quad [\text{Taking out two minus signs}] \\ &= 200 + 30\lambda + \lambda^2 - 100 \quad [\text{Expanding}] \\ &= \lambda^2 + 30\lambda + 100 \end{aligned}$$

Equating this to zero gives the quadratic equation

$$\lambda^2 + 30\lambda + 100 = 0$$

?

**How do we solve this quadratic equation?**

Apply the formula

$$\mathbf{1.16} \quad \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with  $a = 1$ ,  $b = 30$  and  $c = 100$ :

$$\begin{aligned} \lambda &= \frac{-30 \pm \sqrt{(-30)^2 - (4 \times 1 \times 100)}}{2} = \frac{-30 \pm \sqrt{500}}{2} \\ &= -26.18, -3.82 \end{aligned}$$

The two eigenvalues are  $\lambda_1 = -26.18$  and  $\lambda_2 = -3.82$ .Substituting these into the given  $T = \frac{2\pi}{\sqrt{-\lambda}}$ :

$$T_1 = \frac{2\pi}{\sqrt{-(-26.18)}} = 1.23 \text{ s}, \quad T_2 = \frac{2\pi}{\sqrt{-(-3.82)}} = 3.21 \text{ s}$$

Natural periods of vibration are 1.23 s (2 d.p.) and 3.21 s (2 d.p.).

---


$$\mathbf{11.1} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

## F4 3 × 3 Matrices

## Example 34

Determine the eigenvalues of

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 4 & 0 \\ 3 & 5 & -3 \end{pmatrix}$$

Solution

We have

$$A - \lambda I = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 4 & 0 \\ 3 & 5 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 - \lambda & 0 & 4 \\ 0 & 4 - \lambda & 0 \\ 3 & 5 & -3 - \lambda \end{pmatrix}$$

It's easier to remember that  $A - \lambda I$  is actually  $A$  with  $-\lambda$  associated with each entry along the leading diagonal (from top left to bottom right). We need to evaluate  $\det(A - \lambda I)$ .

?

**What is the easiest way to evaluate this?**

From the properties of determinants we know that it will be easier to evaluate the determinant along the second row, containing the elements 0,  $4 - \lambda$  and 0. **Why?**

?

Because it has two zeros, so we do not have to evaluate the  $2 \times 2$  determinants associated with these zeros.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1 - \lambda & 0 & 4 \\ 0 & 4 - \lambda & 0 \\ 3 & 5 & -3 - \lambda \end{pmatrix} \\ &= (4 - \lambda) \left[ \det \begin{pmatrix} 1 - \lambda & 4 \\ 3 & -3 - \lambda \end{pmatrix} \right] && \text{[Expanding second row]} \\ &= (4 - \lambda) [(1 - \lambda)(-3 - \lambda) - (3 \times 4)] && \text{[By 11.1]} \\ &= (4 - \lambda) [(\lambda - 1)(3 + \lambda) - 12] && \text{[Taking out minus signs]} \\ &= (4 - \lambda) [3\lambda + \lambda^2 - 3 - \lambda - 12] \\ &= (4 - \lambda) [\lambda^2 + 2\lambda - 15] && \text{[Simplifying]} \\ &= (4 - \lambda)(\lambda + 5)(\lambda - 3) && \text{[Factorizing]} \end{aligned}$$

This is equal to zero because for eigenvalues  $\det(A - \lambda I) = 0$ :

$$(4 - \lambda)(\lambda + 5)(\lambda - 3) = 0$$

$$\lambda = 4, \lambda = -5, \lambda = 3$$

---

11.1  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$

## Example 35

Find the eigenvectors associated with  $\lambda = 3$  for the matrix  $\mathbf{A}$  in **Example 34**.

Solution

Substituting  $\lambda = 3$  and  $\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 4 & 0 \\ 3 & 5 & -3 \end{pmatrix}$  into  $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{0}$  gives (subtract 3 from the leading diagonal):

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \begin{pmatrix} 1-3 & 0 & 4 \\ 0 & 4-3 & 0 \\ 3 & 5 & -3-3 \end{pmatrix} \mathbf{u} = \mathbf{0}$$

where  $\mathbf{u}$  is the eigenvector corresponding to  $\lambda = 3$ . Remember that  $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$  and let  $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ , so we have

$$\begin{pmatrix} -2 & 0 & 4 \\ 0 & 1 & 0 \\ 3 & 5 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Expanding this gives

$$\begin{array}{l} \dagger \quad -2x + 0 + 4z = 0 \\ \dagger\dagger \quad 0 + y + 0 = 0 \\ \dagger\dagger\dagger \quad 3x + 5y - 6z = 0 \end{array}$$

From middle equation  $\dagger\dagger$  we know that  $y = 0$ . From top equation  $\dagger$  we have

$$2x = 4z \text{ which gives } x = 2z$$

If  $z = 1$  then  $x = 2$ ; or more generally if  $z = a$  then  $x = 2a$  where  $a \neq 0$ .

The general eigenvector  $\mathbf{u} = \begin{pmatrix} 2a \\ 0 \\ a \end{pmatrix} = a \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$  where  $a \neq 0$  corresponds to  $\lambda = 3$ .

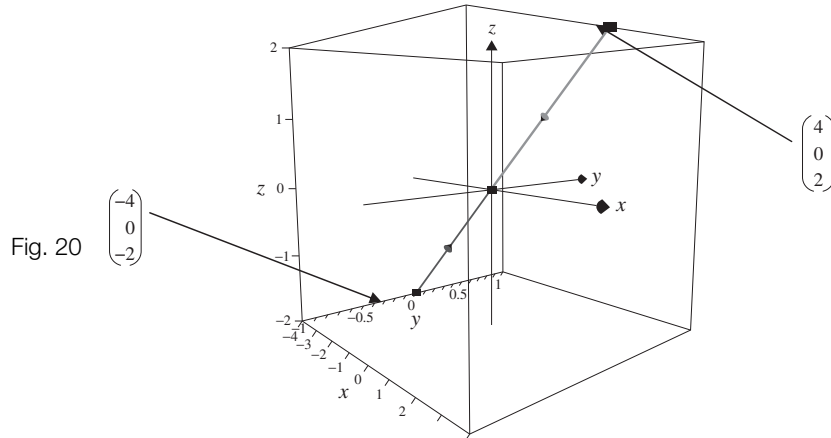
The eigenvectors corresponding to  $\lambda = 4$ ,  $\lambda = -5$  are part of **Exercise 11(f)**, question 4.

## F5 Eigenspace

Note that for the  $\lambda = 3$  in the above **Example 35** we have an infinite number of eigenvectors by substituting various non-zero values of  $a$ :

$$\mathbf{u} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \text{ or } \mathbf{u} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} \text{ or } \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 1/2 \end{pmatrix} \text{ or } \mathbf{u} = \begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix} \text{ etc.}$$

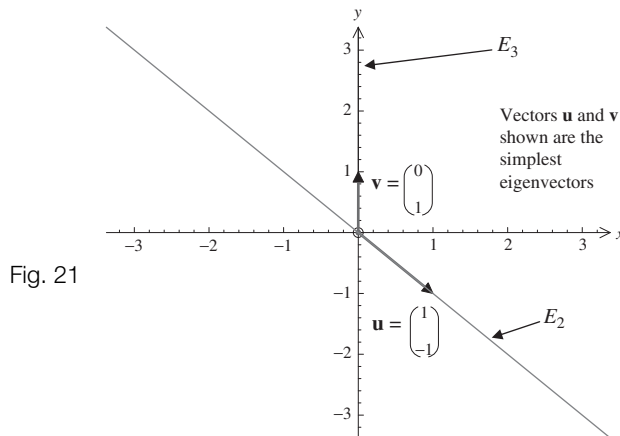
These solutions are given by **all** the points (apart from  $x = y = z = 0$ ) on the line shown below in the Fig. 20:



In general, if  $A$  is a square matrix and  $\lambda$  is an eigenvalue of  $A$  with an eigenvector  $\mathbf{u}$  then every scalar multiple (apart from 0) of the vector  $\mathbf{u}$  is also an eigenvector belonging to the eigenvalue  $\lambda$ .

This space is called an **eigenspace** of  $\lambda$  and is denoted by  $E_\lambda$ .

For example, the eigenspace relating to **Example 32** for the eigenvalue  $\lambda = 2$  is the eigenvector  $\mathbf{u} = a \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and for  $\lambda = 3$  the eigenvector  $\mathbf{v} = a \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , as shown below in Fig. 21:



## SUMMARY

A scalar  $\lambda$  is called an eigenvalue of a square matrix  $A$  if it satisfies

$$11.15 \quad (A - \lambda I)\mathbf{u} = 0$$

and the corresponding  $\mathbf{u} (\neq \mathbf{0})$  is called an eigenvector.

We find  $\lambda$  by solving the resulting equation from

$$11.16 \quad \det(A - \lambda I) = 0$$



## Exercise 11(f)


Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- 1 Find the eigenvalues of the following matrices:

$$\mathbf{a} \quad \mathbf{A} = \begin{pmatrix} 7 & 3 \\ 0 & -4 \end{pmatrix} \quad \mathbf{b} \quad \mathbf{A} = \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}$$

$$\mathbf{c} \quad \mathbf{A} = \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}$$

- 2 Find the particular eigenvectors of the matrices of question 1.

- 3  [control engineering] The system poles,  $\lambda$ , of a system are the eigenvalues of a given matrix  $\mathbf{A}$ . Determine the system poles for

$$\mathbf{a} \quad \mathbf{A} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$$


$$\mathbf{b} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ -4 & -3 \end{pmatrix}$$

$$\mathbf{c} \quad \mathbf{A} = \begin{pmatrix} 5 & -5 \\ 5 & -2 \end{pmatrix}$$

- 4 Obtain the general eigenvectors corresponding to  $\lambda = 4$  and  $\lambda = -5$  for the matrix in **Example 34**. (The eigenvector corresponding to  $\lambda = 3$  was found in **Example 35**.)

- 5 Determine the eigenvalues and particular eigenvectors of

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ -9 & 4 & -1 \\ -6 & 2 & 1 \end{pmatrix}$$


- 6  [vibrations] The natural period,  $T$ , of vibrations of a building is given by

$$T = \frac{2\pi}{\sqrt{-\lambda}}$$

where  $\lambda$  is the eigenvalue of a given matrix  $\mathbf{A}$ . Determine  $T$  for


$$\mathbf{A} = \begin{pmatrix} -15 & 5 \\ 10 & -10 \end{pmatrix}$$

**For the remaining questions use a computer algebra system or a graphical calculator.**

- 7  [control engineering] The state matrix,  $\mathbf{A}$ , of a system is given by

$$\mathbf{A} = \begin{pmatrix} 3.16 & 2.87 & 1.61 \\ 4.93 & 12.03 & 2.64 \\ 5.12 & 3.84 & 4.20 \end{pmatrix}$$

Determine the eigenvalues. The eigenvalues of the system are called system poles.

- 8  [control engineering] Find the system poles for

$$\mathbf{A} = \begin{pmatrix} -0.2 & 5.6 & 7.1 & 3.3 \\ 1.5 & -6.1 & -6.3 & -7.0 \\ -2.2 & 3.6 & 4.8 & 8.3 \\ -9.3 & 8.9 & 6.4 & -6.1 \end{pmatrix}$$

SECTION G **Diagonalization**

By the end of this section you will be able to:

- diagonalize a matrix
- find powers of matrices

## G1 Diagonal matrices

### ? What is meant by a diagonal matrix?

A diagonal matrix is an  $n$  by  $n$  matrix where **all** entries to both sides of the leading diagonal are zero. (The leading diagonal is the diagonal from top left to bottom right of the matrix.)

### ? Can you think of an example of a diagonal matrix?

The identity matrix  $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ . Another example is  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ .

In this section we look at the diagonalization problem for matrices which is the following:

### ? For a square matrix $\mathbf{A}$ , is there a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix?

This process of converting any  $n$  by  $n$  matrix into a diagonal matrix is called **diagonalization**.

In this section we will show that diagonalizing a matrix is equivalent to finding the eigenvalues and corresponding eigenvectors of the matrix. Actually we will demonstrate that the eigenvalues are the leading diagonal entries in the diagonal matrix.

In the next subsection we state a procedure for diagonalization.

## G2 Algorithm for diagonalization

### Example 36

Let  $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$  and  $\mathbf{P} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ . Determine  $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ .

Solution

### ? What do we need to find first?

The inverse of the matrix  $\mathbf{P}$ , denoted  $\mathbf{P}^{-1}$  is given by:

$$\begin{aligned} \mathbf{P}^{-1} &= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{0 - (-1)} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} & \left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right] \\ &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

Carrying out the matrix multiplication

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Note that  $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$  is a diagonal matrix. (The entries of the leading diagonal, 2 and 1, are the eigenvalues of the matrix  $\mathbf{A}$ .)

**11.17** The procedure for diagonalizing an  $n$  by  $n$  matrix  $\mathbf{A}$  with eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \dots$  and  $\lambda_n$  is:

- 1** Determine the eigenvectors of the given matrix  $\mathbf{A}$ . Call these  $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \dots$  and  $\mathbf{p}_n$ .
- 2** Form the matrix  $\mathbf{P}$  by having the eigenvectors of step 1,  $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \dots$  and  $\mathbf{p}_n$ , as its columns. That is, let

$$\mathbf{P} = (\mathbf{p}_1 : \mathbf{p}_2 : \mathbf{p}_3 : \dots : \mathbf{p}_n) \text{ provided } \det(\mathbf{P}) \neq 0$$

If  $\det(\mathbf{P}) = 0$  then we **cannot** diagonalize matrix  $\mathbf{A}$ .

- 3** The diagonal matrix  $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$  will have the eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \dots$  and  $\lambda_n$  along its leading diagonal, that is

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}$$

Eigenvalues of matrix  $\mathbf{A}$

- 4** It is good practice to check that matrices  $\mathbf{P}$  and  $\mathbf{D}$  actually work. For matrices of size **greater than 2** by 2 the evaluation of the inverse matrix  $\mathbf{P}^{-1}$  can be a tedious task. To avoid this evaluation, check that  $\mathbf{P}\mathbf{D} = \mathbf{A}\mathbf{P}$ . **Why?**

Because left-multiplying the above  $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$  by  $\mathbf{P}$  gives

$$\begin{aligned} \mathbf{P}\mathbf{D} &= \mathbf{P}(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \\ &= (\mathbf{P}\mathbf{P}^{-1})\mathbf{A}\mathbf{P} = \mathbf{I}\mathbf{A}\mathbf{P} = \mathbf{A}\mathbf{P} \quad [\text{Remember } \mathbf{P}\mathbf{P}^{-1} = \mathbf{I}] \end{aligned}$$

Hence it is enough to check that  $\mathbf{P}\mathbf{D} = \mathbf{A}\mathbf{P}$ .

Next we illustrate this diagonalization process. We will **not** find the eigenvalues and eigenvectors in the examples below because you should be familiar with determining these by now. However you will need to find these for the questions in **Exercise 11(g)**.

### Example 37

Determine the matrix  $\mathbf{P}$  which diagonalizes the matrix  $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$  given that the eigenvalues of this matrix are  $\lambda_1 = -1$  and  $\lambda_2 = 5$  with the corresponding eigenvectors  $\mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$  and  $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  respectively.

Solution

*Step 1:*

We have been given the eigenvectors in the question.

*Step 2:*

The matrix  $\mathbf{P} = (\mathbf{u} : \mathbf{v}) = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$  where  $\mathbf{u}$  and  $\mathbf{v}$  are the given eigenvectors.

### Example 37 *continued*

*Step 3:*

Before we write down the diagonal matrix we need to make sure that the order of eigenvalues corresponds to the order of the eigenvectors given in matrix  $\mathbf{P}$ . We have  $\mathbf{P} = (\mathbf{u} : \mathbf{v})$  therefore our diagonal matrix  $\mathbf{D}$  is given by

$$\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

*Step 4:*

?

We need to confirm that this matrix  $\mathbf{P}$  does indeed diagonalize the given matrix  $\mathbf{A}$ . **How?**

By checking that  $\mathbf{PD} = \mathbf{AP}$ :

$$\mathbf{PD} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -1 & 5 \end{pmatrix}$$

$$\mathbf{AP} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -1 & 5 \end{pmatrix}$$

This confirms that the matrix  $\mathbf{P}$  does diagonalize the given matrix  $\mathbf{A}$ .

Notice that the eigenvalues,  $\lambda_1 = -1$  and  $\lambda_2 = 5$ , of the given matrix  $\mathbf{A}$  are entries along the leading diagonal in the matrix  $\mathbf{D}$ . They occur in the order  $\lambda_1$  and then  $\lambda_2$  because the matrix  $\mathbf{P}$  is created by  $\mathbf{P} = (\mathbf{u} : \mathbf{v})$  where  $\mathbf{u}$  is the eigenvector belonging to  $\lambda_1$  and  $\mathbf{v}$  is the eigenvector belonging to the other eigenvalue  $\lambda_2$ . **What would be the diagonal matrix if the matrix  $\mathbf{P}$  was created by interchanging  $\mathbf{u}$  and  $\mathbf{v}$ , that is,  $\mathbf{P} = (\mathbf{v} : \mathbf{u})$ ?**

?

Our diagonal matrix would be  $\mathbf{D} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$ . [See **Exercise 11(g)**]

The process of finding the inverse matrix  $\mathbf{P}^{-1}$  of a 3 by 3 matrix can be a tedious task as you will have noticed in **Section 11C**. For the next example, check your result by showing that  $\mathbf{PD} = \mathbf{AP}$ .

### Example 38

For the matrix  $\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{pmatrix}$ , determine the matrix  $\mathbf{P}$  which diagonalizes matrix  $\mathbf{A}$

given that the eigenvalues of  $\mathbf{A}$  are  $\lambda_1 = 1$ ,  $\lambda_2 = 2$  and  $\lambda_3 = 3$  with corresponding

eigenvectors  $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\mathbf{v} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$  and  $\mathbf{w} = \begin{pmatrix} -7 \\ 10 \\ 2 \end{pmatrix}$  respectively.

Check that  $\mathbf{P}$  does indeed diagonalize the given matrix  $\mathbf{A}$ .

Example 38 *continued*

Solution

*Step 1 and Step 2:*We have been given the eigenvalues and corresponding eigenvectors of the matrix  $A$ .**What is our matrix  $P$  equal to?**

We have

$$P = (\mathbf{u} : \mathbf{v} : \mathbf{w}) = \begin{pmatrix} 1 & -2 & -7 \\ 0 & 1 & 10 \\ 0 & 0 & 2 \end{pmatrix} \quad \left[ \text{Given } \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{w} = \begin{pmatrix} -7 \\ 10 \\ 2 \end{pmatrix} \right]$$

*Step 3:*The diagonal matrix has the eigenvalues  $\lambda_1 = 1$ ,  $\lambda_2 = 2$  and  $\lambda_3 = 3$  along the leading diagonal, that is

$$P^{-1}AP = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

*Step 4:*Checking that we have the correct  $P$  and  $D$  matrices by showing  $PD = AP$ :

$$PD = \begin{pmatrix} 1 & -2 & -7 \\ 0 & 1 & 10 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -4 & -21 \\ 0 & 2 & 30 \\ 0 & 0 & 6 \end{pmatrix}$$

$$AP = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & -7 \\ 0 & 1 & 10 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -4 & -21 \\ 0 & 2 & 30 \\ 0 & 0 & 6 \end{pmatrix}$$

Hence this confirms that the matrix  $P$  does indeed diagonalize the given matrix  $A$ .

In general a diagonal matrix is easier to work with because if you are multiplying, solving a system of equations or finding eigenvalues, it is always preferable to have a diagonal matrix.

However **not** all matrices can be diagonalized. For example,  $A = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$  has eigenvalues

**?**  $\lambda_1 = 2$  and  $\lambda_2 = 2$  with corresponding eigenvectors  $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . **What is matrix  $P$  equal to?**

$$P = (\mathbf{u} : \mathbf{v}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

**?** **What is the determinant of matrix  $P$ ?**

$$\det(P) = \det \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = (1 \times 0) - (0 \times 1) = 0$$

We have  $\det(P) = 0$  therefore we **cannot** diagonalize matrix  $A$ .  
Next we look at the applications of diagonalization.

### G3 Powers of matrices

**?** We can use the above diagonalization process to find powers of matrices. **What does diagonalization have to do with powers of matrices?**

If  $A$  is a square matrix which is diagonalizable then there is a matrix  $P$  such that  $P^{-1}AP = D$  where  $D$  is a diagonal matrix. It can be shown that

$$A^m = PD^mP^{-1} \text{ where } m \text{ is a positive whole number}$$

**?** We can use this formula to find  $A^m$  but how do we determine  $D^m$ ?

The matrix  $D^m$  is a diagonal matrix with the leading diagonal entries raised to the power  $m$ , that is

$$\text{If } D = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n \end{pmatrix} \text{ then } D^m = \begin{pmatrix} a_1^m & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_n^m \end{pmatrix}$$

We can apply this to evaluate powers of the square matrix  $A$  because multiplying diagonal matrices or finding powers of diagonal matrices is much easier. (We only need to find the powers of the entries on the **leading diagonal**.)

In general to find  $A^m$  we have to multiply  $m$  copies of the matrix  $A$  and this is a tedious and laborious task. It is less demanding if we use the above, that is  $A^m = PD^mP^{-1}$  even though we need to find  $P$ ,  $P^{-1}$  and  $D$  which is no easy task in itself. This formula means that if you want to work out  $A^m$  without working out lower powers, then using the diagonal matrix is efficient.

**?** Note that in the above subsection when we diagonalized a matrix we could avoid the calculation of  $P^{-1}AP$ . **Why?**

Because we check that  $PD = AP$  which means that  $P^{-1}AP$  is a diagonal matrix with eigenvalues along the leading diagonal. However in evaluating  $A^m$  we need to find  $P^{-1}$  because

$$A^m = PD^mP^{-1}$$

#### Example 39

Let  $A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{pmatrix}$ . Determine  $A^5$  given that

$$P = \begin{pmatrix} 1 & -2 & -7 \\ 0 & 1 & 10 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } P^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 4 & -13 \\ 0 & 2 & -10 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 39 *continued*

Solution

The given matrix  $A$  is the same as the matrix  $A$  given in previous **Example 38**. The

diagonal matrix is  $D = P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ . How do we find  $A^5$ ?

By applying the above proposition  $A^m = PD^mP^{-1}$  with  $m = 5$ :

$$A^5 = PD^5P^{-1}$$

Substituting the matrices  $P = \begin{pmatrix} 1 & -2 & -7 \\ 0 & 1 & 10 \\ 0 & 0 & 2 \end{pmatrix}$ ,  $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$  and

$$P^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 4 & -13 \\ 0 & 2 & -10 \\ 0 & 0 & 1 \end{pmatrix} \text{ into } A^5 = PD^5P^{-1} \text{ gives}$$

$$A^5 = PD^5P^{-1}$$

$$\begin{aligned} &= \begin{pmatrix} 1 & -2 & -7 \\ 0 & 1 & 10 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^5 \frac{1}{2} \begin{pmatrix} 2 & 4 & -13 \\ 0 & 2 & -10 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & -2 & -7 \\ 0 & 1 & 10 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1^5 & 0 & 0 \\ 0 & 2^5 & 0 \\ 0 & 0 & 3^5 \end{pmatrix} \begin{pmatrix} 2 & 4 & -13 \\ 0 & 2 & -10 \\ 0 & 0 & 1 \end{pmatrix} \quad \left[ \text{Taking } \frac{1}{2} \text{ to the front} \right] \\ &= \frac{1}{2} \begin{pmatrix} 1 & -2 & -7 \\ 0 & 1 & 10 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 243 \end{pmatrix} \begin{pmatrix} 2 & 4 & -13 \\ 0 & 2 & -10 \\ 0 & 0 & 1 \end{pmatrix} \quad \left[ \text{Replacing} \right. \\ & \quad \left. 1^5 = 1, 2^5 = 32 \text{ and } 3^5 = 243 \right] \\ &= \frac{1}{2} \begin{pmatrix} 1 & -64 & -1701 \\ 0 & 32 & 2430 \\ 0 & 0 & 486 \end{pmatrix} \begin{pmatrix} 2 & 4 & -13 \\ 0 & 2 & -10 \\ 0 & 0 & 1 \end{pmatrix} \quad \left[ \text{Multiplying the first} \right. \\ & \quad \left. \text{two matrices on the left} \right] \\ &= \frac{1}{2} \begin{pmatrix} 2 & -124 & -1074 \\ 0 & 64 & 2110 \\ 0 & 0 & 486 \end{pmatrix} = \begin{pmatrix} 1 & -62 & -537 \\ 0 & 32 & 1055 \\ 0 & 0 & 243 \end{pmatrix} \quad \left[ \text{Multiplying by} \right. \\ & \quad \left. \text{scalar } 1/2 \right] \end{aligned}$$

You may check this final result by using any appropriate software.

By examining the diagonal matrix  $D$  you can find the determinant of the initial matrix  $A$  by multiplying the entries on the leading diagonal, because the determinant of  $A$  is the product of the eigenvalues which are the entries on the leading diagonal, that is

$$\mathbf{11.18} \quad \det(A) = \lambda_1 \lambda_2 \dots \lambda_n$$

Clearly you don't need to diagonalize, you just need to find the eigenvalues.

**SUMMARY**

If an  $n$  by  $n$  matrix  $A$  is diagonalizable with  $P^{-1}AP = D$  where  $D$  is a diagonal matrix then  $A^m = PD^mP^{-1}$ .

**Exercise 11(g)**

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

In this exercise you may check your numerical answers using mathematical software.

1 For the following matrices find:

- i The eigenvalues and corresponding eigenvectors.
- ii Matrices  $P$  and  $D$  such that  $D = P^{-1}AP$  where  $D$  is a diagonal matrix.

$$\mathbf{a} \ A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \mathbf{b} \ A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{c} \ A = \begin{bmatrix} 3 & 0 \\ 4 & 4 \end{bmatrix} \quad \mathbf{d} \ A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

2 For the matrices in question 1 find  $A^5$ .

3 For the following matrices determine:

- i The eigenvalues and corresponding eigenvectors.
  - ii Matrices  $P$  and  $D$  where  $P$  is an invertible (has an inverse) matrix and  $D = P^{-1}AP$  is a diagonal matrix.
  - iii Determine  $A^4$  in each case by using the results of parts i and ii.
- To find  $P^{-1}$  you may use MATLAB.

$$\mathbf{a} \ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\mathbf{b} \ A = \begin{bmatrix} -1 & 4 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

**Examination questions 11**

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

In this exercise you may check your numerical answers using mathematical software.

1 Let  $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$  and  $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ .

Determine

i  $(A - B)(A + B)$

ii  $A^2 - B^2$

University of Hertfordshire, UK, 2008

2 Given the matrix

$$A = \frac{1}{7} \begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix}$$

a Compute  $A^2$  and  $A^3$ .

b Based on these results, determine the matrices  $A^{-1}$  and  $A^{2004}$ .

University of Wisconsin, USA, 2004

3 Let  $A$ ,  $B$  and  $C$  be matrices defined by

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -2 & 1 & 2 & 3 \end{pmatrix}$$

Which of the following are defined?

$A^T$ ,  $AB$ ,  $B + C$ ,  $A - B$ ,  $CB$ ,  $BC^T$ ,  $A^2$

Compute those matrices which are defined.

Jacobs University, Germany, 2002

4 a If  $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ,

compute  $A^2$ ,  $B^2$ ,  $AB$  and  $BA$ .



## Examination questions 11 continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- b** If  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ ,  
compute  $AB - BA$ .

Queen Mary, University of London,  
UK, 2006

- 5** Let  $A = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ . Determine **i**  $A^2$     **ii**  $A^3$

University of Hertfordshire, UK, 2007

- 6** Let  $A$  be the  $3 \times 3$  matrix determined by

$$A \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix},$$

$$A \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Find  $A$ .

Columbia University, New York, USA,  
2006

The next question has been slightly  
edited.

- 7** Find the determinant of the matrix

$$A = \begin{pmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{pmatrix}$$

University of Ottawa, Canada, 2004

- 8** Compute the inverse of the matrix

$$A = \begin{pmatrix} 2 & -1 & -4 \\ -1 & 1 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

Is  $A$  diagonalizable? Give reasons for  
your answer.

University of New Brunswick, Canada,  
2000

- 9** Find  $\text{adj}A$ ,  $\det A$  and hence  $A^{-1}$  when

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and when

$$A = \begin{pmatrix} 3 & -4 & 1 \\ 1 & -1 & 3 \\ 2 & -2 & 5 \end{pmatrix}$$

University of Sussex, UK, 2008

- 10** If  $A = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$

- a** Find all the eigenvalues of  $A$ .  
**b** Find a non-singular matrix  $Q$  and a  
diagonal matrix  $D$  such that  
 $Q^{-1}AQ = D$  (that is,  $A = QDQ^{-1}$ ).  
**c** For the matrix  $A$  find  $A^5$ .  
(Non-singular means the matrix  $Q$   
has an inverse.)

Purdue University, USA, 2006

- 11** Show that the eigenvalues and  
corresponding eigenvectors of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$
 are given by

$$\lambda_1 = 0, \mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \lambda_2 = 0, \mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

$$\text{and } \lambda_3 = 3, \mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

University of Hertfordshire, UK, 2009

- 12 a** Find the eigenvalues and corresponding  
eigenvectors of  $A = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$ .

- b** Determine which of the following  
vectors

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 2 \end{pmatrix}$$

## Examination questions 11 continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

is an eigenvector of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \text{ and find the}$$

corresponding eigenvalue.

University of New Brunswick,  
Canada, 2004

**13 a** Find the eigenvalues and corresponding eigenvectors for

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

**b** Is  $A$  diagonalizable? Give reasons for your answer.

University of New Brunswick,  
Canada, 2000

## Miscellaneous exercise 11

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

**1** Evaluate the following:

**i**  $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

**ii**  $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} -7 & 0 \\ 0 & -7 \end{pmatrix}$

**iii**  $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$

**2** Find  $a$ ,  $b$ ,  $c$  and  $d$  which satisfy

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

**3** If  $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ , find  $A^2$ ,  $A^3$  and  $A^{10}$ .

**4 i** Find  $x$  and  $y$  which satisfy

$$\begin{pmatrix} 5 & 6 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

**ii** Find  $a$  and  $b$  which satisfy

$$\begin{pmatrix} 5 & 6 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

**iii** Obtain the inverse of  $\begin{pmatrix} 5 & 6 \\ 4 & 5 \end{pmatrix}$ .

**5 a** For  $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$  show that  
 $A^2 - 5A + 6I = 0$ .

**b** For  $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$  show that  
 $A^2 - 3A - 10I = 0$ .

**6** Evaluate


$$\begin{pmatrix} 5 & -1 & -2 \\ 10 & -2 & -4 \\ 15 & -3 & -6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ 2 & 3 & 8 \end{pmatrix}$$

What do you notice?

Could the same thing happen when multiplying two non-zero numbers?

**7** Evaluate the following:

$$\det \begin{bmatrix} 1 & 15 & -6 & 7 & 77 \\ 0 & 2 & -973 & 533 & 207 \\ 0 & 0 & 3 & -66 & 1057 \\ 0 & 0 & 0 & 4 & 855 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**8**  [electrical principles] By using Kirchhoff's laws on a circuit we obtain the equations

$$100I_1 - 30I_2 - 40I_3 = -8$$


$$-30I_1 + 140I_2 - 60I_3 = -2$$

$$-40I_1 - 60I_2 + 170I_3 = 10$$

Write these equations in matrix form and solve for  $I_1$ ,  $I_2$  and  $I_3$  by using Gaussian elimination.


## Miscellaneous exercise 11 continued

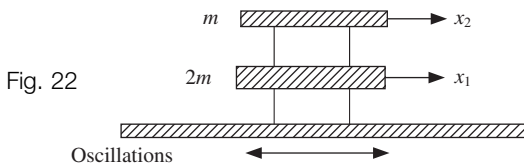
Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- 9  [electrical principles] The equations of a circuit are given by

$$\begin{aligned} 9i_1 - 3i_2 - 5i_3 &= 12 \times 10^{-3} \\ -3i_1 + 16.5i_2 - 0.5i_3 &= 0 \\ -5i_1 - 0.5i_2 + 20.5i_3 &= 15 \times 10^{-3} \end{aligned}$$

Solve for  $i_1$ ,  $i_2$  and  $i_3$  by using matrices. Is the solution unique?

- \*10  [vibrations] Consider the two-storey building shown in Fig. 22.




The equations of motion are expressed as

$$\begin{pmatrix} 3k/2m & k/2m \\ k/m & k/m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where  $k$  is a negative constant and  $m$  is mass. Find the period,  $T$ , given by

$$T = \frac{2\pi}{\sqrt{-\lambda}} \text{ where } \lambda \text{ is an eigenvalue.}$$

- \*11  [control engineering] A system is described by the equations

$$\begin{aligned} \dot{x}_1 &= 2x_1 + 2x_2 + u_1 \\ \dot{x}_2 &= -x_1 + 5x_2 + u_2 \\ y &= x_1 + 4x_2 \end{aligned}$$


Write these equations in the form

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Iu} \\ \mathbf{y} &= \mathbf{Cx} \end{aligned}$$

where  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ,  $\dot{\mathbf{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$  is the time derivative,  $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$  and  $\mathbf{I}$  is the identity matrix.

- i Determine the eigenvalues (system poles) of  $\mathbf{A}$ .

- ii Determine the corresponding general eigenvectors.

- 12  [control engineering] The transfer function, TF, of a system is given by

$$\text{TF} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

where  $\mathbf{A}$  is the state matrix,  $\mathbf{B}$  is the input matrix,  $\mathbf{C}$  is the output matrix and  $\mathbf{I}$  is the identity matrix.

For

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 3 & -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 0.6 \end{pmatrix} \text{ and}$$

$$\mathbf{C} = (1 \ 0)$$

obtain TF.

For the remaining questions use a computer algebra system or a graphical calculator.

- \*13 For a system with

$$\mathbf{A} = \begin{pmatrix} -5 & 2 & 1 \\ -6 & 5 & 9 \\ 7 & -1 & -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix},$$

$$k_0 = 5, k_1 = -3 \text{ and } k_2 = 9$$

find the feedback matrix,  $\mathbf{F}$ , by using Ackermann's formula:

$$\mathbf{F} = (0 \ 0 \ 1)\mathbf{C}_T^{-1}p(\mathbf{A})$$

where  $\mathbf{C}_T = (\mathbf{B} \ \mathbf{AB} \ \mathbf{A}^2\mathbf{B})$  and

$$p(\mathbf{A}) = \mathbf{A}^3 + k_2\mathbf{A}^2 + k_1\mathbf{A} + k_0\mathbf{I}$$

( $\mathbf{I}$  is the  $3 \times 3$  identity matrix).

- 14 By using  $p(\mathbf{A}) = \mathbf{A}^3 + k_2\mathbf{A}^2 + k_1\mathbf{A} + k_0\mathbf{I}$ , find  $p(\mathbf{A})$  for

$$\mathbf{A} = \begin{pmatrix} 6 & 3 & -11 \\ -2 & 15 & 3 \\ 7 & 9 & 12 \end{pmatrix},$$

$$k_2 = 5, k_1 = -7 \text{ and } k_0 = 6$$

## Miscellaneous exercise 11 continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

$$\mathbf{b} \quad \mathbf{A} = \begin{pmatrix} 9.31 & -6.11 & 2.30 \\ -2.75 & 3.93 & -6.51 \\ -8.99 & 12.23 & -1.75 \end{pmatrix},$$

$$k_2 = -6.73, k_1 = 5.29 \text{ and } k_0 = 6$$

- 15** The transfer function, TF, of a system is given by

$$\text{TF} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

where  $\mathbf{A}$  is the state matrix,  $\mathbf{B}$  is the input matrix and  $\mathbf{C}$  is the output matrix.

For

$$\mathbf{A} = \begin{pmatrix} 2.2 & 3.1 & 6.7 & -2.7 \\ 1.1 & -2.8 & 5.6 & -5.5 \\ 3.4 & 2.7 & -1.9 & -4.6 \\ -7.6 & 8.2 & 1.5 & 4.2 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} -1.1 \\ 2.3 \\ 1.7 \\ -4.1 \end{pmatrix} \text{ and}$$

$$\mathbf{C} = (9.1 \quad 6.2 \quad 4.7 \quad 5.5), \text{ find TF.}$$

# Vectors

SECTION A **Vector representation** 637

- ▶ Vector addition and subtraction
- ▶ Apply vectors to engineering problems

SECTION B **Vectors in Cartesian co-ordinates** 645

- ▶ Express vectors in two dimensions by using **i** and **j** components
- ▶ Find the resultant vector

SECTION C **Three-dimensional vectors** 649

- ▶ Express vectors in three dimensions by using **i**, **j** and **k** components
- ▶ Find the magnitude of a vector

SECTION D **Scalar products** 655

- ▶ Find scalar products
- ▶ Solve engineering problems involving scalar products

SECTION E **Vector products** 662

- ▶ Find vector products
- ▶ Solve engineering problems involving vector products

## SECTION A **Vector representation**

By the end of this section you will be able to:

- understand the terms vector and scalar
- apply vector addition and subtraction
- apply vectors to problems in mechanics

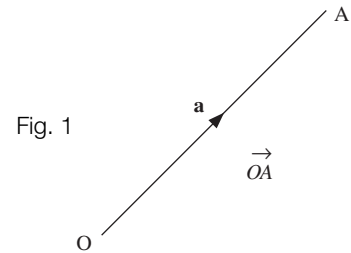
### A1 **Vectors and scalars**

A vector is a quantity that has **size** (magnitude) and **direction**. Examples of vectors are velocity, acceleration, force and displacement. A force of 10 N (newtons) upwards is a vector. **So what are scalars?**

A scalar is a quantity that has size but no direction. Examples of scalars are mass, length, volume, speed and temperature.

**How do we write down vectors and scalars and how can we distinguish between them?**

A vector **from O to A** is denoted by  $\vec{OA}$  or written in bold typeface **a** (Fig. 1). A scalar is denoted by  $a$ , **not** in bold, so that we can distinguish between vectors and scalars.

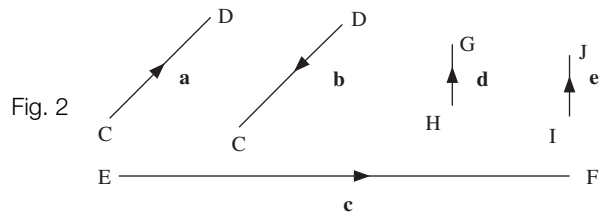


The vectors shown in Fig. 2 are given below in the notation outlined:

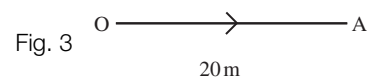
**a** is  $\vec{CD}$    **b** is  $\vec{DC}$    **c** is  $\vec{EF}$

**d** is  $\vec{HG}$    **e** is  $\vec{IJ}$

Two vectors are **equivalent** if they have the same direction and magnitude. In Fig. 2,  $\vec{HG} = \vec{IJ}$  because  $\vec{HG}$  and  $\vec{IJ}$  have the same direction and magnitude (length) but only differ in position.



Also note that the direction of the arrow gives the direction of the vector, that is  $\vec{CD}$  is **different** from  $\vec{DC}$ . The magnitude (or length) of the vector  $\vec{AB}$  is denoted by  $|\vec{AB}|$ .



There are many examples of vectors in engineering, particularly in dynamics:

**a** A displacement of 20 m to the horizontal right of an object from O to A (Fig. 3).

**b** A force on an object acting vertically downwards (Fig. 4).

**c** The velocity and acceleration of an object thrown vertically upwards, as illustrated in Fig. 5. The acceleration is the acceleration due to gravity,  $9.8 \text{ m/s}^2$ .

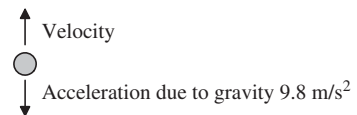
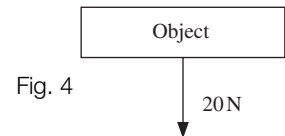


Fig. 5 Ball in upwards motion

**d** Figure 6 shows the various directions of the velocity and acceleration of a particle rotating in a circle. In each case the velocity and acceleration have magnitude and direction so they can be represented by vectors. The acceleration  $\mathbf{a}$  illustrated in Fig. 6 is acting towards the centre of the circle.

The velocity of P is in the direction of the tangent at P.

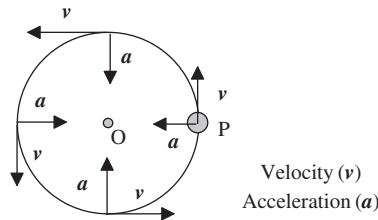
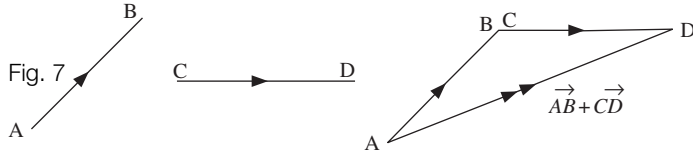


Fig. 6 A ball on a piece of string being swung round the point O. (The expected velocity and direction of the ball in the absence of the string is indicated by the vectors  $\mathbf{v}$ , but at every point P, the string forces the ball into a circular orbit, denoted by the vectors  $\mathbf{a}$ .)

## A2 Vector addition

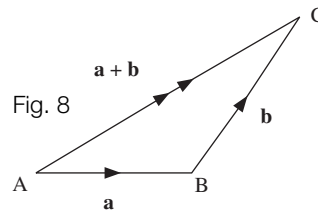
Figure 7 shows how to add two vectors  $\vec{AB}$  and  $\vec{CD}$ .



To add vectors  $\vec{AB}$  and  $\vec{CD}$  we connect the ends B and C which results in the vector  $\vec{AD}$ . In other

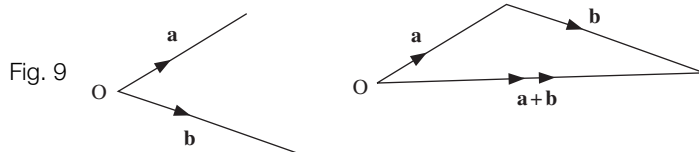
words, we can go directly from A to D or via B and C,  $\vec{AB}$  and along  $\vec{CD}$ . The vectors  $\vec{AB}$  and  $\vec{CD}$  are examples of **free** vectors, which means that the vectors have direction and magnitude but are not at a fixed position.

Consider a robot which moves an object from A to B and then moves it to C. Thus the object can be moved from A to C directly or via B as shown in Fig. 8.



Consider the addition of vectors  $\mathbf{a}$  and  $\mathbf{b}$  in Fig. 9.

We shift vector  $\mathbf{b}$  to the end of vector  $\mathbf{a}$  and the result is going from O to the end of vector  $\mathbf{b}$ . Generally  $\mathbf{a} + \mathbf{b}$  is called the resultant vector of  $\mathbf{a}$  and  $\mathbf{b}$  and is represented by the line with two arrows.



Let's do an example.

**Example 1** *mechanics*

Two forces  $F_1$  and  $F_2$  are applied to an object  $O$ . Determine the magnitude and direction of the resultant force,  $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$ .

**Solution**

Adding vectors  $F_1$  and  $F_2$  in Fig. 10 gives  $\mathbf{R}$  (Fig. 11).



**How can we obtain the magnitude (i.e. the length) of the vector  $\mathbf{R}$ ?**

We can use Pythagoras:

$$|\mathbf{R}| = \sqrt{5^2 + 8^2} = 9.43 \text{ kN}$$

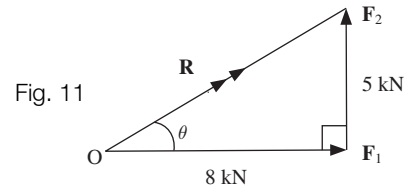
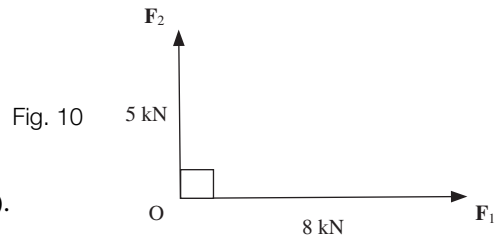
We also need to find the direction of  $\mathbf{R}$  which is given by the angle  $\theta$  in Fig. 11. By using trigonometry we have

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{8}$$

Therefore taking inverse tan gives

$$\theta = \tan^{-1}\left(\frac{5}{8}\right) = 32.01^\circ \quad [\text{By calculator}]$$

The resultant force,  $\mathbf{R}$ , has a magnitude of 9.43 kN and an angle of  $32.01^\circ$  as shown.

**A3 Subtraction of vectors**

We define  $\vec{OA} - \vec{OB}$  as  $\vec{OA} + (-\vec{OB})$ .

**Example 2**

By considering vectors  $\vec{OA}$  and  $\vec{OB}$  as in Fig. 12, show on a different diagram the vectors  
 i  $\vec{OA} - \vec{OB}$  and ii  $\vec{OB} - \vec{OA}$  iii Find  $\vec{OA} - \vec{OA}$ .

**Solution**

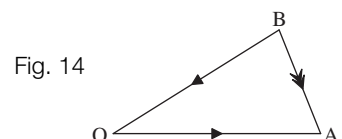
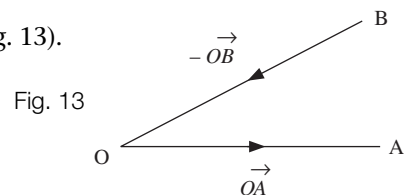
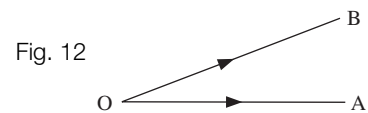
i  $-\vec{OB}$  is the vector with the same length as  $\vec{OB}$  but is in the opposite direction. In fact  $-\vec{OB}$  is the vector  $\vec{BO}$  (Fig. 13).

Therefore  $\vec{OA} - \vec{OB}$  is given by

$$\vec{OA} - \vec{OB} = \vec{BA} \quad (\text{Fig. 14})$$

We can justify this by using algebra:

$$\begin{aligned} \vec{OA} - \vec{OB} &= \vec{OA} + (-\vec{OB}) \\ &= -\vec{OB} + \vec{OA} \\ &= \vec{BO} + \vec{OA} = \vec{BA} \end{aligned}$$





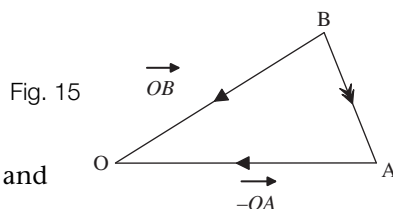
Example 2 *continued*

ii Similarly  $-\vec{OA}$  is the vector  $\vec{AO}$ . Hence

$$\vec{OB} - \vec{OA} = \vec{AB} \quad (\text{Fig. 15})$$

iii Clearly  $\vec{OA} - \vec{OA} = \mathbf{0}$  [Zero vector]

The zero vector is denoted by  $\mathbf{0}$ , and has zero magnitude and direction.



Note that, if  $\vec{AB}$  is not the zero vector then

$$\vec{AB} \neq \vec{BA} \quad [\text{Not equal vectors}]$$

$\vec{AB}$  and  $\vec{BA}$  have the same magnitude but opposite directions, in fact

$$\vec{AB} = -\vec{BA}$$

## Example 3

Write i  $\vec{AB}$  ii  $\vec{BA}$  and iii  $\vec{BC}$  in terms of  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  (see Fig. 16).

Solution

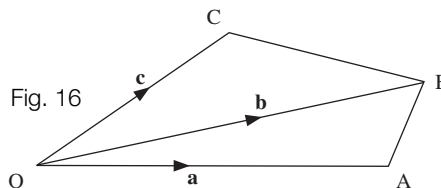
$$\begin{aligned} \text{i } \vec{AB} &= \vec{AO} + \vec{OB} \\ &= -\vec{OA} + \vec{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a} \end{aligned}$$

ii The vector  $\vec{BA}$  is given by

$$\begin{aligned} \vec{BA} &= \vec{BO} + \vec{OA} \\ &= -\vec{OB} + \vec{OA} = -\mathbf{b} + \mathbf{a} = \mathbf{a} - \mathbf{b} \end{aligned}$$

iii The vector  $\vec{BC}$  is

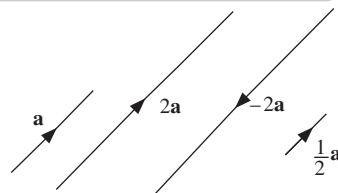
$$\begin{aligned} \vec{BC} &= \vec{BO} + \vec{OC} \\ &= -\vec{OB} + \vec{OC} = -\mathbf{b} + \mathbf{c} = \mathbf{c} - \mathbf{b} \end{aligned}$$



## A4 Scalar multiplication

Figure 17 shows the vector  $\mathbf{a}$  and its scalar multiples  $2\mathbf{a}$ ,  $-2\mathbf{a}$  and  $\frac{1}{2}\mathbf{a}$ . Notice that  $2\mathbf{a}$  has the same direction as vector  $\mathbf{a}$  but is twice as long,

Fig. 17



and  $\frac{1}{2}\mathbf{a}$  is half the length of  $\mathbf{a}$ . Also  $-2\mathbf{a}$  has the same magnitude as  $2\mathbf{a}$  but is in the opposite direction.

### Example 4

Figure 18 shows a triangle OAB with M as the midpoint of the line AB. Determine  $\vec{OM}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .

Solution

Since M is the midpoint of BA we have

$$\vec{OM} = \vec{OB} + \frac{1}{2}\vec{BA}$$

We have  $\vec{OB} = \mathbf{b}$  but we need to express  $\vec{BA}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .

$$\begin{aligned}\vec{BA} &= \vec{BO} + \vec{OA} \\ &= -\vec{OB} + \vec{OA} \\ &= \vec{OA} - \vec{OB} = \mathbf{a} - \mathbf{b}\end{aligned}$$

Substituting  $\vec{OB} = \mathbf{b}$  and  $\vec{BA} = \mathbf{a} - \mathbf{b}$  into  $\vec{OM} = \vec{OB} + \frac{1}{2}\vec{BA}$  gives

$$\begin{aligned}\vec{OM} &= \mathbf{b} + \frac{1}{2}(\mathbf{a} - \mathbf{b}) \\ &= \mathbf{b} + \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b} \\ &= \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{a} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) \quad \left[ \text{Taking out } \frac{1}{2} \right]\end{aligned}$$

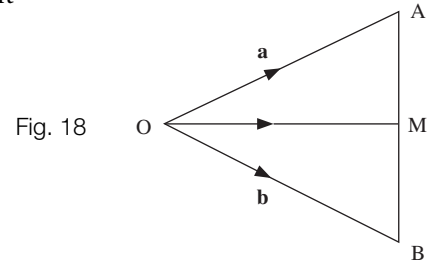


Fig. 18

## A5 Applications of vectors in mechanics



### Example 5 *mechanics*

Find the magnitude,  $|\mathbf{R}|$ , of the resultant force  $\mathbf{R}$  shown in Fig. 19.

Solution



**What is angle B equal to?**

$$\text{angle } B = 180^\circ - (20^\circ + 30^\circ) = 130^\circ$$

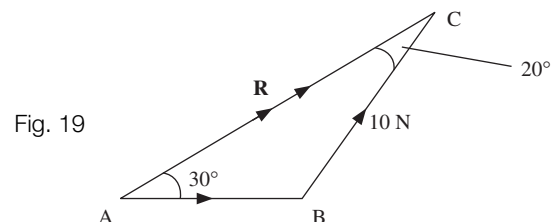


Fig. 19

Example 5 *continued*

To find  $|\mathbf{R}|$  we use the sine rule. Let  $a$  and  $b$  represent the sides opposite angles  $A$  and  $B$  respectively. Thus  $a = 10$  and  $b = |\mathbf{R}|$ . We need to find the length  $b$  by using

$$\frac{b}{\sin(B)} = \frac{a}{\sin(A)}$$

Substituting  $A = 30^\circ$ ,  $a = 10$  and  $B = 130^\circ$ :

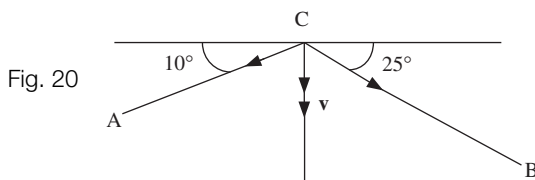
$$\frac{b}{\sin(130^\circ)} = \frac{10}{\sin(30^\circ)} = 20$$

$$b = 20 \times \sin(130^\circ) = 15.32$$

Thus  $|\mathbf{R}| = 15.32$  N.

Example 6 *mechanics*

The point C of Fig. 20 shows a particle rotating in a circle with the spin velocity represented by  $\vec{CA}$  and the forward velocity represented by  $\vec{CB}$ . The resultant velocity,  $\mathbf{v}$ , is vertically downwards. Find  $\mathbf{v}$  if  $|\vec{CB}| = 5$  m/s.

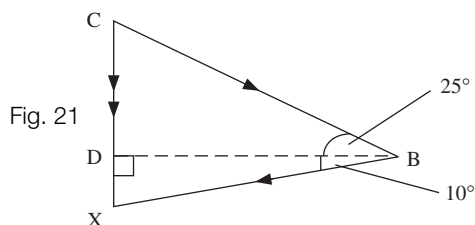


Solution

We shift  $\vec{CA}$  to the end of  $\vec{CB}$  to make the triangle shown in Fig. 21.

To obtain  $\vec{CX}$  of Fig. 21 we use the sine rule:

$$\frac{x}{\sin(X)} = \frac{b}{\sin(B)}$$



where  $x$  and  $b$  are the sides opposite angles  $X$  and  $B$  respectively.



**Considering the triangle BDX, what is angle X equal to?**

Recall that the sum of the angles inside a triangle add up to  $180^\circ$ . Thus

$$\text{angle } X = 180^\circ - (90^\circ + 10^\circ) = 80^\circ$$

In our triangle CBX:

$$\text{angle } X = 80^\circ, \text{ angle } B = 10^\circ + 25^\circ = 35^\circ \text{ and } x = 5 \text{ because } |\vec{CB}| = 5$$

Example 6 *continued*

By substituting these values into  $\frac{x}{\sin(X)} = \frac{b}{\sin(B)}$  we have

$$\frac{5}{\sin(80^\circ)} = \frac{b}{\sin(35^\circ)}$$

$$b = \frac{5}{\sin(80^\circ)} \times \sin(35^\circ) = 2.912$$

Therefore the velocity  $\mathbf{v} = 2.91$  m/s vertically downwards.

**SUMMARY**

A vector has direction and magnitude, and a scalar has magnitude but no direction. Vectors are basic tools used in engineering.

**Exercise 12(a)**

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- 1 [mechanics]

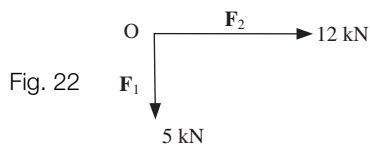
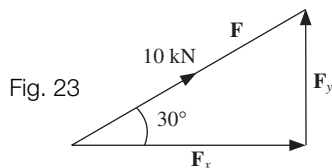


Figure 22 shows two forces  $\mathbf{F}_1$  and  $\mathbf{F}_2$  acting on an object O. Determine the resultant force,  $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$ .

- 2 [mechanics]



Determine the horizontal,  $F_x$ , and the vertical,  $F_y$ , components of the force  $\mathbf{F}$  shown in Fig. 23.

- 3 [mechanics] A force of 12 000 N is applied to a body at an angle of  $45^\circ$  to the horizontal. Determine the vertical and horizontal components of the force.

- 4 [mechanics]

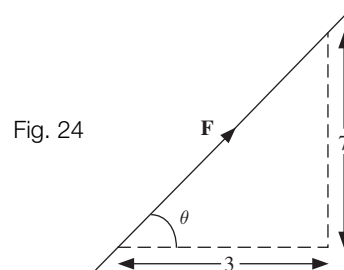


Figure 24 shows a force,  $\mathbf{F}$ , of 10 kN applied at an angle  $\theta$ . Obtain the horizontal and vertical components of the force.

Exercise 12(a) continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

5

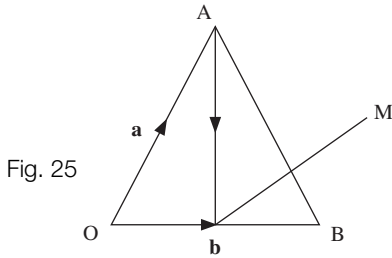


Fig. 25

In the triangle OAB shown in Fig. 25, the point M is the midpoint of OB. Determine  $\vec{AM}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .

6

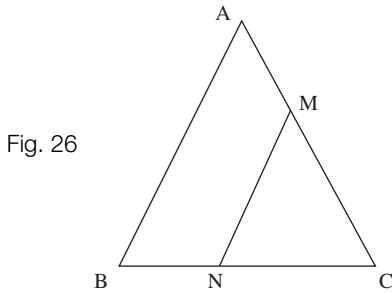


Fig. 26

Figure 26 shows a triangle ABC where M is the midpoint of AC and N is the midpoint of BC. Show that

$$\vec{MN} = \frac{1}{2} \vec{AB}.$$



The remaining questions are in the field of [mechanics].

7 Figure 27 shows forces  $\mathbf{F}_1$  and  $\mathbf{F}_2$  exerted on a particle O. The resultant force,  $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ , is horizontal. Determine  $F$  if  $F_2$  has a magnitude of 10 N.

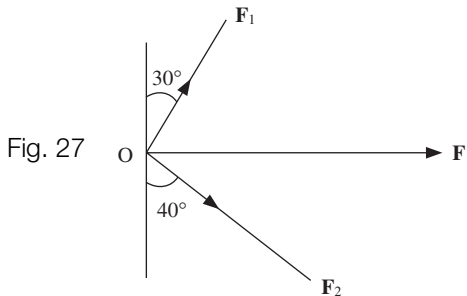
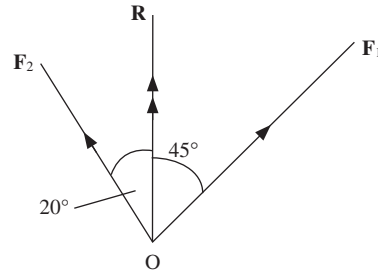


Fig. 27

8 Figure 28 shows forces  $\mathbf{F}_1$  and  $\mathbf{F}_2$  applied to an object O. The resultant force  $\mathbf{R}$  is vertical.

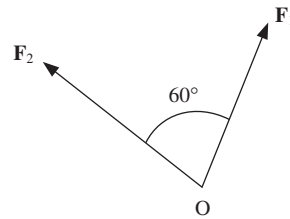
Fig. 28



Determine the magnitude of  $\mathbf{R}$  if  $F_1$  has a magnitude of 12 kN.

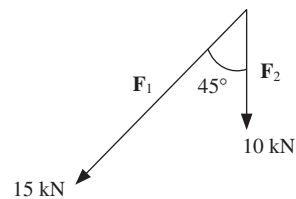
9 Figure 29 shows forces  $\mathbf{F}_1$  and  $\mathbf{F}_2$  of magnitude 7 kN and 20 kN respectively. Determine the resultant force.

Fig. 29



10 Determine the resultant force,  $\mathbf{F}_1 + \mathbf{F}_2$ , of the forces  $\mathbf{F}_1$  and  $\mathbf{F}_2$  shown in Fig. 30.

Fig. 30



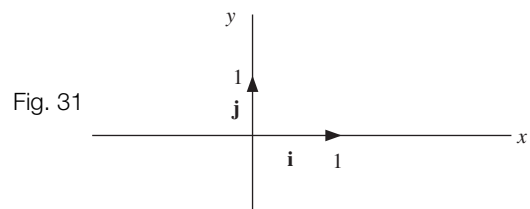
SECTION B **Vectors in Cartesian co-ordinates**

By the end of this section you will be able to:

- write vectors of two dimensions in **i** and **j** notation
- use a calculator to find the resultant vector
- examine engineering applications

B1 **Vectors in two dimensions**

We define vectors **i** and **j** in the  $x$ - $y$  plane as the unit vectors in the  $x$  and  $y$  directions respectively (Fig. 31). The term 'unit vector' means that the magnitude (size) of the vector is equal to 1. So **i** is one unit in the direction of the  $x$  axis and **j** is one unit in the direction of the  $y$  axis. The vectors **i** and **j** are shown in Fig. 31.

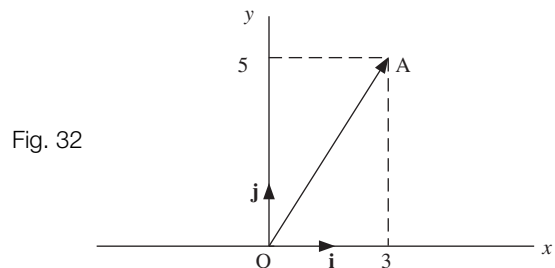


Do not confuse the unit vector **j** with the complex number  $\sqrt{-1}$  of Chapter 10. The unit vector **j** is denoted by bold typeface.

Consider the vector  $\vec{OA}$  shown in Fig. 32.

To get from O to A we can go 3 units along the  $x$  axis and 5 units up the  $y$  axis. We can write this as

$$3\mathbf{i} + 5\mathbf{j} \text{ or } \vec{OA} = 3\mathbf{i} + 5\mathbf{j}$$



Write the vectors  $\vec{OA}$ ,  $\vec{OB}$  and  $\vec{OC}$ , shown in Fig. 33, in terms of the unit vectors **i** and **j**.

The vector  $\vec{OA}$  is 7 units across and 10 units up, so

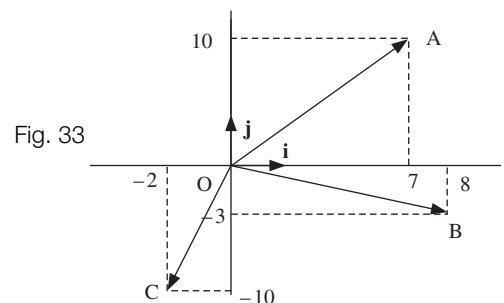
$$\vec{OA} = 7\mathbf{i} + 10\mathbf{j}$$

Similarly

$$\vec{OB} = 8\mathbf{i} - 3\mathbf{j}$$

Also the vector  $\vec{OC}$  is 2 units to the left and 10 units down, therefore

$$\vec{OC} = -2\mathbf{i} - 10\mathbf{j}$$



Generally a vector from the origin, O, to a point, P, is called the position vector of P and is denoted by  $\vec{OP}$ . Moreover the vector  $\vec{OP}$  from the origin to the point  $P = (x, y)$  can be written as

**12.1** 
$$\vec{OP} = x\mathbf{i} + y\mathbf{j}$$

We can solve problems in mechanics by using this new notation. The advantage of this approach is that we can find the resultant vector of two or more vectors without using the sine and cosine rules. Consider the following examples.

The displacement,  $s$ , of an object by 5 m is shown in Fig. 34.

This displacement  $s$  can be represented in  $i$  and  $j$  notation as

$$s = 5i + 0j = 5i$$



### Example 7 *mechanics*

A projectile is fired with an initial velocity of 20 m/s and makes an angle of  $60^\circ$  with the horizontal (Fig. 35).

Write the velocity in  $i$  and  $j$  notation.

**Solution**

We can construct a right-angled triangle (Fig. 36).

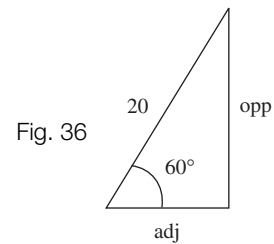
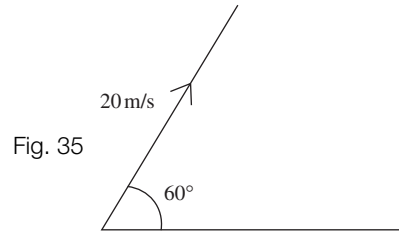
By using trigonometry we have

$$\text{adj} = \text{hyp} \times \cos(\theta) = 20 \cos(60^\circ) = 10$$

$$\text{opp} = \text{hyp} \times \sin(\theta) = 20 \sin(60^\circ) = 17.32$$

Thus the velocity can be written as

$$10i + 17.32j$$



### Example 8 *mechanics*

Figure 37 shows an object O subject to forces  $F_1$  and  $F_2$ . Find the resultant force  $R$ .

**Solution**

Writing  $F_1$  and  $F_2$  in  $i$  and  $j$  components:

$$F_1 = 10i \quad [\text{Because it is only in the } i \text{ direction}]$$

The 12 kN force at an angle of  $45^\circ$  is made up of  $12\cos(45^\circ)$  in the  $i$  direction (the adjacent) and  $12\sin(45^\circ)$  in the  $j$  direction (the opposite), as shown in Fig. 38.

We have

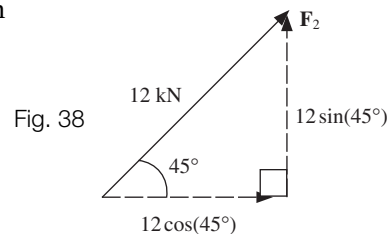
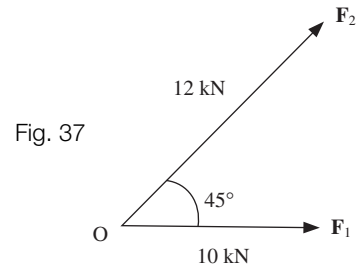
$$F_2 = [12\cos(45^\circ)]i + [12\sin(45^\circ)]j$$

Adding the two vectors gives

$$\begin{aligned} R &= F_1 + F_2 \\ &= 10i + 12\cos(45^\circ)i + 12\sin(45^\circ)j \\ &= [10 + 12\cos(45^\circ)]i + 12\sin(45^\circ)j \\ &= 18.485i + 8.485j \end{aligned}$$

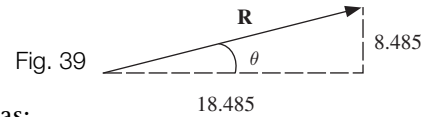
[Collecting like terms]

[Evaluating]



Example 8 *continued*

Hence the vector  $\mathbf{R}$  is 18.485 units in the horizontal direction and 8.485 units up (Fig. 39).



We can find the magnitude (length) by using Pythagoras:

$$|\mathbf{R}| = \sqrt{18.485^2 + 8.485^2} = 20.34 \text{ kN} \quad [\text{Scalar}]$$



Now we need to find the direction of  $\mathbf{R}$ . **What is the value of angle  $\theta$ ?**

By trigonometry

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{8.485}{18.485}$$

Taking inverse tan gives

$$\theta = \tan^{-1}\left(\frac{8.485}{18.485}\right) = 24.66^\circ$$

The resultant force  $\mathbf{R}$  has a magnitude of 20.34 kN at an angle of  $24.66^\circ$  inclined to the horizontal. Remember  $\mathbf{R}$  is a vector but  $|\mathbf{R}|$  is only concerned with its magnitude, **not** its direction, so is a scalar.

Example 9 *mechanics*

Figure 40 shows three forces  $\mathbf{F}_1$ ,  $\mathbf{F}_2$  and  $\mathbf{F}_3$  applied to an object O. Determine the resultant force  $\mathbf{R}$ .

Solution

We can measure each angle from the positive horizontal axis. Therefore the 8 kN force is at an angle of  $135^\circ$  ( $=180^\circ - 45^\circ$ ). Putting each force into its  $\mathbf{i}$  and  $\mathbf{j}$  components by using trigonometry gives

$$\mathbf{F}_1 = [10\cos(60^\circ)]\mathbf{i} + [10\sin(60^\circ)]\mathbf{j}$$

$$\mathbf{F}_2 = [8\cos(135^\circ)]\mathbf{i} + [8\sin(135^\circ)]\mathbf{j}$$

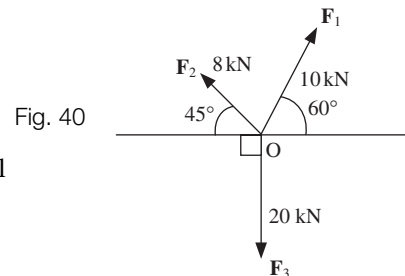
$$\mathbf{F}_3 = -20\mathbf{j}$$

Adding the forces gives the resultant force,  $\mathbf{R}$ :

$$\begin{aligned} \mathbf{R} &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \\ &= [10\cos(60^\circ)]\mathbf{i} + [10\sin(60^\circ)]\mathbf{j} + [8\cos(135^\circ)]\mathbf{i} + [8\sin(135^\circ)]\mathbf{j} - 20\mathbf{j} \\ &= [10\cos(60^\circ) + 8\cos(135^\circ)]\mathbf{i} + [10\sin(60^\circ) + 8\sin(135^\circ) - 20]\mathbf{j} \\ &= -0.657\mathbf{i} - 5.683\mathbf{j} \end{aligned}$$

The magnitude of the resultant force,  $|\mathbf{R}|$ , is found by using Pythagoras:

$$|\mathbf{R}| = \sqrt{(-0.657)^2 + (-5.683)^2} = 5.72 \text{ kN}$$



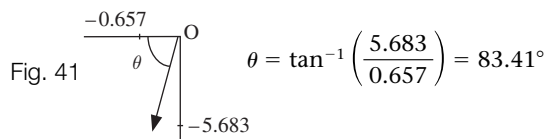




### Example 9 *continued*

The angle can be determined by plotting  $-0.657\mathbf{i} - 5.683\mathbf{j}$  and then using trigonometry and a calculator.

So  $\mathbf{R}$  has magnitude 5.72 kN and is at an angle of  $\theta = 83.41^\circ$ , as shown in Fig. 41.



## B2 Use of calculator

The calculations in examples can be simplified by using a calculator.

Consider **Example 9**. We can also write forces  $\mathbf{F}_1$ ,  $\mathbf{F}_2$  and  $\mathbf{F}_3$  in polar form, that is

$$\mathbf{F}_1 = 10 \angle 60^\circ$$

Measuring the angle anticlockwise from the positive horizontal axis:

$$\mathbf{F}_2 = 8 \angle 135^\circ$$

Measuring the angle clockwise from the positive horizontal axis:

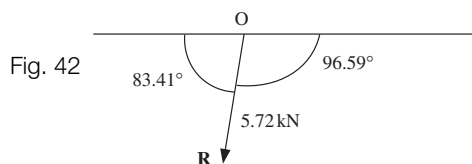
$$\mathbf{F}_3 = 20 \angle (-90^\circ)$$

$10 \angle 60^\circ$  means the vector has a magnitude of 10 units and is at an angle of  $60^\circ$ . We used this polar form notation for complex numbers in Chapter 10.

On some calculators you can add these forces directly in polar form, in others you need to place each force into  $\mathbf{i}$  and  $\mathbf{j}$  notation and then carry out the addition.

See the handbook of your calculator.

Solving **Example 9** on the calculator gives the answer in polar form as  $5.72 \angle -96.59^\circ$ . The minus sign in front of  $96.59^\circ$  means it is measured in a clockwise direction. Note that the calculator gives the angle measured from the positive horizontal axis.



The angle  $96.59^\circ$  in Fig. 42 is the same as the angle  $83.41^\circ$  in Fig. 41.

The resultant vector,  $\mathbf{R}$ , has a magnitude of 5.72 kN and an angle of  $96.59^\circ$  as shown in Fig. 42.

It is important to note that this use of the calculator only works for vectors in two dimensions, and we cannot use this technique for three-dimensional vectors, which are discussed in the next section.

## SUMMARY

Unit vectors in the direction of the  $x$  and  $y$  axes are denoted by  $\mathbf{i}$  and  $\mathbf{j}$  respectively.

We apply trigonometry to find the resultant vector.

We can also use a calculator to find the resultant vector.

## Exercise 12(b)

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)All questions are in the field of [*mechanics*].

1

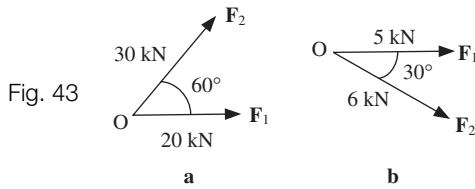


Figure 43, **a** and **b**, shows forces  $F_1$  and  $F_2$  applied to an object  $O$ .

Determine the resultant force,  $R = F_1 + F_2$ , in each case.

2 The following forces have units of kN:

$$F_1 = 4\mathbf{i} - 6\mathbf{j}$$

$$F_2 = 7\mathbf{i} - 12\mathbf{j}$$

Determine the magnitude of the force  $F = 31F_1 - 3F_2$ .

3

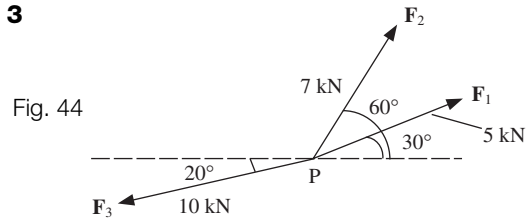
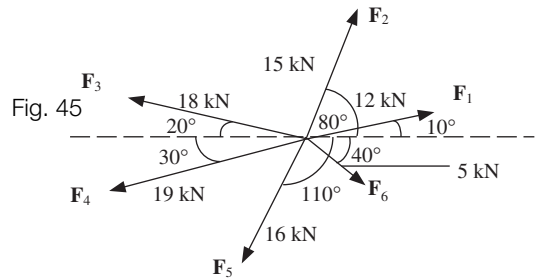


Figure 44 shows forces  $F_1$ ,  $F_2$  and  $F_3$  acting on a particle  $P$ . Determine the resultant force  $R$ .

4 Determine the resultant force of the forces shown in Fig. 45.



## SECTION C Three-dimensional vectors

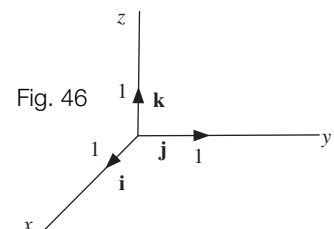
By the end of this section you will be able to:

- write vectors of three dimensions in  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  notation
- find the magnitude of a vector in three dimensions
- evaluate the unit vector

## C1 Vectors in three dimensions

The  $x$ - $y$  plane can be extended to cover three dimensions by including a third axis called the  $z$  axis. This axis is at right angles to the other two,  $x$  and  $y$ , axes. The position of a point in three dimensions is given by three co-ordinates  $(x, y, z)$ . Real life is three dimensional, hence many engineering problems require the application of three dimensions.

Let's examine vectors in three dimensions. We define  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  as the unit vectors in the  $x$ ,  $y$  and  $z$  directions respectively (Fig. 46).



Consider the vector  $\vec{OA}$  in Fig. 47.

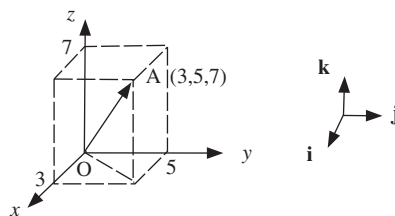
To get from O to A we can go 3 units along the  $x$  axis, 5 units along the  $y$  axis and then 7 units up the  $z$  axis. Therefore

$$\vec{OA} = 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

Thus the position vector of the point with co-ordinates  $(3, 5, 7)$  is

$$3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

Fig. 47



**?** How would you write the vectors from the origin to the points:

$$\mathbf{B} = (-3, 5, 7), \quad \mathbf{C} = (1, 1, -1) \text{ and } \mathbf{D} = (-5, -11, 23)?$$

We have

$$\vec{OB} = -3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}, \quad \vec{OC} = \mathbf{i} + \mathbf{j} - \mathbf{k} \text{ and } \vec{OD} = -5\mathbf{i} - 11\mathbf{j} + 23\mathbf{k}$$

In general the vectors from the origin to the point  $A = (x, y, z)$  are given by

$$\vec{OA} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Let  $\mathbf{p}$  and  $\mathbf{q}$  be vectors from the origin in three-dimensional space such that

$$\mathbf{p} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \text{ and } \mathbf{q} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$$

Then the addition of  $\mathbf{p}$  and  $\mathbf{q}$  is defined by

$$\mathbf{12.2} \quad \mathbf{p} + \mathbf{q} = (a + d)\mathbf{i} + (b + e)\mathbf{j} + (c + f)\mathbf{k}$$

All you need to do is add the  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  components separately.

Consider the vector  $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ .

**?** What is the magnitude (length) of the vector  $\mathbf{r}$ ?

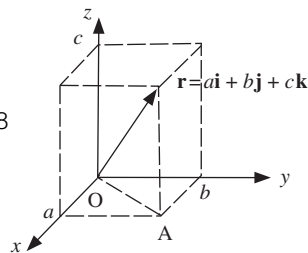
Figure 48 shows the vector  $\mathbf{r}$ . By Pythagoras, we can find the length  $OA^2$ :

$$OA^2 = a^2 + b^2$$

The length of the vector  $|\mathbf{r}|$  is found by applying Pythagoras again:

$$\begin{aligned} |\mathbf{r}|^2 &= OA^2 + c^2 \\ &= a^2 + b^2 + c^2 \end{aligned}$$

Fig. 48



**?** How can we find  $|\mathbf{r}|$ ?

Take the square root of both sides:

$$|\mathbf{r}| = \sqrt{a^2 + b^2 + c^2}$$

Hence the magnitude,  $|\mathbf{r}|$ , of a vector

$$\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

is given by

$$\mathbf{12.3} \quad |\mathbf{r}| = \sqrt{a^2 + b^2 + c^2}$$

In two-dimensional vectors we always found the direction of the vector by evaluating the angle that the vector made with the horizontal axis. In three-dimensional vectors there is **no one** angle between a three-dimensional vector and the horizontal axis or any other axes which specifies the direction.



### Example 10 *mechanics*

Three forces  $\mathbf{F}_1$ ,  $\mathbf{F}_2$  and  $\mathbf{F}_3$  (measured in newton) act on a particle. Obtain the magnitude of the resultant force,  $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ , in each of the following:

**a**  $\mathbf{F}_1 = 5\mathbf{i} - \mathbf{j} - \mathbf{k}$ ,  $\mathbf{F}_2 = 3\mathbf{i} + \mathbf{j}$ ,  $\mathbf{F}_3 = 2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}$

**b**  $\mathbf{F}_1 = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ ,  $\mathbf{F}_2 = \mathbf{i}$ ,  $\mathbf{F}_3 = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$

**c**  $\mathbf{F}_1 = 7\mathbf{i} - \mathbf{j}$ ,  $\mathbf{F}_2 = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ ,  $\mathbf{F}_3 = 7\mathbf{i} - \mathbf{j} - 21\mathbf{k}$

Solution

**a** Adding the forces,  $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ , gives

$$\begin{aligned} \mathbf{R} &= (5\mathbf{i} - \mathbf{j} - \mathbf{k}) + (3\mathbf{i} + \mathbf{j}) + (2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}) \\ &= (5 + 3 + 2)\mathbf{i} + (-1 + 1 - 7)\mathbf{j} + (-1 + 0 + 3)\mathbf{k} \quad [\text{Collecting like terms}] \\ &= 10\mathbf{i} - 7\mathbf{j} + 2\mathbf{k} \end{aligned}$$

To find the magnitude we use  $|\mathbf{r}| = \sqrt{a^2 + b^2 + c^2}$  with  $a = 10$ ,  $b = -7$  and  $c = 2$ :

$$|\mathbf{R}| = \sqrt{10^2 + (-7)^2 + 2^2} = 12.37 \text{ N (2 d.p.)}$$

**b** We have

$$\begin{aligned} \mathbf{R} &= (2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) + \mathbf{i} + (\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \\ &= (2 + 1 + 1)\mathbf{i} + (-3 + 0 + 3)\mathbf{j} + (2 + 0 - 1)\mathbf{k} \\ &= 4\mathbf{i} + \mathbf{k} \quad [\text{No } \mathbf{j} \text{ term}] \end{aligned}$$

The magnitude is

$$|\mathbf{R}| = \sqrt{4^2 + 0^2 + 1^2} = 4.12 \text{ N (2 d.p.)}$$

**c** Similarly we have

$$\begin{aligned} \mathbf{R} &= (7\mathbf{i} - \mathbf{j}) + (2\mathbf{i} + \mathbf{j} - \mathbf{k}) + (7\mathbf{i} - \mathbf{j} - 21\mathbf{k}) \\ &= (7 + 2 + 7)\mathbf{i} + (-1 + 1 - 1)\mathbf{j} + (0 - 1 - 21)\mathbf{k} \\ &= 16\mathbf{i} - \mathbf{j} - 22\mathbf{k} \end{aligned}$$

and the magnitude is

$$|\mathbf{R}| = \sqrt{16^2 + (-1)^2 + (-22)^2} = 27.22 \text{ N (2 d.p.)}$$

## C2 Unit vectors

Unit vectors are vectors which have a magnitude of 1. If a vector,  $\mathbf{r}$ , has a magnitude of 5 then the unit vector, denoted by  $\mathbf{u}$ , in the direction of  $\mathbf{r}$  is given by

$$\mathbf{u} = \frac{1}{5} \mathbf{r} \quad \left[ \text{Sometimes this is denoted by } \hat{\mathbf{r}} = \frac{1}{5} \mathbf{r} \right]$$

For a general vector,  $\mathbf{r}$ , we have the situation shown in Fig. 49.

The unit vector,  $\mathbf{u}$ , along the direction of a vector  $\mathbf{r}$  is given by

$$\text{12.4} \quad \mathbf{u} = \frac{\mathbf{r}}{|\mathbf{r}|} \quad \text{or} \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}$$

provided that  $\mathbf{r}$  is not the zero vector.

A unit vector is a vector which has a magnitude of 1.

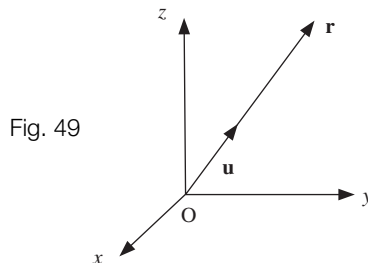


Fig. 49

**?** What is the unit vector,  $\mathbf{u}$ , along the direction of the vector  $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ ?

The magnitude,  $|\mathbf{r}|$ , of the vector  $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$  is given by

$$|\mathbf{r}| = \sqrt{a^2 + b^2 + c^2}$$

Thus

$$\text{12.5} \quad \mathbf{u} = \frac{1}{\sqrt{a^2 + b^2 + c^2}}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

Let's find the unit vector,  $\mathbf{u}$  (or  $\hat{\mathbf{r}}$ ), in the direction of the vector  $\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ .

The magnitude is  $|\mathbf{r}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$ . Therefore the unit vector is given by

$$\mathbf{u} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

**?** What is the magnitude (size) of the vector  $\mathbf{u} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ ?

It's 1 because it is a unit vector as  $|\mathbf{u}| = \frac{1}{3}|\mathbf{r}| = \frac{1}{3}(3) = 1$ .

### Example 11

Determine the unit vector in the direction of  $\vec{AB}$  where  $A = (1, 3, 5)$  and  $B = (7, 1, 2)$ .

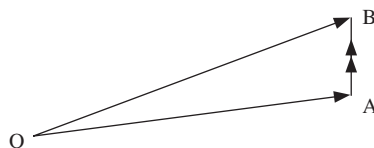
Solution

In Fig. 50:

$$\vec{AB} = \vec{AO} + \vec{OB}$$

$$\vec{AB} = -\vec{OA} + \vec{OB} = \vec{OB} - \vec{OA}$$

Fig. 50



\*

### Example 11 *continued*

The vector,  $\vec{OA}$ , from the origin to  $A = (1, 3, 5)$  is written as

$$\vec{OA} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$$

Similarly for  $B = (7, 1, 2)$  we have

$$\vec{OB} = 7\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

Substituting these into  $\vec{AB} = \vec{OB} - \vec{OA}$  gives

$$\begin{aligned}\vec{AB} &= \vec{OB} - \vec{OA} \\ &= (7\mathbf{i} + \mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) \\ &= (7 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (2 - 5)\mathbf{k} \\ &= 6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}\end{aligned}$$

The unit vector in the direction of  $\vec{AB} = 6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$  is determined by

$$\begin{aligned}\mathbf{u} &= \frac{1}{\sqrt{a^2 + b^2 + c^2}} (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \\ \text{unit vector} &= \frac{1}{\sqrt{6^2 + (-2)^2 + (-3)^2}} (6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) \\ &= \frac{1}{7} (6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k})\end{aligned}$$



### Example 12 *mechanics*

Determine the components of the force,  $\mathbf{F}$ , which have a combined magnitude of 28 kN in the direction of the vector  $\vec{AB}$  where

$$A = (1, 3, 5) \text{ and } B = (7, 1, 2)$$

Solution

We first obtain the unit vector in the direction of  $\vec{AB}$  and then multiply the result by 28 kN because the unit vector gives the direction of  $\vec{AB}$  and 28 kN gives the magnitude.

**How do we find the unit vector?**

$A = (1, 3, 5)$  and  $B = (7, 1, 2)$  are the same co-ordinates as those of **Example 11**, so the

unit vector =  $\frac{1}{7} (6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k})$ . Thus

$$\begin{aligned}\mathbf{F} &= 28 \times \frac{1}{7} (6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) \\ &= 4 (6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) \\ &= (24\mathbf{i} - 8\mathbf{j} - 12\mathbf{k}) \text{ kN}\end{aligned}$$

**SUMMARY**

Unit vectors in the direction of the  $x$ ,  $y$  and  $z$  axes are denoted by  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  respectively. The magnitude,  $|\mathbf{r}|$ , of a vector  $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$  is

$$12.3 \quad |\mathbf{r}| = \sqrt{a^2 + b^2 + c^2}$$

The unit vector,  $\mathbf{u}$  (or  $\hat{\mathbf{r}}$ ), in the direction of the non-zero vector  $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$  is given by

$$12.5 \quad \mathbf{u} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} (a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

**Exercise 12(c)**

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)



All questions belong to the field of [*mechanics*].

- 1 Forces  $\mathbf{F}_1$ ,  $\mathbf{F}_2$ ,  $\mathbf{F}_3$  and  $\mathbf{F}_4$  act on a particle so that it is in equilibrium, that is  $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = \mathbf{0}$ . Determine  $\mathbf{F}_4$  for each of the following:
  - a  $\mathbf{F}_1 = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ ,  $\mathbf{F}_2 = 7\mathbf{i} - 7\mathbf{j} - 7\mathbf{k}$ ,  
 $\mathbf{F}_3 = 9\mathbf{i} - 6\mathbf{j} - \mathbf{k}$
  - b  $\mathbf{F}_1 = 61\mathbf{i} - 64\mathbf{j} - 89\mathbf{k}$ ,  $\mathbf{F}_2 = 93\mathbf{i} - 98\mathbf{j}$ ,  
 $\mathbf{F}_3 = 22\mathbf{i} - 41\mathbf{j} + 43\mathbf{k}$
- 2 Forces  $\mathbf{F}_1$ ,  $\mathbf{F}_2$ ,  $\mathbf{F}_3$  and  $\mathbf{F}_4$  all act through the same point in a body such that the body is in equilibrium. Obtain the force  $\mathbf{F}_1$  given that  $\mathbf{F}_2 = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ ,  
 $\mathbf{F}_3 = -2\mathbf{i} - 3\mathbf{j} - 10\mathbf{k}$  and  
 $\mathbf{F}_4 = -\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$ .
- 3 Forces  
 $\mathbf{F}_1 = 2\mathbf{i} - \mathbf{j} + 10\mathbf{k}$ ,  $\mathbf{F}_2 = 12\mathbf{i} - 3\mathbf{j} + 22\mathbf{k}$ ,  
 $\mathbf{F}_3 = 7\mathbf{i} + \mathbf{j} - 7\mathbf{k}$ ,  $\mathbf{F}_4 = -9\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$   
are applied on a particle. Determine  $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4$  in terms of  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$ .
- 4 Determine the resultant,  $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4$ , of the following forces:  
 $\mathbf{F}_1 = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$   
 $\mathbf{F}_2$  has a magnitude of 26 N in the direction of  $12\mathbf{i} + 5\mathbf{k}$ .  
 $\mathbf{F}_3$  has a magnitude of 18 N in the direction of  $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ .  
 $\mathbf{F}_4$  has a magnitude of 12 N in the direction of  $4\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$ .
- 5 A force  $\mathbf{F}$  of magnitude 36 kN is applied in the direction of the vector  $\vec{AB}$  where  $A = (7, 5, 14)$  and  $B = (-1, 1, 15)$ . Find the force  $\mathbf{F}$ .
- 6 Find the force  $\mathbf{F}$  which has a magnitude of 11.3 kN in the direction of the vector  $\vec{AB}$  where  $A = (1.1, -2.6, -4.3)$  and  $B = (2.2, 5.6, 7.1)$ .
- 7 Let  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$  have co-ordinates  $(-16, 30, 19)$ ,  $(-12, -17, -23)$  and  $(-1, 27, -31)$  respectively. Let  $\mathbf{F}_{AB}$  be the force applied in the direction of  $\vec{AB}$  and  $\mathbf{F}_{AC}$  be the force applied in the direction of  $\vec{AC}$ . Given that the magnitude of  $\mathbf{F}_{AB}$  is 850 kN and the magnitude of  $\mathbf{F}_{AC}$  is 700 kN, determine  
$$\mathbf{F} = \mathbf{F}_{AB} + \mathbf{F}_{AC}$$
  
Also find the magnitude of  $\mathbf{F}$ .

SECTION D **Scalar products**

By the end of this section you will be able to:

- find scalar products
- examine engineering applications of a scalar product

D1 **Scalar product**

Figure 51 shows a constant force  $\mathbf{F}$  applied to an object and as a result the object moves from A to B. The work done by the force is the product of the magnitude of the force in the line of action,  $|\mathbf{F}|\cos(\theta)$ , and the distance the object has moved,  $|\vec{AB}|$ . Hence

$$\text{Work done} = |\mathbf{F}|\cos(\theta)|\vec{AB}|$$

This  $|\mathbf{F}|\cos(\theta)|\vec{AB}|$  gives the scalar product of the force vector  $\mathbf{F}$  and displacement vector  $\vec{AB}$ . Thus the work done is a scalar quantity.

When the angle  $\theta$  between the force  $\mathbf{F}$  and distance  $\vec{AB}$  is 0 then

$$\text{Work done} = |\mathbf{F}|\cos(\theta)|\vec{AB}| = |\mathbf{F}||\vec{AB}| \quad [\text{Because } \cos(0) = 1]$$

This is the least possible force used to push the object because we are pushing in the same direction as we would like the object to move.

In general if a constant force,  $\mathbf{F}$ , moves a particle from A to B through a displacement vector  $\mathbf{r}$ , then the work done by the force  $\mathbf{F}$  is given by

$$\text{Work done} = \mathbf{F} \cdot \mathbf{r} = |\mathbf{F}||\mathbf{r}|\cos(\theta)$$

$\mathbf{F} \cdot \mathbf{r}$  is called the scalar product or the dot product of  $\mathbf{F}$  and  $\mathbf{r}$ . We will return to this application of scalar product later in this section.

First let's consider the more general case of a scalar product of any two vectors,  $\mathbf{p}$  and  $\mathbf{q}$ .

Figure 52 shows two vectors  $\mathbf{p}$  and  $\mathbf{q}$  making an angle  $\theta$ . The scalar product of  $\mathbf{p}$  and  $\mathbf{q}$ , denoted by  $\mathbf{p} \cdot \mathbf{q}$ , is defined by

$$\mathbf{12.6} \quad \mathbf{p} \cdot \mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos(\theta)$$

where  $|\mathbf{p}|$  is the magnitude of  $\mathbf{p}$  and  $|\mathbf{q}|$  is the magnitude of  $\mathbf{q}$ .

The dot in  $\mathbf{p} \cdot \mathbf{q}$  represents the scalar product (or dot product) of  $\mathbf{p}$  and  $\mathbf{q}$ . Note that  $\mathbf{p} \cdot \mathbf{q}$  is a scalar quantity, that is a vector combined with another vector in this way gives a scalar result.

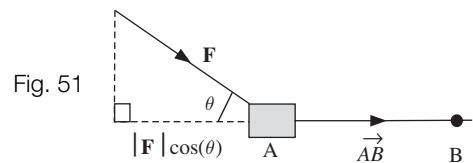


Fig. 51

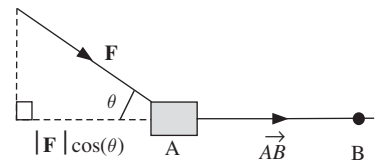


Fig. 52



The scalar product has the following properties:

$$12.7 \quad \mathbf{p} \cdot \mathbf{q} = \mathbf{q} \cdot \mathbf{p}$$

$$12.8 \quad (\mathbf{p} + \mathbf{q}) \cdot \mathbf{r} = \mathbf{p} \cdot \mathbf{r} + \mathbf{q} \cdot \mathbf{r}$$

$$12.9 \quad \mathbf{p} \cdot (\mathbf{q} + \mathbf{r}) = \mathbf{p} \cdot \mathbf{q} + \mathbf{p} \cdot \mathbf{r}$$

$$12.10 \quad k(\mathbf{p} \cdot \mathbf{q}) = k\mathbf{p} \cdot \mathbf{q} = \mathbf{p} \cdot k\mathbf{q} \quad [k \text{ is a constant}]$$

We will apply these properties in evaluating a scalar product of two vectors in component form.

### Example 13

Determine the following scalar products:

$$\mathbf{i} \cdot \mathbf{i}, \mathbf{i} \cdot \mathbf{j}, \mathbf{j} \cdot \mathbf{j}, \mathbf{j} \cdot \mathbf{k}, \mathbf{k} \cdot \mathbf{k} \text{ and } \mathbf{k} \cdot \mathbf{i}$$

where  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  are unit vectors in the  $x$ ,  $y$  and  $z$  directions respectively.

Solution

Since  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  are unit vectors, they all have a magnitude of 1. Furthermore  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  are at an angle of  $90^\circ$  to each other (Fig. 53).

We use the formula

$$12.6 \quad \mathbf{p} \cdot \mathbf{q} = |\mathbf{p}||\mathbf{q}| \cos(\theta)$$

The angle between  $\mathbf{i}$  and  $\mathbf{i}$  is  $0^\circ$  therefore

$$\mathbf{i} \cdot \mathbf{i} = |\mathbf{i}||\mathbf{i}| \underbrace{\cos(0^\circ)}_{=1} = 1 \times 1 \times 1 = 1$$

Similarly  $\mathbf{j} \cdot \mathbf{j} = |\mathbf{j}||\mathbf{j}| \cos(0^\circ) = 1$  and

$$\mathbf{k} \cdot \mathbf{k} = |\mathbf{k}||\mathbf{k}| \cos(0^\circ) = 1.$$

Also

$$\mathbf{i} \cdot \mathbf{j} = |\mathbf{i}||\mathbf{j}| \underbrace{\cos(90^\circ)}_{=0} = 1 \times 1 \times 0 = 0$$

$$\mathbf{j} \cdot \mathbf{k} = |\mathbf{j}||\mathbf{k}| \cos(90^\circ) = 0$$

$$\mathbf{k} \cdot \mathbf{i} = |\mathbf{k}||\mathbf{i}| \cos(90^\circ) = 0$$

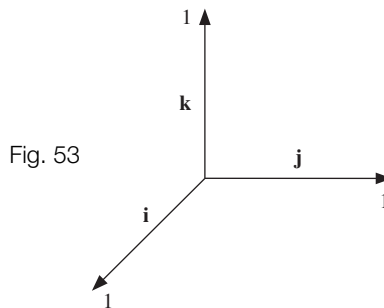


Fig. 53

We have

$$12.11 \quad \mathbf{i} \cdot \mathbf{i} = 1, \mathbf{j} \cdot \mathbf{j} = 1 \text{ and } \mathbf{k} \cdot \mathbf{k} = 1$$

$$12.12 \quad \mathbf{i} \cdot \mathbf{j} = 0, \mathbf{j} \cdot \mathbf{k} = 0 \text{ and } \mathbf{k} \cdot \mathbf{i} = 0$$

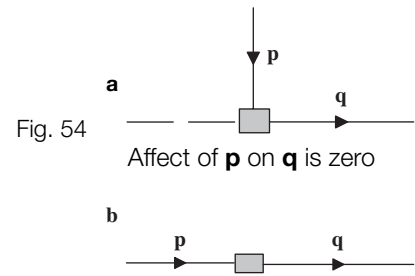
Notice that the scalar product of two different unit vectors which are at right angles to each other is 0, and if the unit vectors are the same then the scalar product is 1.

Any two vectors  $\mathbf{p}$  and  $\mathbf{q}$  which are perpendicular to each other will have a scalar product of zero because the work done by a force perpendicular to displacement is 0 (see Fig. 54a on the following page).

This is why  $\mathbf{i} \cdot \mathbf{j} = 0$ ,  $\mathbf{j} \cdot \mathbf{k} = 0$  and  $\mathbf{k} \cdot \mathbf{i} = 0$ .

Any two vectors  $\mathbf{p}$  and  $\mathbf{q}$  which are in parallel will have a scalar product  $|\mathbf{p}||\mathbf{q}|$  (see Fig. 54b).

This is why  $\mathbf{i} \cdot \mathbf{i} = |\mathbf{i}||\mathbf{i}| = 1 \times 1 = 1$ ,  
 $\mathbf{j} \cdot \mathbf{j} = |\mathbf{j}||\mathbf{j}| = 1 \times 1 = 1$  and  $\mathbf{k} \cdot \mathbf{k} = |\mathbf{k}||\mathbf{k}| = 1 \times 1 = 1$ .



### Example 14

Let  $\mathbf{p}$  and  $\mathbf{q}$  be vectors such that

$$\begin{aligned}\mathbf{p} &= a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \\ \mathbf{q} &= d\mathbf{i} + e\mathbf{j} + f\mathbf{k}\end{aligned}$$

Show that the scalar product of  $\mathbf{p}$  and  $\mathbf{q}$  is given by

$$\mathbf{p} \cdot \mathbf{q} = ad + be + cf$$

Solution

Expanding as algebraic brackets gives

$$\begin{aligned}\mathbf{p} \cdot \mathbf{q} &= (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) \\ &= (ad)\mathbf{i} \cdot \mathbf{i} + (ae)\mathbf{i} \cdot \mathbf{j} + (af)\mathbf{i} \cdot \mathbf{k} + (bd)\mathbf{j} \cdot \mathbf{i} + (be)\mathbf{j} \cdot \mathbf{j} + (bf)\mathbf{j} \cdot \mathbf{k} + (cd)\mathbf{k} \cdot \mathbf{i} \\ &\quad + (ce)\mathbf{k} \cdot \mathbf{j} + (cf)\mathbf{k} \cdot \mathbf{k}\end{aligned}$$

Since the scalar product of two different unit vectors which are at right angles to each other is 0, the only terms that do not vanish in the above are the ones containing  $\mathbf{i} \cdot \mathbf{i}$ ,  $\mathbf{j} \cdot \mathbf{j}$  and  $\mathbf{k} \cdot \mathbf{k}$ , and all these are equal to 1. Thus

$$\mathbf{p} \cdot \mathbf{q} = ad + be + cf$$

**Example 14** shows that the scalar product of two vectors is defined as

$$\mathbf{12.13} \quad (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) = ad + be + cf$$

To find the scalar product of two vectors given in component form you multiply the corresponding coefficients in the  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  directions and then add them together.

### Example 15

Determine  $(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \cdot (-5\mathbf{i} + \mathbf{j} - 4\mathbf{k})$

Solution

Applying

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) = ad + be + cf$$

gives

$$(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \cdot (-5\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = [2 \times (-5)] + [(-3) \times 1] + [1 \times (-4)] = -17$$

Notice that the scalar product of two vectors gives a scalar,  $-17$ .

The scalar product can also be implemented on a calculator. (See the handbook of your calculator.)

We can use the scalar product formula

$$\mathbf{12.6} \quad \mathbf{p} \cdot \mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos(\theta)$$

to find the angle  $\theta$  between vectors  $\mathbf{p}$  and  $\mathbf{q}$ . Rearranging **12.6** gives

$$\mathbf{12.14} \quad \cos(\theta) = \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}||\mathbf{q}|}$$

where  $\mathbf{p}$  and  $\mathbf{q}$  are not the zero vectors.

Remember  $\mathbf{p} \cdot \mathbf{q}$  is the work done by the force  $\mathbf{p}$  in moving an object through a displacement  $\mathbf{q}$  as illustrated in part **a** of Fig. 55.

If  $\mathbf{p}$  and  $\mathbf{q}$  are parallel, then the angle between them is  $0^\circ$ . If  $\mathbf{p}$  and  $\mathbf{q}$  are in opposite directions, then the angle between them is  $180^\circ$ . For example, when a ball is thrown vertically upwards then the angle between the velocity and acceleration is  $180^\circ$  as the ball is moving up, while gravity is attempting to accelerate it vertically downwards as shown in part **b** of Fig. 55.

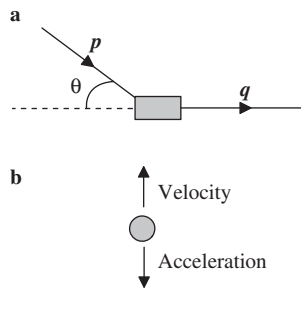


Fig. 55 Ball in upwards motion

## D2 Applications of scalar product



### Example 16 *mechanics*

The velocity,  $\mathbf{v}$ , and acceleration,  $\mathbf{a}$ , of a particle are given by

$$\mathbf{v} = 18\mathbf{i} - 6\mathbf{j} + \mathbf{k}$$

$$\mathbf{a} = 8\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

Determine the angle between the velocity,  $\mathbf{v}$ , and acceleration,  $\mathbf{a}$ .

Solution

We have

$$\dagger \quad \cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}||\mathbf{a}|}$$

We use

$$\mathbf{12.3} \quad |\mathbf{ai} + \mathbf{bj} + \mathbf{ck}| = \sqrt{a^2 + b^2 + c^2}$$

to find the magnitude of the velocity and acceleration:

$$|\mathbf{v}| = |18\mathbf{i} - 6\mathbf{j} + \mathbf{k}| = \sqrt{18^2 + (-6)^2 + 1^2} = 19$$

$$|\mathbf{a}| = |8\mathbf{i} + 4\mathbf{j} + \mathbf{k}| = \sqrt{8^2 + 4^2 + 1^2} = 9$$

$$\mathbf{v} \cdot \mathbf{a} = (18\mathbf{i} - 6\mathbf{j} + \mathbf{k}) \cdot (8\mathbf{i} + 4\mathbf{j} + \mathbf{k}) \quad \text{[Scalar product]}$$

$$= (18 \times 8) + [(-6) \times 4] + [1 \times 1] = 121$$

by **12.13**

**12.13**  $(\mathbf{ai} + \mathbf{bj} + \mathbf{ck}) \cdot (\mathbf{di} + \mathbf{ej} + \mathbf{fk}) = ad + be + cf$

Example 16 *continued*

Substituting  $\mathbf{v} \cdot \mathbf{a} = 121$ ,  $|\mathbf{v}| = 19$  and  $|\mathbf{a}| = 9$  into  $\dagger$  gives

$$\cos(\theta) = \frac{121}{(19 \times 9)}$$

Taking inverse cos gives the angle

$$\theta = \cos^{-1}\left(\frac{121}{(19 \times 9)}\right) = 44.96^\circ$$

The angle between velocity and acceleration is  $44.96^\circ$ .

As mentioned earlier in this section, the scalar product gives the work done by a force  $\mathbf{F}$ . If a constant force  $\mathbf{F}$  moves a particle through a displacement vector  $\mathbf{r}$ , then the work done by the force  $\mathbf{F}$  is given by

$$\mathbf{12.15} \quad \text{Work done} = \mathbf{F} \cdot \mathbf{r}$$

where  $\mathbf{F} \cdot \mathbf{r}$  is the scalar product of  $\mathbf{F}$  and  $\mathbf{r}$ .

Example 17 *mechanics*

Determine the work done by a force,  $\mathbf{F}$ , in moving an object through a displacement  $\mathbf{r}$  in each of the following:

- a**  $\mathbf{F} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}$ ,  $\mathbf{r} = \mathbf{i} - \mathbf{j} - \mathbf{k}$
- b**  $\mathbf{F} = 6\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$ ,  $\mathbf{r} = 9\mathbf{i} + 6\mathbf{j} + \mathbf{k}$
- c**  $\mathbf{F} = 37\mathbf{i} + 35\mathbf{j} - \mathbf{k}$ ,  $\mathbf{r} = 61\mathbf{i} + 64\mathbf{j} + 74\mathbf{k}$

Solution

We need to use

$$\mathbf{12.15} \quad \text{Work done} = \mathbf{F} \cdot \mathbf{r}$$

to find the work done for each of **a**, **b** and **c**.

$$\begin{aligned} \mathbf{a} \quad (3\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} - \mathbf{k}) &= (3 \times 1) + (-1 \times (-1)) + (-1 \times (-1)) \\ &= 5 \text{ J (J = joule is the SI unit of work)} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (6\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}) \cdot (9\mathbf{i} + 6\mathbf{j} + \mathbf{k}) &= (6 \times 9) + (2 \times 6) + (-6 \times 1) \\ &= 60 \text{ J} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad (37\mathbf{i} + 35\mathbf{j} - \mathbf{k}) \cdot (61\mathbf{i} + 64\mathbf{j} + 74\mathbf{k}) &= (37 \times 61) + (35 \times 64) + (-1 \times 74) \\ &= 4423 \text{ J} \end{aligned}$$

## D3 Differentiation of vectors

Remember  $\dot{x}$  means the time derivative,  $\frac{dx}{dt}$ , that is differentiate  $x$  with respect to  $t$ .

Suppose a particle moves in two dimensions, then its position can be defined by the  $x$ ,  $y$  co-ordinates which in turn are functions of time,  $t$ . The position vector,  $\mathbf{r}$ , of a particle can be specified by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

where  $x$  and  $y$  are functions of time,  $t$ . The velocity,  $\mathbf{v}$ , and acceleration,  $\mathbf{a}$ , of a particle can be found by differentiating  $\mathbf{r}$  with respect to  $t$ , that is

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \quad \text{[First derivative]}$$

$$\mathbf{a} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} \quad \text{[Second derivative]}$$

Remember  $\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$ ,  $\dot{x} = \frac{dx}{dt}$ ,  $\dot{y} = \frac{dy}{dt}$ ,  $\ddot{x} = \frac{d^2x}{dt^2}$  and  $\ddot{y} = \frac{d^2y}{dt^2}$ .

Example 18 *mechanics*

An object is dropped and its position vector,  $\mathbf{r}$ , is given by

$$\mathbf{r} = (t^4 - 3t^3)\mathbf{i} + (6t^2 - 2t)\mathbf{j}$$

Find the velocity,  $\mathbf{v}$ , and acceleration,  $\mathbf{a}$ , and the angle between  $\mathbf{v}$  and  $\mathbf{a}$  at  $t = 1$ .

Solution

Differentiating  $\mathbf{r}$  with respect to  $t$ :

$$\mathbf{r} = (t^4 - 3t^3)\mathbf{i} + (6t^2 - 2t)\mathbf{j}$$

$$\mathbf{v} = (4t^3 - 9t^2)\mathbf{i} + (12t - 2)\mathbf{j} \quad \text{[Differentiating]}$$

Differentiating again:

$$\mathbf{a} = (12t^2 - 18t)\mathbf{i} + 12\mathbf{j}$$

Substituting  $t = 1$  into  $\mathbf{v}$  and  $\mathbf{a}$  yields

$$\mathbf{v} = (4 - 9)\mathbf{i} + (12 - 2)\mathbf{j}$$

$$= -5\mathbf{i} + 10\mathbf{j}$$

$$\mathbf{a} = (12 - 18)\mathbf{i} + 12\mathbf{j}$$

$$= -6\mathbf{i} + 12\mathbf{j}$$

To find the angle between the velocity and acceleration we need to use

\*

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{v}}{|\mathbf{a}||\mathbf{v}|}$$

Example 18 *continued*

We have

$$|\mathbf{v}| = |-5\mathbf{i} + 10\mathbf{j}| = \sqrt{(-5)^2 + 10^2} = \sqrt{125}$$

$$|\mathbf{a}| = |-6\mathbf{i} + 12\mathbf{j}| = \sqrt{(-6)^2 + 12^2} = \sqrt{180}$$

$$\begin{aligned}\mathbf{a} \cdot \mathbf{v} &= (-6\mathbf{i} + 12\mathbf{j}) \cdot (-5\mathbf{i} + 10\mathbf{j}) \\ &= [-6 \times (-5)] + (12 \times 10) \\ &= 150\end{aligned}$$

Substituting these into  $\star$  gives

$$\cos(\theta) = \frac{150}{\sqrt{180}\sqrt{125}} = 1$$

Taking inverse cos yields

$$\theta = \cos^{-1}(1) = 0^\circ$$



The angle between velocity and acceleration is  $0^\circ$ . **What does this  $0^\circ$  signify?**

Since we have an angle of  $0^\circ$  between velocity and acceleration then these vectors are parallel (or the same vector). We have

$$\mathbf{v} = -5\mathbf{i} + 10\mathbf{j} \quad \text{and} \quad \mathbf{a} = -6\mathbf{i} + 12\mathbf{j}$$

Observe that  $\mathbf{a} = 1.2\mathbf{v}$  which means  $\mathbf{a}$  has the same direction as  $\mathbf{v}$  but the magnitude is 1.2 times larger.

**SUMMARY**

Let  $\mathbf{p}$  and  $\mathbf{q}$  be vectors with angle  $\theta$  between them. Then the scalar product  $\mathbf{p} \cdot \mathbf{q}$  is given by

$$12.6 \quad \mathbf{p} \cdot \mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos(\theta)$$

Also

$$12.13 \quad (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) = ad + be + cf$$

The scalar product is applied in mechanics to find the work done:

$$12.15 \quad \text{Work done} = \mathbf{F} \cdot \mathbf{r}$$

## Exercise 12(d)


Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)


1 Determine the scalar products of the following:


**a**  $(3\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (5\mathbf{i} + \mathbf{j} + 7\mathbf{k})$

**b**  $(-\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

**c**  $(10\mathbf{i} - 15\mathbf{j} + \mathbf{k}) \cdot (-21\mathbf{i} - \mathbf{k})$

2  [mechanics] A force  $\mathbf{F} = 100\mathbf{i} + 220\mathbf{j} - 250\mathbf{k}$  moves an object through a displacement  $\mathbf{s} = 150\mathbf{i} + 200\mathbf{j}$ . Find the work done by the force  $\mathbf{F}$ .

3  [mechanics] A force  $\mathbf{F} = 5\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$  is applied to an object which moves from  $A = (1, 1, 1)$  to  $B = (5, -1, 2)$ . Determine the work done by the force  $\mathbf{F}$ .


4  [mechanics] A force of magnitude 20 N is applied to an object in the direction  $3\mathbf{i} + \mathbf{j} + \mathbf{k}$ . Determine the work done by the force in moving the object from  $A = (3, -1, 5)$  to  $B = (-1, 7, 10)$ .

5 Let  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  be a vector. Determine

**i**  $\mathbf{r} \cdot \mathbf{r}$       **ii**  $|\mathbf{r}|^2$


where  $|\mathbf{r}|$  is the magnitude of  $\mathbf{r}$ .

What do you notice about your results?

6  [mechanics] The velocity,  $\mathbf{v}$ , and acceleration,  $\mathbf{a}$ , of a particle are given by

$$\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}, \quad \mathbf{a} = 2\mathbf{i} + 2\mathbf{j}$$

Find the angle between these vectors.

7  [mechanics] The position vector,  $\mathbf{r}$ , of a particle moving in curvilinear motion is given by

$$\mathbf{r} = (t^3 - t^2)\mathbf{i} + t^4\mathbf{j}$$

- i** Find expressions for  $\mathbf{v} = \dot{\mathbf{r}}$  and  $\mathbf{a} = \ddot{\mathbf{r}}$ .  
**ii** Determine the angle between  $\mathbf{v}$  and  $\mathbf{a}$  for  $t = 1$ .  
 ( $\mathbf{v}$  is velocity and  $\mathbf{a}$  is acceleration.)

## SECTION E Vector products

By the end of this section you will be able to:

- find vector products
- examine engineering applications of vector products

## E1 Vector product

In the previous **Section D** we defined the scalar product of two vectors denoted by  $\mathbf{p} \cdot \mathbf{q}$ . Here we examine the vector or cross product of the vectors, denoted by  $\mathbf{p} \times \mathbf{q}$ .

The vector product of vectors  $\mathbf{p}$  and  $\mathbf{q}$ , denoted by  $\mathbf{p} \times \mathbf{q}$ , is defined as the vector of size

$$|\mathbf{p}||\mathbf{q}|\sin(\theta)$$

in the direction  $90^\circ$  to the plane containing the vectors  $\mathbf{p}$  and  $\mathbf{q}$ . We have

$$12.16 \quad \mathbf{p} \times \mathbf{q} = |\mathbf{p}||\mathbf{q}|\sin(\theta)\mathbf{u}$$

where  $\theta$  is the angle between the vectors  $\mathbf{p}$  and  $\mathbf{q}$ , and  $\mathbf{u}$  is the unit vector in the direction of  $\mathbf{p} \times \mathbf{q}$  shown in part **a** of Fig. 56. The vector product  $\mathbf{p} \times \mathbf{q}$  is a **vector quantity**, that is the cross product of a vector combined with a vector gives a vector. The direction of the vector  $\mathbf{p} \times \mathbf{q}$  is determined by the direction of a right-handed screw turned from  $\mathbf{p}$  to  $\mathbf{q}$  as shown in part **b** of Fig. 56.

Note that the vector  $\mathbf{q} \times \mathbf{p}$  is in the direction of a right-handed screw turned from  $\mathbf{q}$  to  $\mathbf{p}$  (part **c** of Fig. 56).

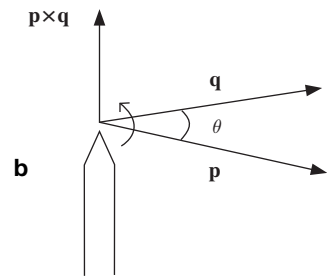
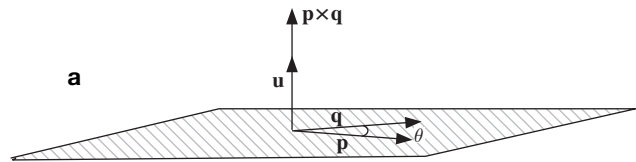


Fig. 56

? What do you notice about  $\mathbf{p} \times \mathbf{q}$  and  $\mathbf{q} \times \mathbf{p}$ ?

$$\mathbf{p} \times \mathbf{q} \neq \mathbf{q} \times \mathbf{p} \quad [\text{Not equal}]$$

In fact

$$12.17 \quad \mathbf{p} \times \mathbf{q} = -(\mathbf{q} \times \mathbf{p})$$

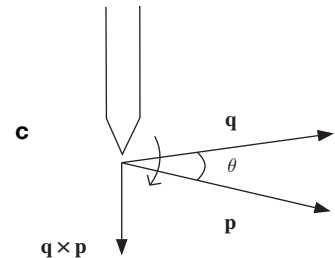
The vector product also has the following properties:

$$12.18 \quad \mathbf{p} \times (\mathbf{q} + \mathbf{r}) = (\mathbf{p} \times \mathbf{q}) + (\mathbf{p} \times \mathbf{r})$$

$$12.19 \quad k(\mathbf{p} \times \mathbf{q}) = (k\mathbf{p}) \times \mathbf{q} = \mathbf{p} \times (k\mathbf{q})$$

[ $k$  is a constant]

$$12.20 \quad \text{If } \mathbf{p} = \mathbf{0} \text{ then } \mathbf{p} \times \mathbf{q} = \mathbf{0}$$



### Example 19

Determine the following vector products:

$$\mathbf{i} \times \mathbf{i}, \mathbf{i} \times \mathbf{j}, \mathbf{k} \times \mathbf{i}, \mathbf{j} \times \mathbf{j}, \mathbf{j} \times \mathbf{k} \text{ and } \mathbf{k} \times \mathbf{k}$$

where  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  are unit vectors in the directions of the  $x$ ,  $y$  and  $z$  axes respectively.

Solution

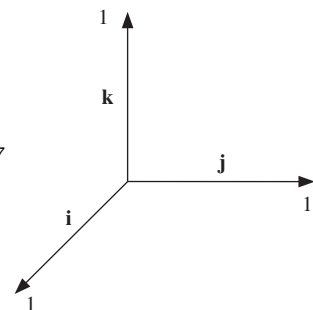
Plot the unit vectors, as shown in Fig. 57.

? Clearly  $\mathbf{i} \times \mathbf{i} = \mathbf{0}$ . Why?

Because the angle between  $\mathbf{i}$  and  $\mathbf{i}$  is  $0^\circ$  and  $\sin(0^\circ) = 0$ . Thus

$$\mathbf{i} \times \mathbf{i} = |\mathbf{i}||\mathbf{i}|\sin(0^\circ)\mathbf{j} = \mathbf{0}$$

Fig. 57





Example 19 *continued*

Similarly  $\mathbf{j} \times \mathbf{j} = \mathbf{0}$  and  $\mathbf{k} \times \mathbf{k} = \mathbf{0}$ .

The unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  are all  $90^\circ$  to each other. We have

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= \underbrace{|\mathbf{i}|}_{=1} \underbrace{|\mathbf{j}|}_{=1} \underbrace{[\sin(90^\circ)]}_{=1} \mathbf{k} \\ &\quad \text{because } \mathbf{k} \text{ is } 90^\circ \text{ to the plane} \\ &\quad \text{containing the vectors } \mathbf{i} \text{ and } \mathbf{j} \\ &= (1)(1)(1)\mathbf{k} = \mathbf{k}\end{aligned}$$

Likewise

$$\mathbf{k} \times \mathbf{i} = \underbrace{|\mathbf{k}|}_{=1} \underbrace{|\mathbf{i}|}_{=1} \underbrace{[\sin(90^\circ)]}_{=1} \mathbf{j} = \mathbf{j}$$

By the same token:

$$\mathbf{j} \times \mathbf{k} = \underbrace{|\mathbf{j}|}_{=1} \underbrace{|\mathbf{k}|}_{=1} \underbrace{[\sin(90^\circ)]}_{=1} \mathbf{i} = \mathbf{i}$$

**?** What are  $\mathbf{j} \times \mathbf{i}$ ,  $\mathbf{i} \times \mathbf{k}$  and  $\mathbf{k} \times \mathbf{j}$  equal to?

By

$$12.17 \quad \mathbf{p} \times \mathbf{q} = -(\mathbf{q} \times \mathbf{p})$$

and **Example 19** we have  $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$ ,  $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$  and  $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ .

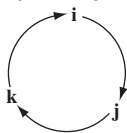
We have the following vector products of the unit vectors:

$$12.21 \quad \mathbf{i} \times \mathbf{i} = \mathbf{0}, \mathbf{j} \times \mathbf{j} = \mathbf{0} \text{ and } \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

$$12.22 \quad \mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i} \text{ and } \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$12.23 \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \mathbf{k} \times \mathbf{j} = -\mathbf{i} \text{ and } \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

We can remember the last two formulae by using:



Moving clockwise you get  $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ ,  $\mathbf{j} \times \mathbf{k} = \mathbf{i}$  and  $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ . Going anticlockwise you get  $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$ ,  $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$  and  $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ .

Compare these results with those given for scalar products in **12.11** and **12.12**.

Let  $\mathbf{p}$  and  $\mathbf{q}$  be vectors such that

$$\begin{aligned}\mathbf{p} &= a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \\ \mathbf{q} &= d\mathbf{i} + e\mathbf{j} + f\mathbf{k}\end{aligned}$$

Then the vector product of  $\mathbf{p}$  and  $\mathbf{q}$  can be shown to be

$$* \quad (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) = (bf - ce)\mathbf{i} + (cd - af)\mathbf{j} + (ae - bd)\mathbf{k}$$

Try verifying this result by expanding brackets and using the preceding results for  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$ .

Rather than applying this formula to find the vector product we can use a determinant as a device for evaluating the vector product:

$$\mathbf{12.24} \quad \mathbf{p} \times \mathbf{q} = (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{pmatrix}$$

This can be shown by expanding the determinant:

$$\begin{aligned} \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{pmatrix} &= \mathbf{i} \left[ \det \begin{pmatrix} b & c \\ e & f \end{pmatrix} \right] - \mathbf{j} \left[ \det \begin{pmatrix} a & c \\ d & f \end{pmatrix} \right] + \mathbf{k} \left[ \det \begin{pmatrix} a & b \\ d & e \end{pmatrix} \right] \\ &= \mathbf{i}(bf - ec) - \mathbf{j}(af - dc) + \mathbf{k}(ae - db) \quad [\text{By } \mathbf{11.1}] \\ &= (bf - ec)\mathbf{i} + (dc - af)\mathbf{j} + (ae - db)\mathbf{k} \\ &= (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) \quad [\text{By } \mathbf{*}] \end{aligned}$$

Hence we evaluate this determinant to find the vector product.

### Example 20

Obtain the vector product of

$$\mathbf{r} = 5\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$$

$$\mathbf{s} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$$

Solution

Using the determinant in **12.24** to find  $\mathbf{r} \times \mathbf{s}$  gives

$$\begin{aligned} (5\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}) \times (3\mathbf{i} - \mathbf{j} + \mathbf{k}) &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 7 & 6 \\ 3 & -1 & 1 \end{pmatrix} \\ &= \mathbf{i} \left[ \det \begin{pmatrix} 7 & 6 \\ -1 & 1 \end{pmatrix} \right] - \mathbf{j} \left[ \det \begin{pmatrix} 5 & 6 \\ 3 & 1 \end{pmatrix} \right] + \mathbf{k} \left[ \det \begin{pmatrix} 5 & 7 \\ 3 & -1 \end{pmatrix} \right] \\ &\stackrel{\text{by } \mathbf{11.1}}{=} \mathbf{i}(7 + 6) - \mathbf{j}[(5 \times 1) - (3 \times 6)] + \mathbf{k}[(5 \times (-1)) - (3 \times 7)] \\ &= 13\mathbf{i} + 13\mathbf{j} - 26\mathbf{k} \end{aligned}$$

We can also calculate the vector product of  $\mathbf{r}$  and  $\mathbf{s}$  of **Example 20** on a calculator.

Of course  $\mathbf{s} \times \mathbf{r} = -13\mathbf{i} - 13\mathbf{j} + 26\mathbf{k}$  (that is  $-(\mathbf{r} \times \mathbf{s})$ ).

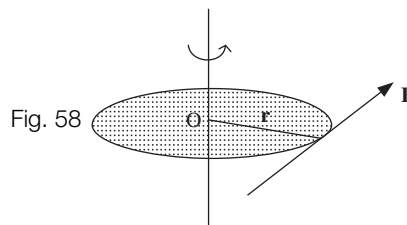
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$$\mathbf{11.1} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$$

## E2 Applications of vector product

Next we look at an application of vector products in mechanics.

A force,  $\mathbf{F}$ , is applied to an object, making it rotate around a fixed axis as shown in Fig. 58 (for example, a spinning top). A moment of force is the tendency of a force to turn or rotate an object. The moment of a vector is a generalization of the moment of force. The moment vector,  $\mathbf{M}$ , of a force,  $\mathbf{F}$ , about the origin  $O$  is defined as



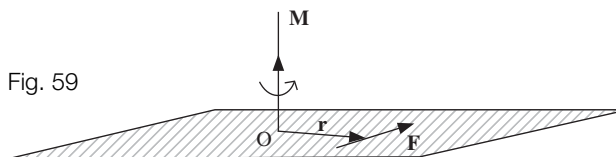
$$\mathbf{12.25} \quad \mathbf{M} = \mathbf{r} \times \mathbf{F}$$

where  $\mathbf{r}$  is the position vector from  $O$  of any point on the line of action of  $\mathbf{F}$ .

Note that the moment,  $\mathbf{M}$ , is a vector. (The moment vector is called the torque in many books on mechanics.)

### ? What is the direction of the vector $\mathbf{M}$ ?

It is  $90^\circ$  to the plane containing the vectors  $\mathbf{r}$  and  $\mathbf{F}$  (Fig. 59).



### Example 21 *mechanics*

Determine the moment vector,  $\mathbf{M}$ , about the origin,  $O$ , of a force  $\mathbf{F} = 10\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ , passing through the point with position vector  $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ .

Solution

By **12.25** we have

$$\begin{aligned} \mathbf{M} &= \mathbf{r} \times \mathbf{F} \\ &= (2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) \times (10\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ &\stackrel{\text{by } \mathbf{12.24}}{=} \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 5 \\ 10 & -1 & 2 \end{pmatrix} \\ &= \mathbf{i} \left[ \det \begin{pmatrix} -3 & 5 \\ -1 & 2 \end{pmatrix} \right] - \mathbf{j} \left[ \det \begin{pmatrix} 2 & 5 \\ 10 & 2 \end{pmatrix} \right] + \mathbf{k} \left[ \det \begin{pmatrix} 2 & -3 \\ 10 & -1 \end{pmatrix} \right] \\ &= -\mathbf{i} + 46\mathbf{j} + 28\mathbf{k} \quad [\text{Evaluating determinant}] \end{aligned}$$

$$\mathbf{12.24} \quad (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{pmatrix}$$

The next example is more complicated than the previous examples. The complication is to determine vector moments about other points rather than just the origin.



### Example 22 *mechanics*

Let A and B have position vectors

$$\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

$$\mathbf{b} = 7\mathbf{i} - 4\mathbf{j} - \mathbf{k}$$

respectively. The force  $\mathbf{F} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$  passes through the point A. Determine the vector moment of  $\mathbf{F}$  about the point B.

Solution

Draw the vectors as in Fig. 60.

Find the resultant vector first,  $\vec{BA}$ :

$$\begin{aligned}\vec{BA} &= \vec{OA} - \vec{OB} \\ &= (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) - (7\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = -5\mathbf{i} + \mathbf{j} + 2\mathbf{k}\end{aligned}$$

Let  $\mathbf{M}$  be the moment of  $\mathbf{F}$ , then by using 12.25 and substituting  $\vec{BA} = -5\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ ,  $\mathbf{F} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$  we have

$$\begin{aligned}\mathbf{M} &= \vec{BA} \times \mathbf{F} \\ &= (-5\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \times (3\mathbf{i} - \mathbf{j} + 5\mathbf{k})\end{aligned}$$

$$\stackrel{\text{by 12.24}}{=} \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & 1 & 2 \\ 3 & -1 & 5 \end{pmatrix}$$

$$= \mathbf{i} \left[ \det \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} \right] - \mathbf{j} \left[ \det \begin{pmatrix} -5 & 2 \\ 3 & 5 \end{pmatrix} \right] + \mathbf{k} \left[ \det \begin{pmatrix} -5 & 1 \\ 3 & -1 \end{pmatrix} \right]$$

$$= \mathbf{i}(5 + 2) - \mathbf{j}(-25 - 6) + \mathbf{k}(5 - 3)$$

$$= 7\mathbf{i} + 31\mathbf{j} + 2\mathbf{k}$$

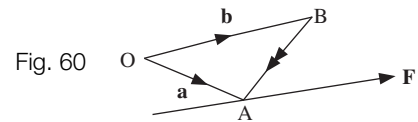


Fig. 60

## SUMMARY

The vector product of  $\mathbf{p}$  and  $\mathbf{q}$  is defined as

$$12.16 \quad \mathbf{p} \times \mathbf{q} = |\mathbf{p}||\mathbf{q}|\sin(\theta) \mathbf{u}$$

$$12.24 \quad (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{pmatrix}$$

**SUMMARY continued**

where  $\mathbf{u}$  is the unit vector in the direction of  $\mathbf{p} \times \mathbf{q}$ . Also

$$12.24 \quad (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k}) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{pmatrix}$$

The moment vector,  $\mathbf{M}$ , about the origin is defined as


$$12.25 \quad \mathbf{M} = \mathbf{r} \times \mathbf{F}$$

where  $\mathbf{r}$  is the position vector from the origin of any point on the line of action of  $\mathbf{F}$ .

**Exercise 12(e)**

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)


- 1 If  $\mathbf{r} = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$  and  $\mathbf{s} = 2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$ , determine the vector products  
**i**  $\mathbf{r} \times \mathbf{s}$       **ii**  $\mathbf{s} \times \mathbf{r}$

- 2  [mechanics] Forces  $\mathbf{F}_1$ ,  $\mathbf{F}_2$  and  $\mathbf{F}_3$  act at a point on a body with the corresponding position vectors:

$$\begin{aligned} \mathbf{F}_1 &= 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}, & \mathbf{r}_1 &= \mathbf{i} + 3\mathbf{j} + \mathbf{k} \\ \mathbf{F}_2 &= 3\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}, & \mathbf{r}_2 &= \mathbf{i} - \mathbf{j} + 2\mathbf{k} \\ \mathbf{F}_3 &= \mathbf{i} + 10\mathbf{j} + 7\mathbf{k}, & \mathbf{r}_3 &= \mathbf{i} + 4\mathbf{j} + 2\mathbf{k} \end{aligned}$$

Show that the body is in equilibrium, that is


$$(\mathbf{r}_1 \times \mathbf{F}_1) + (\mathbf{r}_2 \times \mathbf{F}_2) + (\mathbf{r}_3 \times \mathbf{F}_3) = \mathbf{0}$$


- 3  [mechanics] Forces  $\mathbf{F}_1$ ,  $\mathbf{F}_2$ ,  $\mathbf{F}_3$  and  $\mathbf{F}_4$  act at a point on a body with the corresponding position vectors:

$$\begin{aligned} \mathbf{F}_1 &= 2\mathbf{i} - \mathbf{j} + \mathbf{k}, & \mathbf{r}_1 &= 3\mathbf{i} + \mathbf{j} + 5\mathbf{k} \\ \mathbf{F}_2 &= 8\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}, & \mathbf{r}_2 &= 5\mathbf{i} - 2\mathbf{j} - \mathbf{k} \\ \mathbf{F}_3 &= 4\mathbf{i} + 3\mathbf{k}, & \mathbf{r}_3 &= \mathbf{i} - 6\mathbf{j} + 7\mathbf{k} \\ \mathbf{F}_4 &= \mathbf{i} - 5\mathbf{j}, & \mathbf{r}_4 &= \mathbf{i} + 6\mathbf{k} \end{aligned}$$

Show that the body is in equilibrium, that is

$$\begin{aligned} &(\mathbf{r}_1 \times \mathbf{F}_1) + (\mathbf{r}_2 \times \mathbf{F}_2) \\ &+ (\mathbf{r}_3 \times \mathbf{F}_3) + (\mathbf{r}_4 \times \mathbf{F}_4) = \mathbf{0} \end{aligned}$$

- 4  [mechanics] Obtain the moment,  $\mathbf{M}$ , about the point B of a force,  $\mathbf{F}$ , passing through the point A for the following:  
**a**  $\mathbf{F} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ , A is (1, 2, 3),  
 B is (5, 7, 9)  
**b**  $\mathbf{F} = 4\mathbf{i} - 3\mathbf{j} - 6\mathbf{k}$ ,  
 A is (1, -1, 0), B is (3, -1, -1)

- 5  [mechanics] The points R and S have position vectors

$$\mathbf{r} = \mathbf{i} - \mathbf{j} - \mathbf{k}, \quad \mathbf{s} = 7\mathbf{i} + \mathbf{j} - 9\mathbf{k}$$

respectively. A force  $\mathbf{F} = 14\mathbf{i} + 2\mathbf{j} - 18\mathbf{k}$  passes through R. Calculate the vector moment of  $\mathbf{F}$  about the point S.

- 6 Let  $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$  and  $\mathbf{s} = c\mathbf{i} + d\mathbf{j}$  be any vectors. Show that

$$\mathbf{r} \times \mathbf{s} = (ad - cb)\mathbf{k}$$

- 7 Let  $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$  and  $\mathbf{s} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$  be any vectors. Show that

$$\mathbf{r} \times \mathbf{s} = -(\mathbf{s} \times \mathbf{r})$$

## Examination questions 12

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

In this exercise you may check your numerical answers using mathematical software.

The first two questions have been slightly edited.

- 1 Given  $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$  and  $\mathbf{s} = 4\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$ , find the vector cross product  $\mathbf{r} \times \mathbf{s}$ .
- 2 a Given force  $\mathbf{F} = 6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ , and displacement vector  $\mathbf{r} = 9\mathbf{i} + 6\mathbf{j} + \mathbf{k}$ , find
- The work done by  $\mathbf{F}$  on  $\mathbf{r}$
  - The angle between  $\mathbf{F}$  and  $\mathbf{r}$ .
- b Given the position vector of an object in terms of time is
- $$\mathbf{r} = (t^3 - 3t^2)\mathbf{i} + (6t^2 - 2t)\mathbf{j}$$
- Find the velocity and acceleration in terms of time
  - Find the initial velocity and acceleration
  - Find the time when the horizontal velocity = 0.

Questions 1 and 2 are from University of Portsmouth, UK, 2007

- 3 If  $\mathbf{a} = (3, 5, -2)$  and  $\mathbf{b} = (2, 4, 7)$ , find  $\mathbf{a} \times \mathbf{b}$ .  
University of Sussex, UK, 2007
- 4 An electric charge  $q$  with velocity  $\mathbf{v}$  produces a magnetic field  $\mathbf{B}$  given by:

$$\mathbf{B} = \frac{\mu q}{4\pi} \cdot \frac{\mathbf{v} \times \mathbf{r}}{r^2}$$

Find  $\mathbf{B}$ , in terms of  $\mu q$ , given  
 $\mathbf{r} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$  and  $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ .

University of Portsmouth, UK, 2006

- 5 Given the three vectors  $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ ,  $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  and  $\mathbf{c} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$  find
- $\mathbf{a} \times \mathbf{c}$
  - $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{a})$
  - $(\mathbf{a} \times \mathbf{c}) \times \mathbf{b}$
  - the value of  $x$  which makes the vector  $(x, 1, 0)$  to be at  $60^\circ$  to  $\mathbf{b}$ .

University of Manchester, UK, 2007

- 6 If  $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ ,  $\mathbf{b} = 2\mathbf{i} + \mathbf{k}$ ,  $\mathbf{c} = 3\mathbf{i} - \mathbf{j} - 5\mathbf{k}$  find
- $\mathbf{a} - 2\mathbf{b} + \mathbf{c}$
  - $\mathbf{b} \cdot \mathbf{c}$
  - $\mathbf{b} \times \mathbf{c}$
  - $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) - \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$
  - $|\mathbf{c}| - |\mathbf{a}|$
  - a unit vector in the direction of  $\mathbf{b}$
  - a unit vector perpendicular to  $\mathbf{a}$  and  $\mathbf{b}$ .

Loughborough University, UK, 2007

- 7 Find the cosine of the angle between the vectors  $\mathbf{u} \times \mathbf{v}$  and  $\mathbf{w}$  and between  $\mathbf{u} \times \mathbf{w}$  and  $\mathbf{v}$  where  $\mathbf{u} = (1, -1, 1)$ ,  $\mathbf{v} = (0, 1, 2)$ ,  $\mathbf{w} = (1, 0, 5)$ .

Jacobs University, Germany, 2002

## Miscellaneous exercise 12

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- 1 Find the exact unit vector in the direction of the vector  $\mathbf{F} = 5\mathbf{i} + 7\mathbf{j} + 12\mathbf{k}$ .
- 2 Let  $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ ,  $\mathbf{b} = 7\mathbf{i} + 12\mathbf{k}$ ,  $\mathbf{c} = -\mathbf{i} - \mathbf{j} - 6\mathbf{k}$  and  $\mathbf{d} = 12\mathbf{i} - \mathbf{j} - 11\mathbf{k}$ . Determine:
- $\mathbf{a} + \mathbf{c}$
  - $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$
  - $\mathbf{a} \cdot \mathbf{b}$
  - $\mathbf{b} \cdot \mathbf{a}$
  - $\mathbf{v} \cdot \mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$
  - $(\mathbf{c} \cdot \mathbf{d})\hat{\mathbf{a}}$  where  $\hat{\mathbf{a}}$  is the unit vector in the direction of  $\mathbf{a}$
  - the magnitude of  $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$ .



Questions 3 to 14 are in the field of [mechanics].

- 3 Let the velocity,  $\mathbf{v}$ , and acceleration,  $\mathbf{a}$ , be given by

$$\mathbf{v} = -4\mathbf{i} - 8\mathbf{j} + 4\mathbf{k} \text{ and}$$

$$\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$$

Find the angle between the velocity and acceleration.

## Miscellaneous exercise 12 continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- 4 The velocity,  $\mathbf{v}$ , and acceleration,  $\mathbf{a}$ , of a particle are given by

$$\mathbf{v} = 6\mathbf{i} + 2\mathbf{j}, \quad \mathbf{a} = 3\mathbf{i} + 7\mathbf{j}$$

Find the angle between  $\mathbf{v}$  and  $\mathbf{a}$ .

- 5 Determine the resultant force,  $\mathbf{R}$ , of the forces shown in Fig. 61.

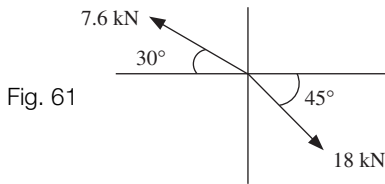


Fig. 61

- 6 Determine the resultant force,  $\mathbf{R}$ , of the forces shown in Fig. 62.

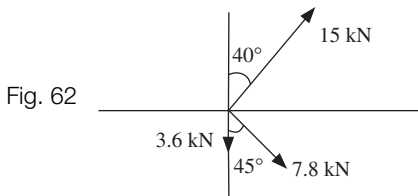


Fig. 62

- 7 Find the horizontal,  $F_x$ , and vertical,  $F_y$ , components of the force  $F$  shown in Fig. 63.

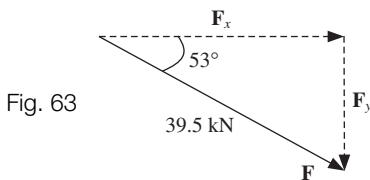


Fig. 63

- 8 Find the resultant force,  $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4$ , in each of the following cases:

**a**  $\mathbf{F}_1 = 2\mathbf{i} - \mathbf{j} + 7\mathbf{k}$ ,  
 $\mathbf{F}_2 = -3\mathbf{i} - 2\mathbf{j} + 11\mathbf{k}$ ,

$$\mathbf{F}_3 = 61\mathbf{i} + 64\mathbf{j} + 19\mathbf{k} \text{ and}$$

$$\mathbf{F}_4 = 3\mathbf{i} + 8\mathbf{j} + 11\mathbf{k}$$

**b**  $\mathbf{F}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ ,  
 $\mathbf{F}_2 = 16\mathbf{i} + 99\mathbf{j} + 19\mathbf{k}$ ,  
 $\mathbf{F}_3 = 47\mathbf{i} + 84\mathbf{j} + 88\mathbf{k}$  and  
 $\mathbf{F}_4 = 22\mathbf{i} + 41\mathbf{j} + 3\mathbf{k}$

- 9 A force  $\mathbf{F} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$  moves an object through a displacement  $\mathbf{s} = 15\mathbf{i} + 20\mathbf{j}$ . Find the work done by the force  $\mathbf{F}$ .
- 10 A force  $\mathbf{F} = 2\mathbf{i} - 7\mathbf{j} + 12\mathbf{k}$  acts on a particle P. Calculate the work done by  $\mathbf{F}$  as the particle P moves from the point A = (2, 3, 7) to the point B = (7, -11, 37).
- 11 Determine the work done by a force  $\mathbf{F}$  on a particle P moving from point A to point B where
- a**  $\mathbf{F} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ ,  
A = (1, 7, 1), B = (3, 4, -1)
- b**  $\mathbf{F} = 2\mathbf{i} + \mathbf{j} - 7\mathbf{k}$ ,  
A = (3, 4, 3), B = (6, 7, -17)
- 12 A force  $\mathbf{F} = 5\mathbf{i} - \mathbf{j} - \mathbf{k}$  passes through the point A whose position vector is  $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + 8\mathbf{k}$ . Determine the vector moment,  $\mathbf{M}$ , of  $\mathbf{F}$  about the point B with position vector  $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$ . Also find the magnitude of  $\mathbf{M}$ .
- 13 Determine the force  $\mathbf{F}$  which has a magnitude of 39 kN in the direction of the vector  $\overrightarrow{AB}$  where A = (-1, 2, 3) and B = (5, 1, -2).
- 14 Forces  $\mathbf{F}_1$ ,  $\mathbf{F}_2$  and  $\mathbf{F}_3$  act at a point in a body with the corresponding position vectors:

$$\mathbf{F}_1 = \mathbf{i} + 3\mathbf{j} - 6\mathbf{k}, \quad \mathbf{r}_1 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{F}_2 = -7\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}, \quad \mathbf{r}_2 = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$$\mathbf{F}_3 = 2\mathbf{j} + 3\mathbf{k}, \quad \mathbf{r}_3 = 2\mathbf{i} + 2\mathbf{j} - 10\mathbf{k}$$

## Miscellaneous exercise 12 continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

Show that the body is in equilibrium, that is

$$(\mathbf{r}_1 \times \mathbf{F}_1) + (\mathbf{r}_2 \times \mathbf{F}_2) + (\mathbf{r}_3 \times \mathbf{F}_3) = \mathbf{0}$$

- 15** Let  $\mathbf{r} = a\mathbf{i} + b\mathbf{k}$  and  $\mathbf{s} = c\mathbf{i} + d\mathbf{k}$  be any vectors. Determine the vector product  $\mathbf{r} \times \mathbf{s}$ .
- 16** Let  $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + x\mathbf{k}$  and  $\mathbf{b} = -3\mathbf{i} + \mathbf{j} + x\mathbf{k}$  be vectors. Find the value(s) of  $x$  for

which the vectors  $\mathbf{a}$  and  $\mathbf{b}$  are at right angles.

- 17** Consider the vectors

$$\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \text{ and } \mathbf{b} = x\mathbf{i} + \mathbf{j} - \mathbf{k}$$

Find the value of  $x$  for which the angle between  $\mathbf{a}$  and  $\mathbf{b}$  is  $45^\circ$ .



# First Order Differential Equations

**SECTION A Solving differential equations 673**

- ▶ Solve first order differential equations by using direct integration
- ▶ Solve first order differential equations by using separation of variables

**SECTION B Using the integrating factor 683**

- ▶ Solve first order differential equations by using the integrating factor

**SECTION C Applications to electrical principles 689**

- ▶ Apply first order differential equations to solve electric circuit problems
- ▶ Use a computer algebra system to solve differential equations

**SECTION D Further engineering applications 695**

- ▶ Apply first order differential equations to solve fluid mechanics problems
- ▶ Apply first order differential equations to solve heat transfer problems

**SECTION E Euler's numerical method 703**

- ▶ Solve first order differential equations by Euler's numerical method
- ▶ Use a computer algebra system to compare numerical and analytical solutions of differential equations

**SECTION F Improved Euler's method 711**

- ▶ Solve first order differential equations by the improved Euler's numerical method
- ▶ Use a computer algebra system to compare numerical and analytical solutions of differential equations

**SECTION G Fourth order Runge–Kutta 718**

- ▶ Solve first order differential equations by the 4th order Runge–Kutta numerical method
- ▶ Use a computer algebra system to compare numerical and analytical solutions of differential equations

SECTION A **Solving differential equations**

By the end of this section you will be able to:

- find the order of a differential equation
- solve first order differential equations by using direct integration
- solve first order differential equations by separating variables
- solve engineering applications with initial conditions

In the next two chapters we discuss differential equations. The entire field of engineering and science – heat, light, sound, gravitation, magnetism, fluid flow, population dynamics and mechanics – can be described by differential equations. Other modern technologies such as radio, television, cars and aircraft all depend on the mathematics of differential equations.

**A1 Introduction****? What is a differential equation?**

An equation which contains derivatives.

For example

$$3\frac{dy}{dx} + x = \cos(y) \text{ is a first order differential equation}$$

$$3\frac{d^2y}{dx^2} + \frac{dy}{dx} + x = \cos(y) \text{ is a second order differential equation}$$

$$3\frac{d^3y}{dx^3} + \frac{dy}{dx} + x = \cos(y) \text{ is a third order differential equation}$$

A differential equation can be classified according to its order. The **order** is the order of the highest derivative appearing in the differential equation.

**Example 1**

If

$$y = e^{5x}$$

Show that

$$\frac{dy}{dx} - 5y = 0$$

Solution

By using

$$\mathbf{6.11} \quad \frac{d}{dx}(e^{kx}) = ke^{kx}$$

Example 1 *continued*

on  $y = e^{5x}$  we have

$$\frac{dy}{dx} = \frac{d}{dx}(e^{5x}) = 5e^{5x}$$

Substituting  $y = e^{5x}$ , so  $5y = 5e^{5x}$  and we have

$$5e^{5x} - 5y = 5e^{5x} - 5e^{5x} = 0$$

Hence  $\frac{dy}{dx} - 5y = 0$  holds.

Up until now we have dealt with the situation of **Example 1**, where we are given a solution,  $y = e^{5x}$ , and asked to verify that it satisfies the differential equation,  $\frac{dy}{dx} - 5y = 0$ .

In this chapter we are given the differential equation,  $\frac{dy}{dx} - 5y = 0$ , and we are asked to find a

**?** solution,  $y = e^{5x}$ . **If we are asked to solve  $\frac{dy}{dx} - 5y = 0$  then what are we trying to find?**

A function of  $x$ ,  $y = y(x)$ , which satisfies  $\frac{dy}{dx} - 5y = 0$ .

Note that  $x$  is the independent variable and  $y$  is the dependent variable.

Consider an investment  $P$  which earns an interest rate of  $r$ . These two quantities are related by the differential equation.

$$\frac{dP}{dt} = rP$$

This differential equation also describes an exponential growth or decaying system, such as radioactive decay and the current through a series  $RL$  circuit. This type of equation is an important first order differential equation which crops up in finance, engineering and science.

A2 **Direct integration**

If we are given the general first order differential equation

$$13.1 \quad \frac{dy}{dx} = f(x) \quad [\text{Function of } x]$$

then we can use direct integration to find  $y$ .

**Example 2**

Solve

$$\frac{dy}{dx} = 4x$$

Solution

We have

$$dy = 4x \, dx$$

because  $dy = \frac{dy}{dx} dx$ . Integrating gives

$$\int dy = \int 4x \, dx$$

$$y = \frac{4x^2}{2} + C$$

$$y = 2x^2 + C$$

$y = 2x^2 + C$  is called the general solution of  $\frac{dy}{dx} = 4x$ .

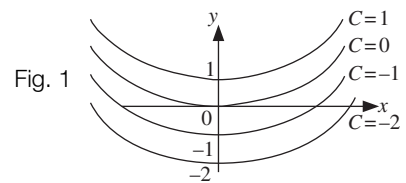
**? Why is it called the general solution?**

Because the solution has a constant,  $C$ , which does not have a particular value. The solution,  $y = 2x^2 + C$ , satisfies the differential equation,  $\frac{dy}{dx} = 4x$ , for any value of  $C$ .

We can sketch these solutions,  $y = 2x^2 + C$ , for different values of  $C$ .

Figure 1 shows the graphs of  $y = 2x^2 + C$ .

The differential equation,  $\frac{dy}{dx} = 4x$ , has an infinite number of solutions which are called the **family of solutions**.

**Example 3 materials**

The bending moment,  $M$ , of a beam of length  $l$  is given by

$$\frac{dM}{dx} = w(x + l)$$

where  $w$  is the constant load. Find an expression for  $M$  in terms of  $x$ .

Example 3 *continued*

Solution

For the given differential equation,  $M$  is the dependent variable and  $x$  is the independent variable. We have

$$\begin{aligned} dM &= w(x + l) dx \\ &= (wx + wl) dx \quad \text{[Expanding brackets]} \end{aligned}$$

Integrating both sides:

$$\begin{aligned} \int dM &= \int (wx + wl) dx \\ M &= \frac{wx^2}{2} + wlx + C \end{aligned}$$

This  $M = \frac{wx^2}{2} + wlx + C$  is the general solution of the differential equation  $\frac{dM}{dx} = w(x + l)$ .

Example 4 *mechanics*

The velocity,  $v = u + at$ , of an object is defined as

$$* \quad v = \frac{ds}{dt}$$

where  $s$  is the displacement,  $u$  is the initial velocity and  $a$  is the acceleration. Given that when  $t = 0$ ,  $s = 0$  show that

$$s = ut + \frac{1}{2}at^2$$

Solution

Substituting the given equation  $v = u + at$  into  $*$  yields

$$\begin{aligned} \frac{ds}{dt} &= u + at \\ ds &= (u + at) dt \end{aligned}$$

Integrating:

$$\begin{aligned} \int ds &= \int (u + at) dt \\ s &= ut + \frac{at^2}{2} + C \end{aligned}$$

Substituting the given conditions  $t = 0$ ,  $s = 0$ :

$$0 = 0 + 0 + C \text{ gives } C = 0$$

Hence we have

$$s = ut + \frac{1}{2}at^2$$

Notice that  $s = ut + \frac{1}{2}at^2$  is a **particular solution** of  $\frac{ds}{dt} = u + at$  because we have a particular value for  $C$  ( $= 0$ ). A particular solution is **one** member of the family of general solutions. Generally if we can find a value for  $C$  by using some given conditions then the solution is called a particular solution and the conditions are called the **initial conditions** or **boundary conditions** of the differential equation.

### A3 Separating variables

A general first order differential equation of the form

$$\text{13.2} \quad \frac{dy}{dx} = f(x)g(y)$$

is called a separable equation because we can separate the variables by dividing through by  $g(y)$ . In **13.2**,  $f(x)$  is a function of  $x$  only and  $g(y)$  is a function of  $y$  only.

#### Example 5

Solve

$$\frac{dy}{dx} = (1+x)(1+y) \quad (y > -1)$$

Solution

Since

$$dy = \frac{dy}{dx} dx$$

substituting  $\frac{dy}{dx} = (1+x)(1+y)$  gives

$$dy = (1+x)(1+y) dx$$

Dividing through by  $1+y$  so that the  $y$ 's and  $dy$ 's are on one side (separating variables) gives

$$\frac{dy}{1+y} = (1+x) dx$$



**What are we trying to find?**

We need to find  $y$  as a function of  $x$  which satisfies the differential equation.



Therefore we eliminate  $dy$  and  $dx$ . **How?**

Example 5 *continued*

We integrate both sides:

$$\int \frac{dy}{1+y} = \int (1+x) dx$$

$$\underbrace{\ln(1+y)}_{\text{by 8.42}} = x + \frac{x^2}{2} + C$$

?

We need to extract  $y$  from the Left-Hand Side. **How?**

Take exponentials of both sides:

$$e^{\ln(1+y)} = e^{x+(x^2/2)+C}$$

By using the rules of indices and exponentials we have

$$\underbrace{(1+y)}_{\text{by 5.17}} = e^x e^{x^2/2} e^C$$

$$= Ae^x e^{x^2/2} \text{ where } A = e^C \text{ is a constant}$$

Subtracting 1 from both sides gives

$$y = Ae^x e^{x^2/2} - 1$$

Again this  $y = Ae^x e^{x^2/2} - 1$  is the general solution of  $\frac{dy}{dx} = (1+x)(1+y)$  because we do not have a particular value for  $A$ .

Sometimes it can be difficult to extract  $y$  from the resulting integration.

In the above example we have used exponential and logarithmic properties from **Chapter 5**. It is critical that you are comfortable in using the following properties of logs and exponentials because they are used throughout the first four sections of this chapter:

$$\mathbf{5.12} \quad \ln(A) + \ln(B) = \ln(AB)$$

$$\mathbf{5.13} \quad \ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right)$$

$$\mathbf{5.14} \quad \ln(A^n) = n \ln(A)$$

$$\mathbf{5.16} \quad \ln(e^x) = x$$

$$\mathbf{5.17} \quad e^{\ln x} = x$$

---


$$\mathbf{5.17} \quad e^{\ln(x)} = x \qquad \mathbf{8.42} \quad \int \frac{f'(x)}{f(x)} dx = \ln|f(x)|$$



### Example 6 *fluid mechanics*

The streamlines of a fluid flow are given by the first order differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

Solve the differential equation.

Solution

Separating the variables of the given differential equation:

$$\begin{aligned} y \, dy &= -x \, dx \\ \int y \, dy &= \int -x \, dx \\ \frac{y^2}{2} &= -\frac{x^2}{2} + C \quad \text{[Integrating]} \end{aligned}$$

Multiplying both sides by 2:

$$\begin{aligned} y^2 &= -x^2 + 2C \\ y^2 + x^2 &= r^2 \text{ where } r^2 = 2C \end{aligned}$$

The equation

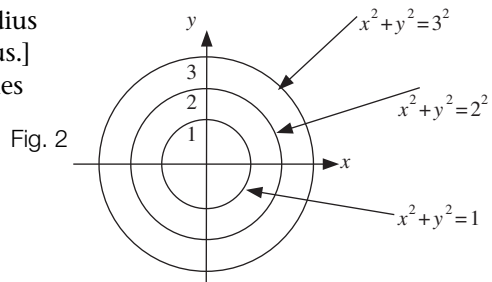
$$x^2 + y^2 = r^2$$

is an equation of a circle with centre origin and radius  $r$ . [In the above we chose  $r^2 = 2C$  because  $r = \text{radius}$ .] By using this equation we can sketch the streamlines of the above differential equation.

[A streamline is a curve in the  $x$ - $y$  plane given by  $y = f(x)$ .]

Here the streamlines are circles with centre at the origin.

Figure 2 shows the streamlines for  $2C = 1$ ,  $2C = 4$  and  $2C = 9$  where  $2C = r^2$ . There are an infinite number of these streamlines.



### Example 7 *fluid mechanics*

Consider a tank full of water which is being drained out through an outlet. The height  $H$  (m) of water in the tank at time  $t$  (s) is given by

$$\frac{dH}{dt} = -(2.8 \times 10^{-3})\sqrt{H}$$

Given that when  $t = 0$ ,  $H = 4$ , find an expression for  $H$  in terms of  $t$ .





### Example 7 *continued*

Solution

Separating variables:

$$\frac{dH}{\sqrt{H}} = -(2.8 \times 10^{-3}) dt$$

$$H^{-1/2} dH = -(2.8 \times 10^{-3}) dt \quad [\text{Because } 1/\sqrt{H} = H^{-1/2}]$$

Integrating:

$$\int H^{-1/2} dH = -\int (2.8 \times 10^{-3}) dt$$

$$\frac{H^{1/2}}{\underbrace{1/2}_{\text{by 8.1}}} = -(2.8 \times 10^{-3})t + C$$

$$2H^{1/2} = -(2.8 \times 10^{-3})t + C$$

Substituting the given initial condition  $t = 0, H = 4$  yields

$$2 \times 4^{1/2} = 0 + C \text{ which gives } C = 4$$

Substituting  $C = 4$  and dividing by 2 gives

$$\begin{aligned} H^{1/2} &= \frac{4 - (2.8 \times 10^{-3})t}{2} \\ &= 2 - (1.4 \times 10^{-3})t \end{aligned}$$

Squaring both sides:

$$H = [2 - (1.4 \times 10^{-3})t]^2$$


## SUMMARY

Differential equations are equations which contain a derivative. Solving a differential equation containing  $\frac{dy}{dx}$  involves finding an expression for the dependent variable,  $y$ , in terms of the independent variable,  $x$ . Two methods were discussed in this section – direct integration and separation of variables.

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8.1  $\int x^n dx = \frac{x^{n+1}}{n+1}$

## Exercise 13(a)

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)
 Questions 1 to 4 are in the field of [**fluid mechanics**].

- 1 The pressure,  $p$ , on an object under a fluid of density  $\rho$  (Greek letter rho) is given by

$$\frac{dp}{dz} = -\rho g$$

where  $z$  represents depth and  $g$  is the acceleration due to gravity. Find an expression for  $p$ .

- 2 Consider a fluid rotating about a vertical axis. The pressure  $p$  at radius  $r$  from the axis is given by

$$\frac{dp}{dr} = -\rho\omega^2 r$$

where  $\rho$  is the density and  $\omega$  is the angular velocity of the fluid. Find an expression for  $p$ .

- 3 The streamlines of a fluid flow are given by


$$\frac{dy}{dx} = C$$

where  $C$  is a constant. Plot the streamline that passes through the origin and  $C = 5$ .

- 4 The streamlines of a fluid flow are given by

$$\frac{dy}{dx} = e^x$$

Solve the differential equation and sketch the streamlines.

 Questions 5 to 7 are in the field of [**mechanics**].

- 5 The acceleration,  $a$ , of an object is defined as

$$a = \frac{dv}{dt}$$

where  $v$  is the velocity of the object at time  $t$ . Given that when  $t = 0$ ,  $v = u$ , and the acceleration,  $a$ , is constant, show that  $v = u + at$ .

- 6 The acceleration,  $a$ , of a particle is given by

$$a = 5 - 3t$$

Given that the initial displacement at  $t = 0$  is  $-2.1$  m and the initial velocity is 8 m/s, find expressions for the velocity and displacement.

[Hint: We have,  $a = \frac{dv}{dt}$ ,  $v = \frac{ds}{dt}$  where  $s$  is the displacement]

- 7 An object rotates with angular acceleration,  $\alpha$ , defined as

$$\alpha = \frac{d\omega}{dt}$$

where  $\omega$  is the angular velocity. Given that when  $t = 0$ ,  $\omega = \omega_0$ , show that


$$\omega = \omega_0 + \alpha t$$

Also the angular velocity  $\omega$  is defined as

$$\omega = \frac{d\theta}{dt}$$

where  $\theta$  is the angular displacement. Given that when  $t = 0$ ,  $\theta = 0$  show that

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$$

 Questions 8 to 10 belong to [**fluid mechanics**].

- 8 The streamlines of a fluid flow are given by

$$\frac{dy}{dx} = -\frac{x}{y}$$

## Exercise 13(a) continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

Show that  $x^2 + y^2 = A$  where  $A$  is a constant. Sketch the streamlines for  $A = 1, 25$  and  $100$ .

- 9 The streamlines of a fluid flow are given by


$$\frac{dy}{dx} = -\frac{y}{x} \quad (y > 0 \text{ and } x > 0)$$

Show that  $y = \frac{A}{x}$  and sketch on the same axes the streamlines for  $A=1, 5$  and  $8$  ( $A$  is a constant).

- 10 The streamlines of a fluid flow are given by the first order differential equation

$$\frac{dy}{dx} = \frac{y+1}{x+1} \quad (y > -1 \text{ and } x > -1)$$

Show that  $y = Ax + A - 1$  where  $A$  is a constant.

- 11  [heat transfer] The temperature gradient,  $\frac{d\theta}{dx}$ , of a slab of thickness  $t$  and with thermal conductivity  $k$  is given by

$$\frac{d\theta}{dx} = C$$

where  $C$  is a constant and  $\theta$  is a function of  $x$ .


By using the conditions  $\theta(0)=\theta_1$ ,  $\theta(t)=\theta_2$  and Fourier's law

$$Q = -kA \frac{d\theta}{dx}$$

show that

$$Q = -kA \left( \frac{\theta_2 - \theta_1}{t} \right)$$

[ $\theta(0)=\theta_1$  means that when  $x$  is 0 then  $\theta = \theta_1$ , and  $\theta(t)=\theta_2$  means that when  $x$  is  $t$  then  $\theta = \theta_2$ .]


- 12  [aerodynamics] The pressure,  $p > 0$ , of a gas in isothermal condition is given by

$$\frac{dp}{dz} = -\frac{pmg}{RT}$$

where  $z$  is the altitude,  $T$  (constant) is the temperature,  $m$  is the molar mass and  $R$  is a gas constant. Show that

$$p = Ae^{-\frac{mg}{RT}z}$$

(where  $A$  is a constant).

- \*13  [aerodynamics] The pressure,  $p$ , of the atmosphere at an altitude  $z$  is given by

$$\frac{dp}{dz} = -kp^{\frac{1}{\gamma}} \quad (k \neq 0)$$

where  $\gamma$  is the specific heat constant ( $\gamma > 1$ ) and  $k$  is a constant. Show that

$$z = \frac{\gamma p^{\frac{\gamma-1}{\gamma}}}{k(1-\gamma)} + A$$

(where  $A$  is a constant).

**SECTION B Using the integrating factor**

By the end of this section you will be able to:

- understand the integrating factor method for solving first order differential equations
- apply the integrating factor method to examples

**B1 Integrating factor**

**Example 8** is on differentiation and should ideally be placed in **Chapter 6** but we establish a result in this example which is used to explain the integrating factor.

**Example 8**

Find  $\frac{d}{dx}(ye^{Px})$  where  $y$  is a function of  $x$  and  $P$  is a constant.

Solution

We can differentiate  $ye^{Px}$  by using the product rule:

$$\mathbf{6.31} \quad \frac{d}{dx}(uv) = u'v + uv'$$

with

$$\begin{aligned} u &= y & v &= e^{Px} \\ u' &= \frac{dy}{dx} & v' &= Pe^{Px} \end{aligned} \quad \text{[Differentiating]}$$

Substituting these into **6.31** gives

$$\frac{d}{dx}(ye^{Px}) = \frac{dy}{dx}e^{Px} + Pye^{Px}$$

Suppose we have a general first order differential equation of the form

$$\mathbf{13.3} \quad \frac{dy}{dx} + Py = Q(x)$$

which we need to solve, that is find  $y$  which satisfies the differential equation. Multiply both sides of **13.3** by  $e^{Px}$ :

$$\frac{dy}{dx}e^{Px} + Pye^{Px} = e^{Px} Q(x)$$

**? What do you notice about the Left-Hand Side?**

By the result of **Example 8**, the Left-Hand Side  $\frac{dy}{dx} e^{Px} + P y e^{Px} = \frac{d}{dx}(y e^{Px})$ . Hence we have

$$\frac{d}{dx}(y e^{Px}) = e^{Px} Q(x)$$

**? How can we obtain  $y$  from this?**

Integrate both sides with respect to  $x$ .

Integration removes the derivative,  $\frac{d}{dx}$ , on the Left-Hand Side:

$$y e^{Px} = \int e^{Px} Q(x) dx$$

Since  $P$  is constant,  $Px = \int P dx$ . Thus replacing  $Px$  with  $\int P dx$  we have

$$y e^{\int P dx} = \int e^{\int P dx} Q(x) dx$$

$e^{\int P dx}$  is called the integrating factor (IF). Summarizing the above:

The solution of the differential equation

$$\text{13.3} \quad \frac{dy}{dx} + P y = Q(x)$$

is given by

$$\text{13.4} \quad y(\text{IF}) = \int [(\text{IF})Q(x)] dx \quad \text{where IF} = e^{\int P dx}$$

In the above we illustrated the integrating factor method for a constant,  $P$ , but it may be generalized to cover  $P$  as a function of  $x$ , that is,  $P = P(x)$ . We still obtain formula **13.4**. We will not go over the same treatment for a non-constant  $P$  but just apply **13.4** in the appropriate case.

**Example 9**

Solve

$$\frac{dy}{dx} + 3y = e^x$$

with the initial condition that when  $x = 0$ ,  $y = 0$ .

Solution

The given equation is of the same format as **13.3**, so we can use **13.4** with  $P = 3$ :

$$\text{IF} = e^{\int P dx} = e^{\int 3 dx} = e^{3x}$$

Applying  $y(\text{IF}) = \int [(\text{IF})Q(x)] dx$  with  $\text{IF} = e^{3x}$  and  $Q(x) = e^x$  we have

**Example 9** *continued*

$$\begin{aligned}
 ye^{3x} &= \int [e^{3x} e^x] dx \\
 &= \int [e^{3x+x}] dx \\
 &= \int e^{4x} dx \\
 &= \frac{e^{4x}}{4} + C \quad \left[ \text{Integrating by } \int e^{kx} dx = \frac{e^{kx}}{k} \right]
 \end{aligned}$$

Substituting the initial condition  $x = 0, y = 0$  into  $ye^{3x} = \frac{e^{4x}}{4} + C$  gives

$$0 = \frac{1}{4} + C \text{ and so } C = -\frac{1}{4} \quad [\text{Remember that } e^0 = 1]$$

Putting  $C = -1/4$ :

$$ye^{3x} = \frac{e^{4x}}{4} - \frac{1}{4}$$



Remember we need to find  $y$ . **How?**

Divide through by  $e^{3x}$  and then use the rules of indices to simplify:

$$\begin{aligned}
 y &= \frac{e^{4x}}{4e^{3x}} - \frac{1}{4e^{3x}} \\
 &= \frac{e^{4x-3x}}{4} - \frac{e^{-3x}}{4} \quad \left[ \text{Using } \frac{a^m}{a^n} = a^{m-n} \text{ and } \frac{1}{a^n} = a^{-n} \right] \\
 &= \frac{e^x}{4} - \frac{e^{-3x}}{4} \\
 &= \frac{1}{4}(e^x - e^{-3x}) \quad \left[ \text{Taking out } \frac{1}{4} \right]
 \end{aligned}$$

**B2 Engineering applications**

In this section we examine electrical engineering applications. In electrical principles  $R, L$  and  $C$  are constants representing resistance, inductance and capacitance respectively.  $E$  represents the e.m.f. and  $v = v(t), i = i(t)$  represent voltage and current respectively at time  $t$ .



### Example 10 *electrical principles*

The first order differential equation

$$RC \frac{dv}{dt} + v = Ee^{-\frac{t}{RC}}$$

describes a series  $RC$  circuit. By using the initial condition  $v(0) = 1$ , show that

$$v = e^{-\frac{t}{RC}} \left( \frac{Et}{RC} + 1 \right)$$

Solution

Divide the given differential equation by  $RC$ :

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{E}{RC} e^{-\frac{t}{RC}}$$

**?** For this differential equation, can we use the integrating factor method?

Yes, because we can write the differential equation as

$$\frac{dv}{dt} + Pv = Q(t)$$

where  $P = \frac{1}{RC}$  because  $\frac{v}{RC} = \frac{1}{RC}v$  and  $Q(t) = \frac{E}{RC} e^{-\frac{t}{RC}}$

With  $P = \frac{1}{RC}$  and integrating with respect to  $t$ :

$$\text{IF} = e^{\int P dt} = e^{\int \frac{1}{RC} dt} = e^{\frac{t}{RC}}$$

Applying

$$\text{13.4} \quad v(\text{IF}) = \int [(\text{IF})Q(t)] dt$$

with  $\text{IF} = e^{t/RC}$  and  $Q(t) = \frac{E}{RC} e^{-t/RC}$  we have

$$\begin{aligned} ve^{\frac{t}{RC}} &= \int \left( \frac{E}{RC} e^{\frac{t}{RC}} e^{-\frac{t}{RC}} \right) dt \\ &= \frac{E}{RC} \int \left( e^{\frac{t}{RC} - \frac{t}{RC}} \right) dt && \left[ \text{Taking out } \frac{E}{RC} \text{ and using rules of indices} \right] \\ &= \frac{E}{RC} \int \underset{=1}{(e^0)} dt \\ &= \frac{E}{RC} \int dt \\ ve^{\frac{t}{RC}} &= \frac{Et}{RC} + D && [\text{Integrating}] \end{aligned}$$



### Example 10 *continued*

where  $D$  is the constant of integration.

The initial condition  $v(0) = 1$  means that when  $t = 0$ ,  $v = 1$ . Substituting this condition,  $t = 0$ ,  $v = 1$ , gives

$$1 = 0 + D \quad \text{[Remember that } e^0 = 1\text{]}$$

Hence  $D = 1$  and therefore we have

$$ve^{\frac{t}{RC}} = \frac{Et}{RC} + 1$$

Dividing through by  $e^{t/RC}$  to find  $v$  on its own gives

$$\begin{aligned} v &= \frac{1}{e^{\frac{t}{RC}}} \left( \frac{Et}{RC} + 1 \right) \\ &= e^{-\frac{t}{RC}} \left( \frac{Et}{RC} + 1 \right) \end{aligned}$$

So we have the required result.

In the above example we used the initial condition  $v(0) = 1$  which means when  $t = 0$ ,  $v = 1$ . In general the initial condition  $y(x_0) = y_0$  means when  $x = x_0$  then  $y = y_0$ . It is important that you understand this notation because it is used extensively throughout this chapter.

## SUMMARY

The solution of the general first order differential equation of the form

$$13.3 \quad \frac{dy}{dx} + Py = Q(x)$$

where  $P = P(x)$  is given by

$$13.4 \quad y(\text{IF}) = \int [(\text{IF})Q(x)] dx \quad \text{where IF} = e^{\int P dx}$$

## Exercise 13(b)

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

**1** Solve the differential equation

$$\frac{dy}{dx} + 2y = e^{-x}$$

with the initial condition  $x = 0$ ,  $y = 0$ .

**2** Solve the differential equation

$$\frac{dy}{dx} - y = e^{2x}$$


**3** Solve the differential equation

$$(1 - x^2)\frac{dy}{dx} - 2xy = 1 \quad (x < 1)$$

with the initial condition  $x = 0$ ,  $y = 0$ .



## Exercise 13(b) continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)
 Questions 4 and 5 belong to the field of **[mechanics]**.

- 4 The velocity,  $v$ , of an object of mass  $m$  in free fall can be described by the first order differential equation

$$m \frac{dv}{dt} = mg - kv$$

where  $k$  is a positive constant and  $g$  is the acceleration due to gravity. For the initial condition, when  $t = 0$ ,  $v = 0$ , show by using the integrating factor method that

$$v = \frac{mg}{k} \left( 1 - e^{-\frac{k}{m}t} \right)$$


- 5 The velocity,  $v$ , of an object is given by the differential equation

$$\frac{dv}{dt} = \frac{1}{t}(1 - v) \quad (t > 0)$$

By using the integrating factor method, determine an expression for  $v$  with the initial condition that when  $t = 1$ ,  $v = 0$ .

What is the terminal velocity (the velocity as  $t \rightarrow \infty$ )?

Sketch the graph of  $v$  versus  $t$  for  $t > 0$ .

 The remaining questions are in the field of **[electrical principles]**.

- 6 By applying Kirchhoff's voltage law to a series  $RL$  circuit we obtain the first order differential equation

$$L \frac{di}{dt} + Ri = E(t)$$

where  $E(t)$  is the applied voltage and  $i = i(t)$  is the current through the circuit.

i Show that  $i = \frac{e^{-\frac{R}{L}t}}{L} \int \left( E(t)e^{\frac{R}{L}t} \right) dt$ .

- ii If  $E(t)$  is constant,  $E(t) = E$ , and  $i(0) = i_0$  then show that

$$i = \frac{E}{R} + \left( i_0 - \frac{E}{R} \right) e^{-\frac{R}{L}t}$$

- iii Sketch the graph of  $i$  versus  $t$  (assuming  $i_0 < E/R$ ). What value does  $i$  tend to as  $t$  becomes large?

- 7 A circuit consists of a resistor of resistance  $R$ , and a capacitor of capacitance  $C$ , connected in series, and is described by the first order differential equation

$$RC \frac{dv}{dt} + v = E$$

where  $E$  is the constant e.m.f. and  $v$  is the voltage across the capacitor. Given that  $v(0) = 0$ , show by using the integrating factor method that

$$v = E(1 - e^{-t/(RC)})$$

- 8 By applying Kirchhoff's voltage law to a series  $RL$  circuit we obtain the differential equation

$$(1 \times 10^{-3}) \frac{di}{dt} + (3 \times 10^3)i = 10e^t$$

where  $i = i(t)$  is the current through the circuit. Given the initial condition  $i(0) = 0$ , determine an expression for the current  $i$ .

- 9 The charge,  $q$ , on a capacitor of capacitance  $C$ , of a series  $RC$  circuit is given by

$$R \frac{dq}{dt} + \frac{q}{C} = Ve^{-t}$$

Show that when  $t = 0$ ,  $q = 0$  gives

$$q = \frac{VC}{1 - RC} \left( e^{-t} - e^{-\frac{t}{RC}} \right)$$

## SECTION C Applications to electrical principles

By the end of this section you will be able to:

- model electrical circuits which give rise to a first order differential equation
- solve the resulting first order differential equation
- use a computer algebra system to solve a first order differential equation

### C1 Electrical laws

Below are some of the laws regarding basic electrical components such as a resistor, a capacitor and an inductor. Kirchhoff's voltage law says:

In a closed path

**13.5** sum of the voltage rises = sum of the voltage drops

With reference to Fig. 3,  
Kirchhoff's law says

$$v = v_1 + v_2 + v_3 \quad \text{Fig. 3}$$

The voltage,  $v$ , across a resistor of resistance  $R$  (Fig. 4), is given by

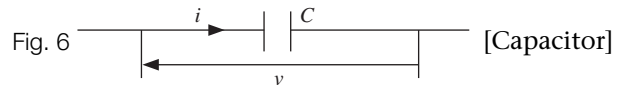
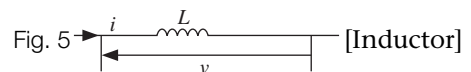
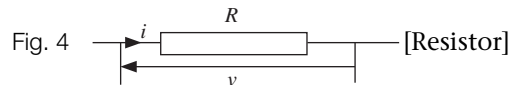
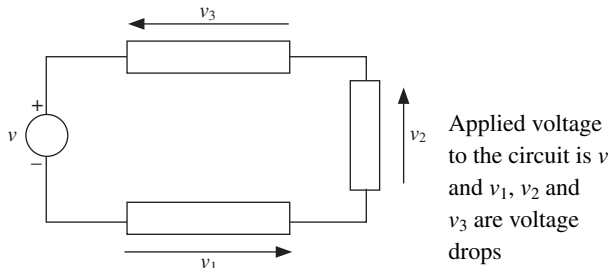
**13.6**  $v = iR$  (Ohm's law)

The voltage,  $v$ , across an inductor of inductance  $L$  (Fig. 5), is given by

$$\mathbf{13.7} \quad v = L \frac{di}{dt}$$

The voltage,  $v$ , across a capacitor of capacitance  $C$  (Fig. 6), is related to the current by

$$\mathbf{13.8} \quad i = C \frac{dv}{dt}$$



Remember that  $R$ ,  $L$  and  $C$  are positive constants.

Next we use these rules to form differential equations of electrical circuits.

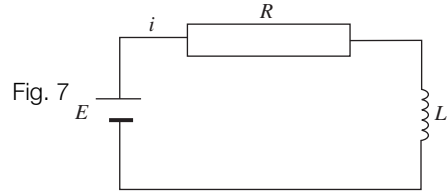
The following example is complicated because it involves a number of concepts such as forming and solving a differential equation, using algebraic techniques and graphing functions.

## C2 Electrical circuits

Example 11 *electrical principles*

Figure 7 shows a resistor of resistance  $R$ , connected in series with an inductor of inductance  $L$ , and an applied constant voltage  $E$ .

- i Obtain a first order differential equation for the current  $i$  at time  $t$ .
- ii Solve this differential equation for the initial condition, when  $t = 0$ ,  $i = 0$ .
- iii What is the value of  $i$  as  $t \rightarrow \infty$ ?
- iv Sketch the graph of  $i$  versus  $t$  for  $t \geq 0$ .



Solution

- i By applying Kirchhoff's law to the circuit of Fig. 7 we have:

$$E = (\text{voltage drop across resistance, } R) + (\text{voltage drop across inductance, } L)$$

$$* \quad = \underbrace{iR}_{\text{by 13.6}} + L \underbrace{\frac{di}{dt}}_{\text{by 13.7}}$$

This,  $*$ , is a first order differential equation.

- ii **What does 'solve' mean in this context?**

It means find an expression for  $i$  in terms of  $E$ ,  $R$ ,  $L$  and  $t$ . This differential equation can be solved by separating variables as discussed in **Section 13A**. By subtracting  $iR$  from both sides of  $*$  we have

$$L \frac{di}{dt} = E - iR$$

Separating variables:

$$L \frac{di}{E - iR} = dt$$

Integrating both sides:

$$L \int \frac{di}{E - iR} = \int dt$$

$$\dagger \quad -\underbrace{\frac{L}{R} \ln(E - iR)}_{\text{by 8.42}} = t + C \quad [\text{Because the derivative of } E - iR \text{ is } -R]$$



### Example 11 *continued*

Substituting the given initial condition  $t = 0, i = 0$  into  $\dagger$ :

$$-\frac{L}{R}\ln(E) = C$$

Placing  $C = -\frac{L}{R}\ln(E)$  into  $\dagger$  gives

$$\begin{aligned} -\frac{L}{R}\ln(E - iR) &= t - \frac{L}{R}\ln(E) \\ -t &= \frac{L}{R}\left[\ln(E - iR) - \ln(E)\right] \end{aligned}$$

Using the properties of logs inside the square brackets and multiplying both sides by  $\frac{R}{L}$ :

$$-\frac{Rt}{L} = \underbrace{\ln\left(\frac{E - iR}{E}\right)}_{\text{by 5.13}}$$

Taking exponentials of both sides:

$$\begin{aligned} e^{-\frac{Rt}{L}} &= e^{\ln\left(\frac{E - iR}{E}\right)} \stackrel{\text{by 5.17}}{=} \frac{E - iR}{E} \\ Ee^{-\frac{Rt}{L}} &= E - iR \quad [\text{Multiplying by } E] \end{aligned}$$

Rearranging gives

$$\begin{aligned} iR &= E - Ee^{-\frac{Rt}{L}} \\ &= E\left(1 - e^{-\frac{Rt}{L}}\right) \quad [\text{Taking out } E] \end{aligned}$$

Dividing through by  $R$  yields  $i$ :

$$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

**iii** Since  $R$  and  $L$  are positive, we have  $\frac{R}{L} > 0$ ,  $e^{-\frac{Rt}{L}} \rightarrow 0$  as  $t \rightarrow \infty$  and so  $i \rightarrow \frac{E}{R}$ .

**iv** At  $t = 0, i = 0$  therefore the graph goes through

the origin and is asymptotic to the line

$$i = \frac{E}{R} \quad (\text{Fig. 8}).$$

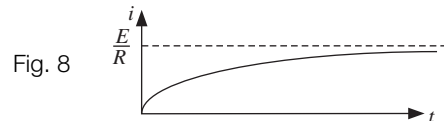


Fig. 8

The next example uses the application of a computer algebra system such as MAPLE.

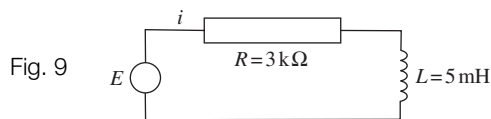
$$\text{5.13} \quad \ln(A) - \ln(B) = \ln\left(\frac{A}{B}\right)$$

$$\text{5.17} \quad e^{\ln(x)} = x$$



### Example 12 *electrical principles*

Figure 9 shows a series  $RL$  circuit with an applied voltage,  $E = 240(1 - e^{-t/3})$ .



- i** Obtain a differential equation giving the current  $i$  at time  $t$ .
- ii** Solve the differential equation for the initial condition, when  $t = 0$ ,  $i = 0$ .
- iii** Plot the graph of  $i$  versus  $t$ . [Use MAPLE for parts **ii** and **iii**.]

Solution

- i** By applying Kirchhoff's law we have

$$E = (\text{voltage drop across resistor}) + (\text{voltage drop across inductor})$$

$$= \underbrace{iR}_{\text{by 13.6}} + \underbrace{L \frac{di}{dt}}_{\text{by 13.7}}$$

Substituting the given values [remember that k = kilo =  $10^3$  and m = milli =  $10^{-3}$ ],  $R = 3 \times 10^3$ ,  $L = 5 \times 10^{-3}$ ,  $E = 240(1 - e^{-t/3})$  into the differential equation:

$$240(1 - e^{-t/3}) = (3 \times 10^3)i + (5 \times 10^{-3}) \frac{di}{dt}$$

- ii** We solve this first order differential equation, by using MAPLE:



```
> de_1 := (3*10^3) * i(t) + (5*10^(-3)) * diff(i(t), t) = 240
  * (1 - exp(-t/3));
```

$$de_1: = 3000 i(t) + \frac{1}{200} \left( \frac{\partial}{\partial t} i(t) \right) = 240 - 240 e^{(-1/3)t}$$



```
> soln := dsolve({de_1, i(0)=0}, i(t));
```

```
* soln: = i(t) = 2/25 - 144000/1799999 e^{(-1/3)t} + 2/44999975 e^{(-600000)t}
```



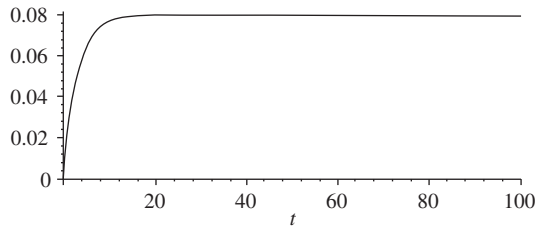
### Example 12 *continued*

iii

```
plot(rhs(soln), t=0..100);
```



Fig. 10



In **\*** of **Example 12**, the extreme right term is called the transient term. Transient is a brief disturbance in a circuit. Because it happens to have a large negative exponent, it dies away very rapidly, and cannot be shown on the scale of Fig. 10. The first two terms show the time variation of the applied voltage. The current is eventually  $E/R$ , or  $2/25 = 0.08$  in this case, which is shown in Fig. 10.

## SUMMARY

**13.6** The voltage,  $v$ , across a resistor of resistance  $R$ , is given by  $v = iR$  (Ohm's law)

**13.7** The voltage,  $v$ , across an inductor of inductance  $L$ , is given by  $v = L \frac{di}{dt}$

**13.8** The voltage,  $v$ , across a capacitor of capacitance  $C$ , is related to the current by  $i = C \frac{dv}{dt}$

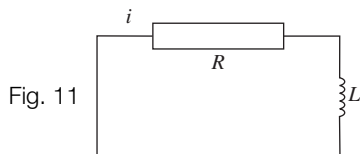
## Exercise 13(c)

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)



All questions belong to the field of **[electrical principles]**.

**1** Figure 11 shows a series  $RL$  circuit.



**i** Show that

$$\frac{di}{dt} = -\frac{iR}{L}$$

where  $i$  represents the current through the circuit.

**ii** Show that for the initial condition,  $t = 0$  s,  $i = 1$  A, then

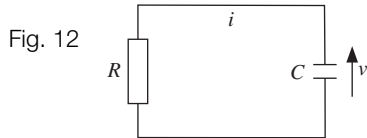
$$i = e^{-\frac{R}{L}t}$$

## Exercise 13(c) continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- iii Sketch the graph of  $i$  against  $t$ .
- iv What is the value of  $i$  at  $t = \tau$  where  $\tau = \frac{L}{R}$  ( $\tau$  is called the time constant for this circuit)?

2 Figure 12 shows a series  $RC$  circuit.



- i Show that the differential equation relating the voltage,  $v$ , across the capacitor of capacitance  $C$  at time  $t$  is given by

$$\frac{dv}{dt} = -\frac{v}{RC}$$

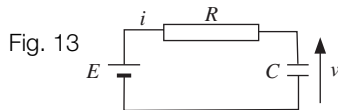
- ii Show that for the initial condition when  $t = 0$ ,  $v = E$  ( $> 0$ ) that

$$v = Ee^{-t/RC}$$

- iii Sketch the graph of  $v$  against  $t$  for  $t \geq 0$ .

- iv What is the value of  $v$  at  $t = \tau$  where  $\tau = RC$  ( $\tau$  is called the time constant)?

3 Figure 13 shows a series  $RC$  circuit with a constant applied e.m.f. =  $E$ .

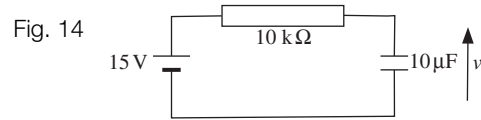


Show that the differential equation of this circuit is given by

$$\frac{dv}{dt} = \frac{1}{RC}(E - v)$$

where  $v$  is the voltage across the capacitor. Solve this differential equation for the initial condition,  $t = 0$ ,  $v = 0$ .

4



- i For the circuit of Fig. 14 show that

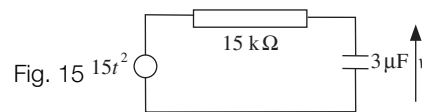
$$0.1 \frac{dv}{dt} = 15 - v$$

where  $v$  is the voltage across the capacitor at time  $t$ .

- ii Solve this differential equation for the initial condition, when  $t = 0$ ,  $v = 0$ .
- iii Sketch the graph of  $v$  against  $t$  for  $t \geq 0$ .
- iv What percentage of the final voltage is gained at each of the following stages:  $t = 0.1$ ,  $0.2$  and  $0.3$ ? ( $0.1$  is the time constant for this circuit and  $0.2$ ,  $0.3$  are 2 and 3 time constants respectively.)

Use a computer algebra system for the remaining questions.

5



- i Show that the differential equation of the circuit in Fig. 15 is given by

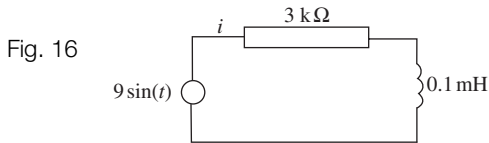
$$0.045 \frac{dv}{dt} = 15t^2 - v$$

- ii Solve this differential equation for the initial condition,  $t = 0$ ,  $v = 1$ .
- iii Plot the graph of  $v$  against  $t$  for  $0 \leq t \leq 0.5$ .

## Exercise 13(c) continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- 6 Figure 16 shows a series resistor–inductor circuit.



- i Show that

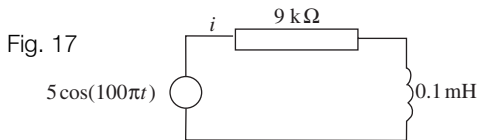
$$\frac{di}{dt} = (90 \times 10^3)\sin(t) - (30 \times 10^6)i$$

where  $i$  is the current through the circuit.

- ii Solve for the initial condition,  $t = 0, i = 1 \times 10^{-3}$ .
- iii Plot, on different axes, the graphs of  $i$  against  $t$  for
- $0 \leq t \leq 20$
  - $0 \leq t \leq 2 \times 10^{-7}$ .

Comment upon your results.

7



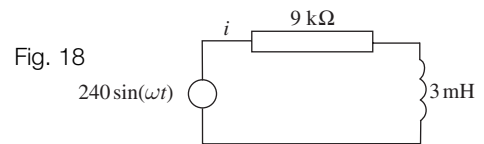
- i For the circuit of Fig. 17, show that

$$\frac{di}{dt} = (50 \times 10^3)\cos(100\pi t) - (90 \times 10^6)i$$

- ii Solve the differential equation for the initial condition,  $t = 0, i = 0$ .
- iii Plot the graph of  $i$  against  $t$ . Choose a range that includes 5 complete cycles. Also plot the range  $0 \leq t \leq 10^{-7}$ .

Comment upon your results.

8



- i Show that the differential equation of Fig. 18 is given by

$$\frac{di}{dt} = (80 \times 10^3)\sin(\omega t) - (3 \times 10^6)i$$

- ii Solve this differential equation for the initial condition,  $t = 0, i = 0$ .
- iii Plot the graph of  $i$  against  $\omega t$  for  $\omega t \geq 0$ . Initially choose  $\omega = 10\,000$  and  $\omega = 10^6$ . Plot about 5 to 10 complete cycles. From the results you get, choose other values of  $\omega$  which give interesting results.

## SECTION D Further engineering applications

By the end of this section you will be able to:

- apply first order differential equations to other fields of engineering such as fluid mechanics and heat transfer
- understand Torricelli's law and derive and solve the relevant first order differential equation
- apply Newton's law of cooling



### D1 Torricelli's law (fluid mechanics)

Figure 19 shows a water tank with an outlet of cross-sectional area  $A_o$  ( $\text{m}^2$ ) and water in the tank filled to a height of  $h$  (m). Torricelli's law states that the velocity,  $v$  (m/s), of water exiting through the outlet is given by

$$v = \sqrt{2gh}$$

where  $g$  is the acceleration due to gravity

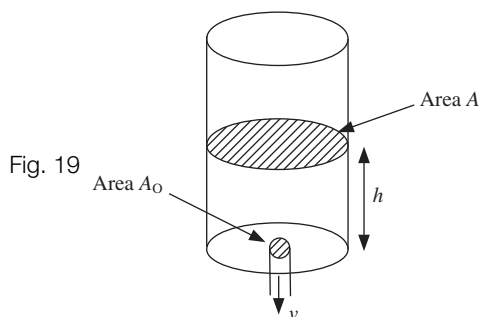


Fig. 19

(This is obtained by equating energies,  $\frac{1}{2}mv^2 = mgh$  and solving for  $v$ .)

Let the area of the outlet be  $A_o$  ( $\text{m}^2$ ) then the volume of water leaving the tank per second is  $A_o\sqrt{2gh}$ . The volume of water leaving the tank in time  $\Delta t$  is

$$(A_o\sqrt{2gh})\Delta t$$

Hence the change of volume,  $\Delta V$ , of water in the tank in time  $\Delta t$  is given by

$$\Delta V = -(A_o\sqrt{2gh})\Delta t$$

where the negative sign indicates that the volume  $V$  is decreasing. We have

$$\frac{\Delta V}{\Delta t} = -(A_o\sqrt{2gh})$$

Taking limits gives

$$\text{13.9} \quad \frac{dV}{dt} = -A_o\sqrt{2gh}$$

If the volume of water in the tank can be written as  $Ah$  where  $A$  is the constant cross-sectional area of the tank then

$$\frac{dV}{dt} = \frac{d}{dt}(Ah) = A\frac{dh}{dt} \quad [A \text{ is constant}]$$

Substituting  $\frac{dV}{dt} = A\frac{dh}{dt}$  into **13.9** gives

$$A\frac{dh}{dt} = -A_o\sqrt{2gh}$$

Dividing both sides by  $A$ :

$$\text{13.10} \quad \frac{dh}{dt} = -\frac{A_o}{A}\sqrt{2gh}$$

We take  $g$  to be  $9.81 \text{ m/s}^2$ .



### Example 13 *fluid mechanics*

A cubical tank of side 3 m is filled with water to a height of  $h$  (m). The water is drained through a circular hole of diameter 0.1 m.

- i** Obtain the differential equation relating the height  $h$  of water at time  $t$ .
- ii** Solve this differential equation for the initial condition,  $t = 0$ ,  $h = 2$ .
- iii** How long (in minutes) does it take to empty the tank from 2 m full?
- iv** By using a computer algebra system or a graphical calculator, plot the graph of  $h$  against  $t$  for  $t \geq 0$ . Comment upon your graph.

**Solution**

- i** The outlet area,  $A_o$ , is a circle of diameter 0.1 m and so the radius  $r = 0.1/2$ :

$$\begin{aligned} A_o &= \pi \times r^2 \\ &= \pi \times \left(\frac{0.1}{2}\right)^2 \\ &= \pi \times 0.05^2 = (2.5 \times 10^{-3})\pi \end{aligned}$$

Since we have a cubical tank of side 3 m, therefore

$$\text{cross-sectional area of tank } A = 3^2 = 9$$

Substituting  $g = 9.81$ ,  $A_o = (2.5 \times 10^{-3})\pi$  and  $A = 9$  into

$$\begin{aligned} \text{13.10} \quad \frac{dh}{dt} &= -\frac{A_o}{A} \sqrt{2gh} \\ \frac{dh}{dt} &= -\frac{(2.5 \times 10^{-3})\pi}{9} \sqrt{2 \times 9.81 \times h} \\ &= -\frac{(2.5 \times 10^{-3})\pi \sqrt{2 \times 9.81}}{9} \sqrt{h} \\ &= -(3.87 \times 10^{-3})h^{1/2} \quad [\text{Evaluating and writing } \sqrt{h} = h^{1/2}] \end{aligned}$$

- ii** The remaining evaluation is very similar to **Example 7**. Separating variables of

$$\frac{dh}{dt} = -(3.87 \times 10^{-3})h^{1/2}$$

gives

$$\frac{dh}{h^{1/2}} = -(3.87 \times 10^{-3}) dt$$

Integrating:

$$\int h^{-1/2} dh = \int -(3.87 \times 10^{-3}) dt \quad [\text{Because } 1/h^{1/2} = h^{-1/2}]$$

$$\frac{h^{1/2}}{1/2} = -(3.87 \times 10^{-3})t + C \quad [\text{Integrating}]$$

$$2h^{1/2} = -(3.87 \times 10^{-3})t + C$$

Substituting the given initial condition,  $t = 0$ ,  $h = 2$ :

$$2(2)^{1/2} = 0 + C$$

Example 13 *continued*

Putting  $C = 2(2)^{1/2}$ :

$$2h^{1/2} = -(3.87 \times 10^{-3})t + 2(2)^{1/2}$$

$$h^{1/2} = -(1.935 \times 10^{-3})t + 2^{1/2} \quad [\text{Dividing by } 2]$$



**How can we find  $h$ ?**

Square both sides:

$$\dagger \quad h = [-(1.935 \times 10^{-3})t + 2^{1/2}]^2$$



**iii What are we trying to find?**

The value of  $t$  when  $h = 0$  because the tank is empty at  $h = 0$ .

Substituting  $h = 0$  and then taking the square root of  $\dagger$  gives

$$0 = -(1.935 \times 10^{-3})t + 2^{1/2}$$

$$t = \frac{2^{1/2}}{(1.935 \times 10^{-3})} = 731 \text{ s} \quad (\text{s} = \text{seconds})$$



**What is  $t$  in minutes (mins)?**

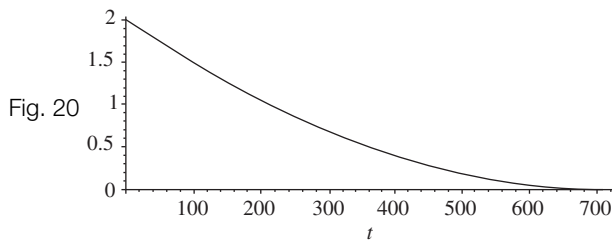
$$t = \frac{731}{60} = 12.18 \text{ mins}$$

It takes just over 12 mins to drain the water in the tank.

**iv** We plot  $h = [-(1.935 \times 10^{-3})t + 2^{1/2}]^2$  against  $t$  using MAPLE:



```
> h:=t->((( -1.935*10^(-3))*t+sqrt(2))^2);
h: = t -> (-.001935000000 t + sqrt(2))^2
> plot(h(t), t=0..731);
```



The graph of Fig. 20 is only valid for  $0 \leq t \leq 731$ . When  $t > 731$  the tank is empty and hence the height of the water in the tank is **zero**. Moreover the expression plotted is only valid for the tank which is 2 m full.

From the graph it is clear that the height of the water is decreasing with time  $t$ .

We can use MAPLE to solve the differential equation in **Example 13** and find how long it takes to empty the tank. It is in forming the differential equation [part i] where we cannot use MAPLE.

## D2 Newton's law of cooling (heat transfer)

Newton's law of cooling states that the rate at which a body cools is proportional to the difference in temperature between the body and its surrounding environment. If  $\theta = \theta(t)$  is the temperature of the body at time  $t$  and  $T$  is the constant surrounding temperature then

$$\frac{d\theta}{dt} \propto \theta - T \quad (\propto \text{ means directly proportional to})$$

Note that  $\theta$  is a function of time  $t$ . We have

$$13.11 \quad \frac{d\theta}{dt} = k(\theta - T)$$

where  $k$  is called the constant of proportionality.



### Example 14 *heat transfer*

A body is exposed to a constant temperature of 280 K. After 1 minute the temperature of the body is 350 K and after 5 minutes it is 310 K. Find an expression for the temperature  $\theta$  at time  $t$ . Sketch the graph of  $\theta$  against  $t$  for  $t \geq 0$ . Comment upon your graph.

Solution

Using

$$13.11 \quad \frac{d\theta}{dt} = k(\theta - T)$$

with  $T = 280$  gives

$$\frac{d\theta}{dt} = k(\theta - 280)$$

Separating variables:

$$\frac{d\theta}{\theta - 280} = k dt$$

Integrating both sides:

$$\int \frac{d\theta}{\theta - 280} = \int k dt$$

$$\underbrace{\ln(\theta - 280)}_{\text{by 8.42}} = kt + C$$

Since  $t$  is measured in seconds we have  $t = 1 \times 60 = 60$  and  $\theta = 350$ .

Substituting these values into  $\ln(\theta - 280) = kt + C$  gives

---


$$8.42 \quad \int \frac{f'(\theta)}{f(\theta)} d\theta = \ln|f(\theta)|$$



### Example 14 *continued*

$$\dagger \quad \ln(350 - 280) = 60k + C$$

We are also given  $t = 5 \times 60 = 300$  and  $\theta = 310$ . Similarly we have

$$\dagger\dagger \quad \ln(310 - 280) = 300k + C$$

Solving the simultaneous equations  $\dagger$  and  $\dagger\dagger$  gives  $k = -3.53 \times 10^{-3}$  and  $C = 4.46$ .

Hence substituting these values into  $\ln(\theta - 280) = kt + C$  yields

$$\ln(\theta - 280) = -(3.53 \times 10^{-3})t + 4.46$$

Taking exponentials of both sides to find  $\theta$  we have

$$\theta - 280 = e^{-(3.53 \times 10^{-3})t + 4.46} \quad [\text{Because } e^{\ln(\theta - 280)} = \theta - 280]$$

$$\theta = 280 + e^{4.46} e^{-(3.53 \times 10^{-3})t}$$

$$\theta = 280 + \frac{86.49}{e^{4.46}} e^{-(3.53 \times 10^{-3})t}$$

For sketching the graph we need to find the value of  $\theta$  at  $t = 0$ :

$$\begin{aligned} \theta &= 280 + 86.49e^0 \\ &= 280 + 86.49 = 366.49 \quad [\text{Because } e^0 = 1] \end{aligned}$$

Also as  $t \rightarrow \infty$ ,  $e^{-(3.53 \times 10^{-3})t} \rightarrow 0$ , thus  $\theta \rightarrow 280$ .

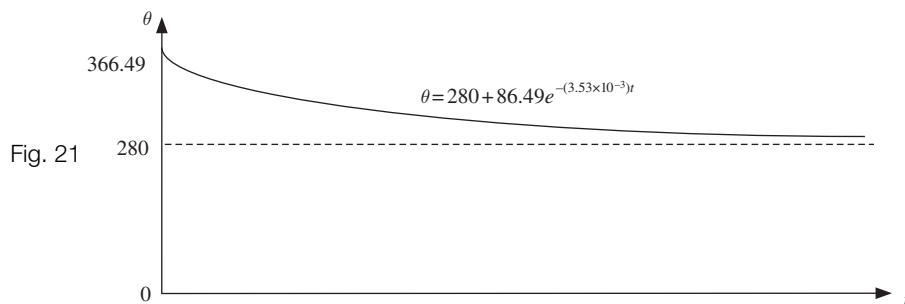


Fig. 21

The graph in Fig. 21 shows that the temperature of the body will eventually reach very close to the temperature of the surroundings at 280 K. This is because  $86.49e^{-(3.53 \times 10^{-3})t}$  is the **transient term** and decays to zero as  $t$  gets large.

**SUMMARY**

The differential equation describing the tank filled to a height  $h$  (m) at time  $t$  is given by

$$13.10 \quad \frac{dh}{dt} = -\frac{A_o}{A} \sqrt{2gh}$$

where  $A_o$  and  $A$  are the cross-sectional areas of the outlet and the tank respectively. Newton's law of cooling gives a first order differential equation which describes the temperature  $\theta$  at time  $t$  by

$$13.11 \quad \frac{d\theta}{dt} = k(\theta - T)$$

where  $k$  is a constant and  $T$  is the constant surrounding temperature.

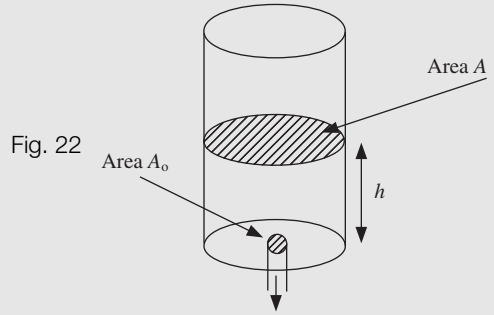



Fig. 22

**Exercise 13(d)**

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

Plot the graphs in this exercise by using a computer algebra system or a graphical calculator.

 Questions 1 to 4 are in the field of [fluid mechanics].

- 1 A cubical tank of side 2 m is filled with water to a height of  $h$  (m). The water is drained through a circular hole of diameter 0.2 m.
  - i Obtain a differential equation giving the height  $h$  at any time  $t$ .
  - ii Given that when  $t = 0$ ,  $h = 1.3$ , solve this differential equation to give an expression for  $h$  in terms of  $t$ .
  - iii How long (in min) does it take to empty the tank if it is 1.3 m full?
  - iv Plot the graph of  $h$  against  $t$  for  $t \geq 0$ . Comment upon your graph.
- 2 A cylindrical tank of diameter 1.5 m contains water of height  $h$ . The water is drained through a square hole of length 0.05 m.

- i Obtain a differential equation representing the height  $h$  at any time  $t$ .
- ii Given that when  $t = 0$ ,  $h = 1.6$ , solve this differential equation to give an expression for  $h$  in terms of  $t$ .
- iii How long (in min) does it take to empty the tank if it is 1.6 m full?
- iv Plot the graph of  $h$  against  $t$  for  $t \geq 0$ .

3

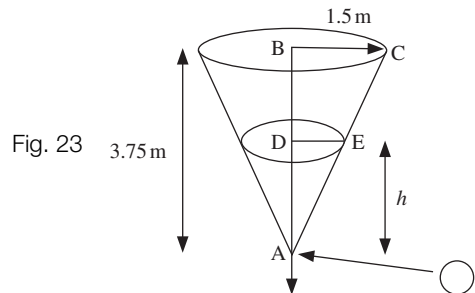


Fig. 23

Figure 23 shows a conical tank containing water of height  $h$ . The outlet is a circular hole of diameter 0.05 m.

- i Obtain the differential equation relating the height  $h$  of water at time  $t$ .

## Exercise 13(d) continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/science/engineering/singh](http://www.palgrave.com/science/engineering/singh)

[Hint: Consider similar triangles, ABC and ADE. Use the property

$$\frac{AB}{BC} = \frac{AD}{DE}$$

This is a similarity property of triangles]

- ii Solve this differential equation for the initial condition, when  $t = 0$ ,  $h = 3$ .
  - iii How long (in min) does it take for the tank to empty if it is initially 3 m full?
  - iv Plot the graph of  $h$  against  $t$  for  $t \geq 0$ . Comment upon your graph.
  - v How long does it take to empty the tank if it is 3.8 m full?
- 4 The general differential equation giving the height,  $h$ , of water in a tank at time  $t$  is given by

$$\frac{dh}{dt} = -k\sqrt{2gh}$$

where  $k$  is a constant. Solve this differential equation for the initial condition,  $h(0) = 1$ .

Plot the graphs of the solutions of height,  $h$ , against time,  $t$ , for

$$k = 0.1, 0.01 \text{ and } 0.001$$

Take  $g$  to be  $9.81 \text{ m/s}^2$ .

What effect does  $k$  have on the rate at which water is drained from the tank?

**U** The remaining questions are in the field of [heat transfer].

- 5 A body is at a temperature of 373 K. After 5 minutes the temperature of the body is 330 K. Find an expression for  $\theta = \theta(t)$  given that the constant surrounding temperature is 300 K. Sketch the graph of  $\theta$  against  $t$  for  $t \geq 0$ .

What does  $\theta$  tend to as  $t \rightarrow \infty$ ?

- 6 An object is initially at 400 K, and the constant surrounding temperature is 300 K. Determine an expression that gives the temperature  $\theta = \theta(t)$  at time  $t$ .

- 7 Newton's law of cooling gives

$$\frac{d\theta}{dt} = k(\theta - T)$$

where  $\theta$  is the temperature at time  $t$ ,  $T$  is the constant surrounding temperature and  $k$  is a constant. Given that  $\theta(0) = T_0$ , show that

$$\theta = (T_0 - T)e^{kt} + T$$

- 8 By applying Newton's law of cooling to an object we obtain

$$\frac{d\theta}{dt} = k(\theta - 320)$$

where  $\theta$  is the temperature at time  $t$  and  $k$  is a constant. Given that when  $t = 0$ ,  $\theta = 348 \text{ K}$ , find an expression for  $\theta$ . Plot, on the same axes, the graphs of  $\theta$  against  $t$  for  $t \geq 0$  and  $k = -0.001, -0.01$  and  $-0.1$ .

How does the temperature,  $\theta$ , change with  $k$ ?

- \*9 Stefan's law of radiation gives the rate of change of temperature,  $\frac{d\theta}{dt}$ , of a body by

$$\frac{d\theta}{dt} = k(\theta^4 - T^4)$$

where  $k$  is a constant,  $\theta (= \theta(t))$  is the temperature of the body at time  $t$  and  $T$  is the constant ambient temperature. Show, by solving the differential equation, that

$$\frac{1}{4T^3} \left[ \ln \left( \frac{\theta - T}{\theta + T} \right) - 2 \tan^{-1} \left( \frac{\theta}{T} \right) \right] = kt + C$$

where  $C$  is the constant of integration.

## SECTION E Euler's numerical method

By the end of this section you will be able to:

- apply Euler's numerical formula
- use a computer algebra system to find the analytical and numerical solutions of differential equations
- use a spreadsheet to find numerical solutions

Up until now we have solved differential equations by employing analytical methods such as separating variables and the integrating factor. However many differential equations **cannot** be solved by using these analytical techniques. Sometimes it is difficult, or impossible, to find a solution to a given differential equation using these methods. With the advancement in technology it is becoming easier to use numerical methods. These numerical methods involve using a repetitive process, hence the use of computers is beneficial in eliminating the repetitive calculations. Each repetitive mathematical process is called an **iteration**.

You can use a spreadsheet, such as EXCEL, or mathematical software, such as MAPLE, MATLAB or MATHEMATICA, to perform these processes.

In this chapter we cover three main numerical methods: Euler's, improved Euler's and 4th order Runge–Kutta. There are others but we will confine ourselves to these three. In this section we examine Euler's method.

### E1 Euler's method

Euler's method approximates the solution of the differential equation by using tangent lines. Remember that the solution of a differential equation is a function of  $x$  which we can plot on a graph.

The gradient,  $m$ , of the line in the graph of Fig. 24 is given by

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

$$y_1 - y_0 = m(x_1 - x_0)$$

Adding  $y_0$  to both sides makes  $y_1$  the subject:

$$\dagger \quad y_1 = y_0 + m(x_1 - x_0)$$

We use this result,  $\dagger$ , in deriving Euler's numerical formula.

Suppose we want to solve the first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

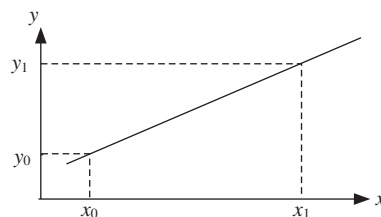


Fig. 24

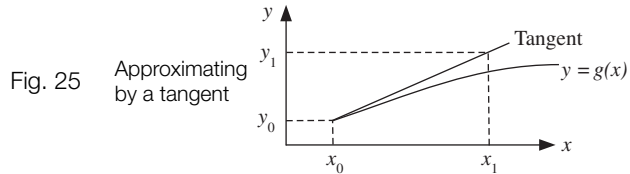


which satisfies the initial condition, at  $x = x_0$  we have  $y = y_0$ . The notation  $f(x, y)$  means a function of  $x$  and  $y$ . **What do we mean by 'solve'?**

Ideally we need to find a solution of the form,  $y = \text{function of } x$ .

Suppose we know the solution,  $y = g(x)$ , and it has the graph shown in Fig. 25.

We know that  $g(x_0) = y_0$  because this is the given initial condition.



**?** Can we find the equation of the tangent line at  $(x_0, y_0)$  shown in Fig. 25?

**?** It is a straight line so we can use the above  $y_1 = y_0 + m(x_1 - x_0)$  **†**. How do we find the gradient,  $m$ ?

**?** Remember the derivative gives the gradient. Hence the gradient at  $x = x_0$  is  $\frac{dy}{dx}$ . What is  $\frac{dy}{dx}$  equal to at  $x = x_0$ ?

From the given differential equation we have

$$\frac{dy}{dx} = f(x, y)$$

and at  $x = x_0$

$$\frac{dy}{dx} = f(x_0, y_0) = m \quad [\text{Gradient}]$$

That means by the equation of a tangent

$$\text{†} \quad y_1 = y_0 + m(x_1 - x_0)$$

we have

$$y_1 = y_0 + f(x_0, y_0)[x_1 - x_0] \quad [\text{Substituting for } m]$$

Thus  $y_1$  is an approximate value of the actual solution,  $g(x_1)$ .

We can repeat this procedure to find  $y_2$  and  $y_3$ , the  $y$  values at  $x_2$  and  $x_3$  respectively. Moreover we can similarly show that

$$y_2 = y_1 + f(x_1, y_1)[x_2 - x_1]$$

and

$$y_3 = y_2 + f(x_2, y_2)[x_3 - x_2]$$

**?** What would be the formula for  $y_{n+1}$ ?

$$\text{*} \quad y_{n+1} = y_n + f(x_n, y_n)[x_{n+1} - x_n]$$

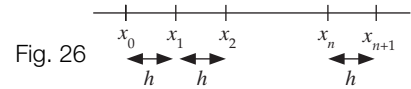
If we assume a uniform step size of  $h$  (Fig. 26)

then  $x_{n+1} - x_n = h$  and replacing this in formula **\*** we have

$$13.12 \quad y_{n+1} = y_n + h[f(x_n, y_n)]$$

or

$$y_{n+1} = y_n + hK_1 \quad \text{where } K_1 = f(x_n, y_n)$$



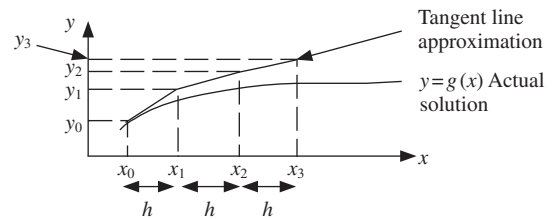
13.12 is known as Euler's numerical formula. This formula approximates the actual solution by tangent lines. By using this formula we can find an approximate solution to a differential equation. Note that

$$x_{n+1} = x_n + h \quad y_{n+1} = y_n + h[f(x_n, y_n)]$$

which means

$$(\text{new } x) = (\text{old } x) + h \quad (\text{new } y) = (\text{old } y) + h[f(\text{old } x, \text{old } y)]$$

We use the old  $x$  and  $y$  values to find the new  $x$  and  $y$  values. By repeating this procedure we obtain the collection of lines shown in Fig. 27.



**?** How can we obtain a more accurate solution?

Well, the smaller the step size,  $h$ , the better the approximation to the actual solution. Let's do an example.

### Example 15

Consider the differential equation

$$\frac{dy}{dx} = x^2 + y^2 \quad \text{with } y(0) = 1 \quad [\text{which means that when } x = 0 \text{ then } y = 1]$$

Use Euler's method to find the approximate values of  $y$  at  $x = 0.1, 0.2$  and  $0.3$  with step size  $h = 0.1$ .

Solution

Putting  $h = 0.1$  into formula 13.12 gives

$$y_{n+1} = y_n + 0.1[f(x_n, y_n)]$$

**?** What is  $f(x_n, y_n)$ ?

Remember that  $f(x_n, y_n)$  is the given function of  $x$  and  $y$ :

$$f(x_n, y_n) = x_n^2 + y_n^2$$

Substituting this into  $y_{n+1}$ :

$$* \quad y_{n+1} = y_n + 0.1(x_n^2 + y_n^2)$$

Step 1:

Starting with  $n = 0$  gives

$$y_1 = y_0 + 0.1(x_0^2 + y_0^2)$$

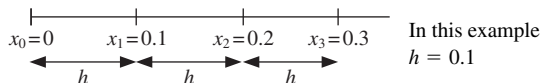
Example 15 *continued***?** What are  $x_0$  and  $y_0$ ?

These values are given by the initial condition,  $y(0) = 1$ , which means that  $x_0 = 0$  and  $y_0 = 1$ . Substituting these into  $y_1$  gives

$$\begin{aligned} y_1 &= y_0 + 0.1(x_0^2 + y_0^2) \\ &= 1 + 0.1(0^2 + 1^2) = 1.1 \end{aligned}$$

The  $y_1$  value at  $x = 0.1$  is 1.1.

Fig. 28



We need to evaluate  $y_2$  and  $y_3$ , the  $y$  values at  $x = 0.2$  and  $0.3$  respectively (Fig. 28).

*Step 2:*

**?** How do we calculate  $y_2$ ?

By substituting  $n = 1$  into  $y_{n+1} = y_n + 0.1(x_n^2 + y_n^2)$ , which gives

$$y_2 = y_1 + 0.1(x_1^2 + y_1^2)$$

**?** What are  $x_1$  and  $y_1$  equal to?

$$x_1 = 0.1 \text{ and } y_1 = 1.1 \quad [\text{Evaluated above}]$$

Substituting these into  $y_2$  gives

$$\begin{aligned} y_2 &= y_1 + 0.1(x_1^2 + y_1^2) \\ &= 1.1 + 0.1(0.1^2 + 1.1^2) = 1.222 \end{aligned}$$

*Step 3:*

Similarly, substituting  $n = 2$  into  $y_{n+1} = y_n + 0.1(x_n^2 + y_n^2)$  gives  $y_3$ :

$$y_3 = y_2 + 0.1(x_2^2 + y_2^2)$$

**?** What are  $x_2$  and  $y_2$  equal to?

$$x_2 = 0.2 \text{ and } y_2 = 1.222$$

Putting these into  $y_3$  gives

$$\begin{aligned} y_3 &= y_2 + 0.1(x_2^2 + y_2^2) \\ &= 1.222 + 0.1(0.2^2 + 1.222^2) = 1.375\ 328\ 4 \end{aligned}$$

We have  $y_1 = 1.1$ ,  $y_2 = 1.222$  and  $y_3 = 1.375\ 328\ 4$ , and these are the approximate values of  $y$  at  $x = 0.1$ ,  $0.2$  and  $0.3$  respectively.

Note that the **old** values of  $x$  and  $y$  are used to find the **new** values. For example, to find  $y_3$  we need to use  $x_2$  and  $y_2$ .

We can apply MAPLE or any other computer algebra system to find the exact solution of **Example 15** and determine the error in using Euler's method at  $y_1$ ,  $y_2$  and  $y_3$ . The commands for using MAPLE are given in the next example. If you find difficulty in following these commands, have a look at the help menu. You can copy and paste examples from the help menu and make the appropriate changes for your example.

### Example 16

By using a computer algebra system, solve the differential equation

$$\frac{dy}{dx} = x^2 + y^2 \quad \text{with } y(0) = 1$$

Determine  $y$  at  $x = 0.1$ ,  $x = 0.2$  and  $x = 0.3$ , and plot your solution. Compare your answers with  $y_1$ ,  $y_2$  and  $y_3$  respectively, obtained in **Example 15**.

Solution

By using MAPLE we have:

```

> de_1:=diff(y(x),x)=(x^2)+(y(x)^2);
de_1:=  $\frac{\partial}{\partial x}y(x) = x^2 + y(x)^2$ 

> soln:=dsolve({de_1,y(0)=1},y(x));
> evalf(subs(x=0.1,soln));
y(.1) = 1.111463377
> evalf(subs(x=0.2,soln));
y(.2) = 1.253016934
> evalf(subs(x=0.3,soln));
y(.3) = 1.439670778

> e_soln:=dsolve({de_1,y(0)=1},y(x),type=numeric,
method=classical,output=array([0.1,0.2,0.3]),stepsize=0.1);

e_soln:=  $\begin{bmatrix} [x,y(x)] \\ [.1 & 1.100000000000000008] \\ [.2 & 1.222000000000000020] \\ [.3 & 1.375328400000000034] \end{bmatrix}$ 

> with(plots):
> p_1:=plots[odeplot](e_soln,[x,y(x)],linestyle=6):
> p_2:=plot(rhs(soln),x=0.1..0.3):
> display({p_1,p_2});

```

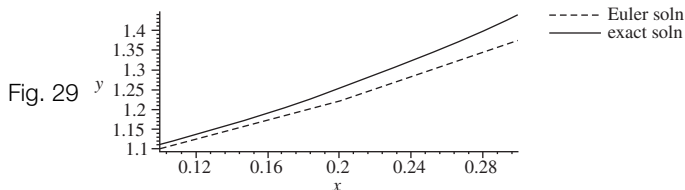


Figure 29 shows the Euler solution and the actual solution; notice the discrepancy. We explore this discrepancy by evaluating the percentage error, % error, which is defined as

$$\mathbf{13.13} \quad \% \text{ error} = \frac{(\text{actual value}) - (\text{approximate value})}{\text{actual value}} \times 100$$

The percentage error for  $y_3$  for **Example 16** is

$$\left( \frac{1.439670778 - 1.3753284}{1.439670778} \right) \times 100 = 4.47\%$$

Note that the actual value in this example is only correct to 9 decimal places.

So Euler's method gives an error of more than 4%.

**?** **How can we reduce this error?**

By considering a smaller value of  $h$ .

**?** **Are there any penalties if we consider a smaller value of  $h$ ?**

We will need more iterations to evaluate  $y_1$ ,  $y_2$  and  $y_3$ . We would need 30 iterations to evaluate  $y$  at  $x = 0.3$  if  $h = 0.01$  and  $x_0 = 0$ .

**?** **How many iterations are required for  $y$  at  $x = 0.3$  if  $h = 0.001$  and  $x_0 = 0$ ?**

300. Thus to do justice to Euler's method, or any other numerical method for differential equations, we need to use mathematical software or write a procedure in a programmable calculator. Nobody would want to perform 300 iterations by hand.

By using MAPLE with different values of  $h$  we obtain Table 1 for **Example 16**.

TABLE 1	$x$	$h = 0.1$	$h = 0.01$	$h = 0.001$	<i>Solution (correct to 9 d.p.)</i>
	0.1	1.1	1.110129764	1.111327762	1.111463377
	0.2	1.222	1.249332251	1.252641232	1.253016934
	0.3	1.3753284	1.431804205	1.438865618	1.439670778

The percentage error for  $h = 0.01$  in evaluating  $y$  at  $x = 0.3$  is

$$\left( \frac{1.439670778 - 1.431804205}{1.439670778} \right) \times 100 = 0.55\%$$

The percentage error for  $h = 0.001$  in evaluating  $y$  at  $x = 0.3$  is

$$\left( \frac{1.439670778 - 1.438865618}{1.439670778} \right) \times 100 = 0.056\%$$

Note how the error decreases as  $h$  gets smaller.

Most students are more familiar with spreadsheets than with a computer algebra system. Finding numerical solutions of differential equations is a straightforward task for spreadsheets such as EXCEL. These numerical solutions are evaluated in a worksheet document. A worksheet is an array of cells labelled A1, A2, ..., B666, etc.

We will not give detailed instructions on how to use a spreadsheet but just assume that the reader is familiar with the process.

Table 2 shows an EXCEL spreadsheet for **Example 16**:

$$\frac{dy}{dx} = x^2 + y^2 \quad \text{with } y(0) = 1$$

and  $h = 0.1$  for  $x = 0.1$  to  $x = 1$ .

TABLE 2	$x_n$	$y_n$	$K_1 (=x_n^2 + y_n^2)$	$y_{n+1}$
	0	1	1	1.1
	0.1	1.1	1.22	1.222
	0.2	1.222	1.533284	1.3753284
	0.3	1.3753284	1.98152821	1.57348122
	0.4	1.57348122	2.63584315	1.83706554
	0.5	1.83706554	3.62480978	2.19954651
	0.6	2.19954651	5.19800487	2.719347
	0.7	2.719347	7.88484811	3.50783181
	0.8	3.50783181	12.944884	4.80232022
	0.9	4.80232022	23.8722794	7.18954816
1	7.18954816	52.6896027	12.4585084	

The  $y_{n+1}$  column shows the  $y$  values by using Euler's method with the step size of 0.1.

## SUMMARY

Euler's numerical method gives an approximate solution to the first order differential equation

$$\frac{dy}{dx} = f(x, y) \quad \text{with } y(x_0) = y_0$$

This is obtained by using the formula

$$13.12 \quad y_{n+1} = y_n + h [f(x_n, y_n)]$$

The smaller the value of  $h$  the better the accuracy, but then we need more iterations.

## Exercise 13(e)

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- 1 Consider the differential equation

$$\frac{dy}{dx} = x + y^2 \quad \text{with } y(0) = 1$$

Use Euler's numerical formula with  $h = 0.1$  to determine approximate solutions of  $y$  at  $x = 0.1, 0.2$  and  $0.3$ .

- 2 Consider the initial value problem

$$\frac{dy}{dx} = x^2 + y^2 \quad \text{with } y(0) = 1$$


Use Euler's numerical formula with  $h = 0.1$  to determine the approximate solution of  $y$  at  $x = 0.5$  (work to 7 d.p.).

- 3 For the differential equation

$$\frac{dy}{dt} = t \quad \text{with } y(0) = 0$$

complete Table 3.


TABLE 3	$t$	Euler $y$ with $h = 1$	Exact $y = 0.5t^2$
	0	0	0
	1		
	2		
	3		
	4		
5			

- 4
- 
- [mechanics] The velocity,
- $v$
- , of an object falling under gravity is given by

$$\frac{dv}{dt} = 10 - 0.1v^2 \quad \text{with } v(0) = 0$$

Determine  $v(0.3)$  by using Euler's numerical formula with  $h = 0.05$  (work to 5 d.p.).

**Use a computer algebra system or a spreadsheet for the remaining questions.**

- 5
- 
- [mechanics] The velocity,
- $v$
- , of a sphere falling under gravity is given by

$$\frac{dv}{dt} = 10 - v^2 \quad \text{with } v(0) = 0$$

- Use Euler's numerical formula with  $h = 0.1$  and  $h = 0.01$  to obtain approximate values of  $v$  at  $t = 0.5, 1, 1.5$  and  $2$ .
  - Determine the solution of the given differential equation.
  - Plot, on the same axes, the graphs for  $h = 0.1, h = 0.01$  and the solution of part ii.
- 6 Consider the initial value problem

$$\frac{dy}{dx} = \sqrt{x + y} \quad \text{with } y(0) = 0$$

Complete Table 4.

TABLE 4	$x$	0	0.1	0.2	0.3	0.4	0.5
	Euler $y$ with $h = 0.1$						
	Euler $y$ with $h = 0.01$						
	Euler $y$ with $h = 0.005$						

- 7 Consider the differential equation

$$\frac{dx}{dt} = t^2 + xe^{-t} \quad \text{with } x(0.1) = -1$$

**Exercise 13(e) continued**Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- |   |  |
|---|--|
| <p><b>i</b> Determine the exact solution of the given differential equation.</p> <p><b>ii</b> Use Euler's formula with <math>h = 0.25</math> and <math>h = 0.005</math> to obtain approximate values of <math>x</math> at <math>t = 0.5, 1, 1.5, 2</math> and <math>2.5</math>.</p> | <p><b>iii</b> Plot, on the same axes, the graphs for <math>h = 0.25, h = 0.005</math> and the solution in part <b>i</b>.</p> <p><b>iv</b> Determine the percentage error in <math>x</math> for <math>h = 0.25</math> and <math>h = 0.005</math> at <math>t = 2</math>.</p> |
|---|--|

**SECTION F Improved Euler's method**

By the end of this section you will be able to:

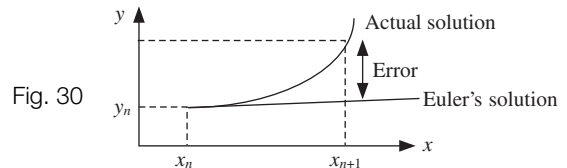
- use the improved Euler's formula
- compare errors between the improved Euler's and Euler's methods
- use a computer algebra system and a spreadsheet to evaluate and compare solutions

**F1 Improved Euler's method (Heun's method)**

Say we want to solve the general first order differential equation

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_n) = y_n \quad [\text{when } x = x_n \text{ then } y = y_n]$$

using a numerical method. With Euler's method we have the situation shown in Fig. 30.

Euler's numerical formula from **Section E** is

$$\text{13.12} \quad y_{n+1} = y_n + h[f(x_n, y_n)]$$

and at  $x = x_n$  we have  $f(x_n, y_n) = \frac{dy}{dx} = y'_n$ . This is the slope at  $x = x_n$ . Thus substituting  $f(x_n, y_n) = y'_n$  into **13.12** yields

$$\text{*} \quad y_{n+1} = y_n + hy'_n$$

Euler's method involves finding the slope,  $y'_n$ , at the beginning of the interval,  $[x_n, x_{n+1}]$ . Generally this leads to a large error unless  $h$  is very small.

**?** Where can we consider the slope in order to reduce this error?

We consider two slopes, one at the beginning and the other at the end of the interval. Then taking the average of these two slopes gives an improved estimate of the slope for that interval.



Let  $y'_n$  be the slope at  $x_n$  and  $y'_{n+1}$  be the slope at  $x_{n+1}$ . We can replace the slope,  $y'_m$  by the average slope

$$\frac{y'_n + y'_{n+1}}{2}$$

We thus have the situation shown in Fig. 31.

Replacing  $y'_n$  with  $\frac{y'_n + y'_{n+1}}{2}$  in

Euler's numerical formula, **\***, gives

$$\dagger \quad y_{n+1} = y_n + h \underbrace{\left( \frac{y'_n + y'_{n+1}}{2} \right)}_{\text{average slope}}$$

We need to replace the  $y'_{n+1}$  term on the Right-Hand Side because we can only find  $y'_{n+1}$  by using the old  $x$  and  $y$  values ( $x_n$  and  $y_n$ ). We have

$$y'_{n+1} = f(x_{n+1}, y_{n+1}) = f\left(x_{n+1}, \underbrace{y_n + hy'_n}_{\text{by } \mathbf{*}}\right)$$

Remember that  $x_{n+1} = x_n + h$ . Substituting this,  $x_{n+1} = x_n + h$ , into  $y'_{n+1}$ :

$$y'_{n+1} = f(x_n + h, y_n + hy'_n)$$

We have now established (new)  $y'_{n+1}$  in terms of the  $n$ th (old) iteration. Replacing this term in **†** and taking out a factor of  $1/2$  gives

$$\mathbf{13.14} \quad y_{n+1} = y_n + \frac{h}{2} \left[ y'_n + f(x_n + h, y_n + hy'_n) \right]$$

**13.14** is called the improved Euler's formula or Heun's formula.

This formula is complicated compared to Euler's formula of **Section E**.

### Example 17

For the differential equation

$$\frac{dy}{dx} = x^2 + y^2 \quad \text{with } y(0) = 1$$

find the approximate values of  $y$  at  $x = 0.1$  and  $x = 0.2$  by using the improved Euler's method with  $h = 0.1$  (work to 6 d.p.).

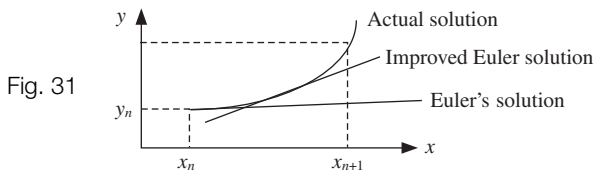


Fig. 31

Example 17 *continued*

Solution

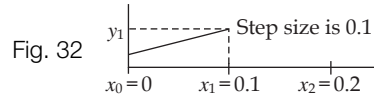
Applying formula 13.14 with  $h = 0.1$  gives

$$y_{n+1} = y_n + \frac{0.1}{2} [y'_n + f(x_n + 0.1, y_n + 0.1y'_n)]$$

$$\dagger \quad y_{n+1} = y_n + 0.05[y'_n + f(x_n + 0.1, y_n + 0.1y'_n)]$$

*Step 1:*

?

**What are we trying to find?**First we need to obtain  $y_1$ , the  $y$  value at  $x = 0.1$  (Fig. 32).Substituting  $n = 0$  into  $\dagger$  gives

$$y_1 = y_0 + 0.05[y'_0 + f(x_0 + 0.1, y_0 + 0.1y'_0)]$$

?

**What are the values of  $x_0$  and  $y_0$ ?**These are the given initial condition,  $y(0) = 1$ , which means  $x_0 = 0$  and  $y_0 = 1$ .Substituting these into  $y_1$ :

$$* \quad y_1 = 1 + 0.05[y'_0 + f(0.1, 1 + 0.1y'_0)]$$

?

**What is  $y'_0$  equal to?**

From the question we have

$$y' = \frac{dy}{dx} = x^2 + y^2 = f(x, y)$$

$$\begin{aligned} y'_0 &= x_0^2 + y_0^2 \\ &= 0^2 + 1^2 = 1 \end{aligned}$$

Substituting  $y'_0 = 1$  into  $*$  gives

$$\begin{aligned} y_1 &= 1 + 0.05[y'_0 + f(0.1, 1 + 0.1y'_0)] \\ &= 1 + 0.05[1 + f(0.1, 1 + (0.1 \times 1))] \\ &= 1 + 0.05[1 + f(0.1, 1.1)] \\ &= 1 + 0.05 \left[ 1 + \underbrace{(0.1^2 + 1.1^2)}_{\text{by using } f(x,y)=x^2+y^2} \right] = 1.111 \end{aligned}$$

*Step 2:*Next we determine  $y_2$ , the  $y$  value at  $x = 0.2$ . Substituting  $n = 1$  into  $\dagger$  gives

$$y_2 = y_1 + 0.05[y'_1 + f(x_1 + 0.1, y_1 + 0.1y'_1)]$$

?

**What are the values of  $x_1$  and  $y_1$ ?**

$$x_1 = x_0 + 0.1 = 0 + 0.1 = 0.1$$

$$y_1 = 1.111 \quad [\text{Evaluated above}]$$

Example 17 *continued*

Putting these into  $y_2 = y_1 + 0.05[y'_1 + f(x_1 + 0.1, y_1 + 0.1y'_1)]$  gives

$$** \quad y_2 = 1.111 + 0.05[y'_1 + f(0.2, 1.111 + 0.1y'_1)]$$

?

**What is the value of  $y'_1$ ?**

$$\begin{aligned} y'_1 &= x_1^2 + y_1^2 \quad [\text{Given function of } x, y] \\ &= 0.1^2 + 1.111^2 = 1.244321 \end{aligned}$$

Substituting  $y'_1 = 1.244321$  into **\*\*** gives

$$\begin{aligned} y_2 &= 1.111 + 0.05[1.244321 + f(0.2, 1.111 + (0.1 \times 1.244321))] \\ &= 1.111 + 0.05[1.244321 + f(0.2, 1.235432)] \\ &= 1.111 + 0.05[1.244321 + (0.2^2 + 1.235432^2)] \quad [\text{Using } f(x, y) = x^2 + y^2] \\ y_2 &= 1.251531 \end{aligned}$$

We have calculated the following values: at  $x = 0.1$ ,  $y = 1.111$  and at  $x = 0.2$ ,  $y = 1.251531$ .

Normally we would not work to so many significant figures but because of the effect of cumulative rounding errors, we must keep many more significant figures as guard digits.

The solutions to the differential equation of **Example 17** at  $x = 0.1$  and  $x = 0.2$  are

$$y(0.1) = 1.111463 \text{ and } y(0.2) = 1.253017$$

These were found by using MAPLE as shown in **Example 16**. The solutions are given correct to 6 d.p. Hence the percentage error at  $x = 0.2$  is

$$\left( \frac{\text{actual value} - \text{approximate value}}{\text{actual value}} \right) \times 100 = \left( \frac{1.253017 - 1.251531}{1.253017} \right) \times 100 = 0.12\%$$

The error produced by the improved method is approximately 0.12%.

?

**What is the size of the error developed by Euler's technique?**

Euler's technique gave  $y_2 = 1.222$  as evaluated in **Example 15**. The percentage error with this process at  $x = 0.2$  is

$$\left( \frac{1.253017 - 1.222}{1.253017} \right) \times 100 = 2.48\%$$

Hence the improved method gives an error of 0.12% while Euler's numerical formula produces an error of 2.48%. For this example, Euler's formula gives an error of more than 20 times the error produced by the improved Euler's method. Generally there is a substantial reduction in error by using the improved approach.

Next we use MAPLE to compare exact, Euler's and improved Euler's solutions. You could use any other mathematical software such as MATHEMATICA, MATLAB, etc. Moreover you could employ a spreadsheet such as EXCEL to find the numerical solution.

### Example 18

Consider the initial value problem

$$\frac{dy}{dx} = x^2 + y^2 \quad \text{with } y(0) = 1$$

- i** Solve this differential equation.
- ii** By employing Euler's numerical formula with  $h = 0.1$ , determine  $y$  at  $x = 0.1, 0.2, 0.3, 0.4$  and  $0.5$ .
- iii** By using the improved Euler's formula with  $h = 0.1$ , determine  $y$  at  $x = 0.1, 0.2, 0.3, 0.4$  and  $0.5$ .
- iv** Plot, on the same axes, the graphs for Euler's, the improved Euler's and the solution in part **i**.


Some commands might be difficult to follow, have a look at the help menu in MAPLE.

Solution

Using MAPLE we have

**i**

```
> de_18:=diff(y(x),x)=(y(x))^2+x^(2);
```



$$de\_18: = \frac{\partial}{\partial x} y(x) = y(x)^2 + x^2$$

```
> soln:= dsolve({de_18,y(0)=1},y(x));
```

**ii**

```
> soln_Euler:=evalf(dsolve({de_18,y(0)=1},y(x),type=numeric,
method=classical,output=array([0.1,0.2,0.3,0.4,0.5]),
stepsize=0.1));
```



$$soln\_Euler := \begin{bmatrix} [x, y(x)] \\ .1 & 1.100000000 \\ .2 & 1.222000000 \\ .3 & 1.375328400 \\ .4 & 1.573481221 \\ .5 & 1.837065536 \end{bmatrix}$$

**iii**

```
> soln_IMPROVEul:=evalf(dsolve({de_18, y(0)=1}, y(x),
type=numeric,method=classical[heunform],output=array([0.1,0.2,
0.3,0.4,0.5]),stepsize=0.1));
```



$$Soln\_IMPROVEul := \begin{bmatrix} [x, y(x)] \\ .1 & 1.111000000 \\ .2 & 1.251530674 \\ .3 & 1.436057423 \\ .4 & 1.688007332 \\ .5 & 2.048770723 \end{bmatrix}$$

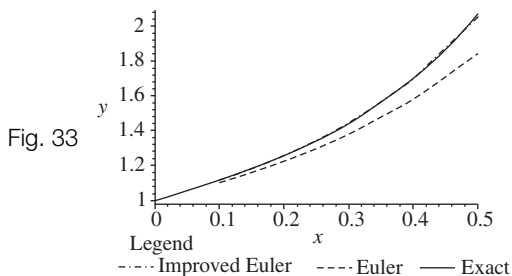
Example 18 *continued*

iv

```

> with(plots):
> p_1:=plots[odeplot](soln_Euler,[x,y(x)],linestyle=6,
  color=blue):
> p_2:=plots[odeplot](soln_IMPROVEul,[x,y(x)],linestyle=4,
  color=black):
> p_3:=plot(rhs(soln),x=0..0.5):
> display({p_1,p_2,p_3});

```



The numerical results in the graph of Fig. 33 are plotted as points which are joined by straight lines.

The improved Euler's formula is sometimes written as

$$y_{n+1} = y_n + \frac{h}{2} [K_1 + K_2]$$

where  $K_1 = f(x_n, y_n)$  and  $K_2 = f(x_n + h, y_n + hK_1)$ .

We can use a spreadsheet such as EXCEL to obtain the numerical solution of the above

example:  $\frac{dy}{dx} = x^2 + y^2$ ,  $y(0) = 1$ .

TABLE 5	$x_n$	$y_n$	$x_n+h$	$K_1$	$y_n + hK_1$	$K_2$	$y_{n+1}$
	0	1	0.1	1	1.1	1.22	1.111
	0.1	1.111	0.2	1.244321	1.2354321	1.56629247	1.25153067
	0.2	1.25153067	0.3	1.60632903	1.41216358	2.08420597	1.43605742
	0.3	1.43605742	0.4	2.15226092	1.65128352	2.88673725	1.68800733
	0.4	1.68800733	0.5	3.00936875	1.98894421	4.20589906	2.04877072
	0.5	2.04877072	0.6	4.44746147	2.49351687	6.57762638	2.60002512
	0.6	2.60002512	0.7	7.1201306	3.31203818	11.4595969	3.52901149
	0.7	3.52901149	0.8	12.9439221	4.8234037	23.9052232	5.37146876
	0.8	5.37146876	0.9	29.4926766	8.32073641	70.0446545	10.3483353
0.9	10.3483353	1	107.898044	21.1381397	447.820949	38.1342849	
1	38.1342849	1.1	1455.22369	183.656654	33730.9765	1797.44429	

The  $y_{n+1}$  column in Table 5 shows the approximate  $y$  values for  $x = 0.1, 0.2, \dots, 1.1$  using the improved Euler's method.

Look at the last figure in the  $y_{n+1}$  column, 1797.44429. This figure is too large. The reason is that there is a zero in the denominator of the analytical solution at about  $x = 0.96$ . If we plot the graph (in MAPLE) of the analytical solution between 0.9 and 1 we can see that there is a pole at  $x = 0.96$ .

Euler and other numerical methods step over the pole, so its effect could be missed. The moral is, don't believe everything your computer says. Euler's methods are intrinsically unstable. If you go far enough, the error in the Euler methods will always diverge. Runge–Kutta methods, discussed in the next section, are stable; errors are finite.

## SUMMARY

The Improved Euler's method gives an approximate solution to the first order differential equation

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_n) = y_n$$

This is attained by using the formula

$$13.14 \quad y_{n+1} = y_n + \frac{h}{2}[y'_n + f(x_n + h, y_n + hy'_n)]$$

This formula produces a substantial reduction in error compared with Euler's numerical formula, 13.12.

## Exercise 13(f)

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- 1** Consider the differential equation

$$\frac{dy}{dx} = x + y \text{ with } y(0) = 2$$

Use the improved Euler's formula with  $h = 0.1$  to obtain approximate solutions of  $y$  at  $x = 0.1$  and  $x = 0.2$ .

- 2** For the initial value problem

$$\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 1$$

approximate  $y(0.3)$  by using the improved Euler's method with  $h = 0.1$  (work to 6 d.p.).

- 3** For the differential equation

$$\frac{dy}{dx} = \ln|x + y|, \quad y(1.2) = 1$$

approximate  $y(1.6)$  by employing the improved Euler's formula with  $h = 0.2$  (work to 4 d.p.).

**Use a computer algebra system or a spreadsheet for the remaining questions.**

- 4** Consider the initial value problem

$$\frac{dx}{dt} = t^2 + xe^{-t} \text{ with } x(0.1) = -1$$

- i** Solve this differential equation.
- ii** Determine the approximate values of  $x$  at  $t = 0.5, 1, 1.5, 2$  and  $2.5$  by using the Euler's numerical formula with  $h = 0.25$ .
- iii** Determine the approximate values of the  $x$  at  $t = 0.5, 1, 1.5, 2$  and  $2.5$  by using the improved Euler's formula with  $h = 0.25$ .
- iv** Plot, on the same axes, the graphs of Euler's, the improved Euler's and the exact solutions.

## Exercise 13(f) continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

5 Consider the initial value problem

$$\frac{dy}{dx} = x^3 + y^2, \quad y(0) = -1$$

- i Solve this differential equation.
- ii Determine the approximate values of  $y$  at  $x = 0.5, 1, 1.5$  and  $2$  by using Euler's numerical formula with  $h = 0.01$ .
- iii Determine the approximate values of  $y$  at  $x = 0.5, 1, 1.5$  and  $2$  by using the improved Euler's formula with  $h = 0.01$ .
- iv Plot, on the same axes, the graphs of Euler's, the improved Euler's and the solution of part i.

6 Consider the initial value problem

$$\frac{dx}{dt} = 10 - x^2 \text{ with } x(0) = 0$$

- i Solve this differential equation.
- ii Determine the approximate values of  $x$  at  $t = 0.1, 0.2, 0.3, 0.4$  and  $0.5$  by using Euler's numerical formula with  $h = 0.09$ .
- iii Determine the approximate values of  $x$  at  $t = 0.1, 0.2, 0.3, 0.4$  and  $0.5$  by using the improved Euler's formula with  $h = 0.09$ .
- iv Plot, on the same axes, the graphs of Euler's, the improved Euler's and the exact solutions.
- v Determine the percentage error at  $t = 0.5$  for Euler's and the improved Euler's methods. What do you notice?

## SECTION G Fourth order Runge–Kutta

By the end of this section you will be able to:

- apply the 4th order Runge–Kutta method
- use a computer algebra system to find the Runge–Kutta solutions
- compare errors between Euler's, the improved Euler's and the Runge–Kutta methods

We discussed Euler's and the improved Euler's method in previous sections. In this part we examine another numerical method called 4th order Runge–Kutta. 1st, 2nd and 3rd order Runge–Kutta methods also exist but the 4th order is most commonly used and is one of the most accurate numerical methods.

**Carle Runge** was born in Germany in 1856. He studied mathematics at the University of Munich. He obtained a chair at Hannover and 18 years later he was offered a chair of Applied Mathematics at the University of Göttingen.

**Martin Kutta** was born in Poland in 1867 and also studied at the University of Munich. He became professor at Stuttgart in 1911.

The Runge–Kutta method was developed in 1901.

## G1 Fourth order Runge–Kutta

The 4th order Runge–Kutta method is used to determine approximate solutions to a first order differential equation. Consider the **initial value problem**:

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_n) = y_n$$

Sometimes a differential equation with initial conditions is called an initial value problem.

The 4th order Runge–Kutta method is defined as

$$\mathbf{13.15} \quad y_{n+1} = y_n + \frac{h}{6} [K_1 + 2(K_2 + K_3) + K_4]$$

where

$$K_1 = f(x_n, y_n)$$

$$K_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hK_1\right)$$

$$K_3 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hK_2\right)$$

$$K_4 = f(x_n + h, y_n + hK_3)$$

### Example 19

Consider the differential equation

$$\frac{dy}{dx} = x^2 + y^2 \text{ with } y(0) = 1$$

Use the 4th order Runge–Kutta method to determine the approximate values of  $y$  at  $x = 0.1$  and  $x = 0.2$  with  $h = 0.1$  (work to 6 d.p.).

**Solution**

We need to ascertain  $y_1$  and  $y_2$ , the  $y$  values at  $x = 0.1$  and  $x = 0.2$  respectively. Substituting  $n = 0$ ,  $h = 0.1$  into formula **13.15** produces

$$\ast \quad y_1 = y_0 + \frac{0.1}{6} [K_1 + 2(K_2 + K_3) + K_4]$$

For  $n = 0$  we have  $K_1 = f(x_0, y_0)$ .

*Step 1:*



**What are the values of  $x_0$  and  $y_0$ ?**

These are given by the initial condition,  $y(0) = 1$ , which means  $x_0 = 0$  and  $y_0 = 1$ . Putting these into  $K_1 = f(x_0, y_0)$  gives

$$\dagger \quad K_1 = f(0, 1)$$



**What does  $f$  represent?**

This is the function of  $x$  and  $y$  on the Right-Hand Side of the given differential equation:

$$f(x, y) = x^2 + y^2$$

$$f(0, 1) = 0^2 + 1^2 = 1$$



Example 19 *continued*

It follows from **†** that  $K_1 = 1$ . We need to find  $K_2$ . Putting  $h = 0.1$ ,  $n = 0$  and  $K_1 = 1$  into

$$K_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hK_1\right) \text{ gives}$$

$$\begin{aligned} K_2 &= f\left(x_0 + \frac{1}{2}(0.1), y_0 + \frac{1}{2}[0.1 \times 1]\right) \\ &= f(0 + 0.05, 1 + 0.05) \quad [\text{Substituting } x_0 = 0 \text{ and } y_0 = 1] \\ &= f(0.05, 1.05) \\ K_2 &= 0.05^2 + 1.05^2 = 1.105 \end{aligned}$$

**?** What else do we need to find to use **\***?

$K_3$  and  $K_4$ . Evaluating  $K_3$  is similar to finding  $K_2$ . Substituting  $h = 0.1$ ,  $n = 0$  and  $K_2 = 1.105$  into  $K_3 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hK_2\right)$  gives

$$\begin{aligned} K_3 &= f\left[x_0 + \frac{1}{2}(0.1), y_0 + \frac{1}{2}(0.1 \times 1.105)\right] \\ &= f(0 + 0.05, 1 + 0.05525) \quad [\text{Substituting } x_0 = 0 \text{ and } y_0 = 1] \\ &= f(0.05, 1.05525) \\ K_3 &= 0.05^2 + 1.05525^2 = 1.116053 \end{aligned}$$

**?** How do we calculate  $K_4$ ?

By placing  $n = 0$ ,  $h = 0.1$  and  $K_3 = 1.116053$  into  $K_4 = f(x_n + h, y_n + hK_3)$ :

$$\begin{aligned} K_4 &= f(x_0 + 0.1, y_0 + [0.1 \times 1.116053]) \\ &= f(0 + 0.1, 1 + 0.111605) \\ &= f(0.1, 1.111605) \\ K_4 &= 0.1^2 + 1.111605^2 = 1.245666 \end{aligned}$$

Substituting the calculated values,  $y_0 = 1$ ,  $K_1 = 1$ ,  $K_2 = 1.105$ ,  $K_3 = 1.116053$  and  $K_4 = 1.245666$ , into **\*** gives

$$\begin{aligned} y_1 &= y_0 + \frac{0.1}{6} [K_1 + 2(K_2 + K_3) + K_4] \\ &= 1 + \frac{0.1}{6} [1 + 2(1.105 + 1.116053) + 1.245666] \\ &= 1.111463 \end{aligned}$$

Step 2:

Similarly we evaluate  $y_2$ . Putting  $n = 1$  and  $h = 0.1$  into **13.15** results in

$$\mathbf{**} \quad y_2 = y_1 + \frac{0.1}{6} [K_1 + 2(K_2 + K_3) + K_4]$$

**?** What is  $K_1$  equal to in this case?

$$K_1 = f(x_1, y_1)$$

Example 19 *continued***What are the values of  $x_1$  and  $y_1$ ?**

$$x_1 = 0.1 \text{ and } y_1 = 1.111463 \quad [\text{Evaluated above}]$$

Substituting these into  $K_1 = f(x_1, y_1)$  yields

$$\begin{aligned} K_1 &= f(0.1, 1.111463) \\ &= 0.1^2 + 1.111463^2 = 1.245350 \end{aligned}$$

We need to determine  $K_2$  next. Putting  $n = 1$ ,  $h = 0.1$  and  $K_1 = 1.245350$  into

$$\begin{aligned} K_2 &= f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hK_1\right) \text{ gives} \\ K_2 &= f\left[x_1 + \frac{1}{2}(0.1), y_1 + \frac{1}{2}(0.1 \times 1.245350)\right] \\ &= f(0.1 + 0.05, 1.111463 + 0.062268) \quad [\text{Substituting } x_1 = 0.1, y_1 = 1.111463] \\ &= f(0.15, 1.173731) \\ K_2 &= 0.15^2 + 1.173731^2 = 1.400144 \end{aligned}$$

Similarly by the definition of  $K_3$  we have

$$\begin{aligned} K_3 &= f\left[x_1 + \frac{1}{2}(0.1), y_1 + \frac{1}{2}(0.1 \times 1.400144)\right] \\ &= f(0.1 + 0.05, 1.111463 + 0.070007) \\ &= f(0.15, 1.181470) \\ K_3 &= 0.15^2 + 1.181470^2 = 1.418371 \end{aligned}$$

Next we determine  $K_4$ :

$$\begin{aligned} K_4 &= f[x_1 + 0.1, y_1 + (0.1 \times 1.418371)] \\ &= f(0.1 + 0.1, 1.111463 + 0.141837) \\ &= f(0.2, 1.2533) \\ K_4 &= 0.2^2 + 1.2533^2 = 1.610761 \end{aligned}$$

Substituting  $y_1 = 1.111463$ ,  $K_1 = 1.245350$ ,  $K_2 = 1.400144$ ,  $K_3 = 1.418371$  and  $K_4 = 1.610761$  into **\*\*** gives

$$\begin{aligned} y_2 &= 1.111463 + \frac{0.1}{6} [1.245350 + 2(1.400144 + 1.418371) + 1.610761] \\ &= 1.253015 \end{aligned}$$

The calculated values are  $x = 0.1$ ,  $y = 1.111463$  and at  $x = 0.2$ ,  $y = 1.253015$ .

Next we compare the errors for the differential equation of **Example 19** between Euler's, the improved Euler's and the 4th order Runge–Kutta solutions.

The  $y$  values at  $x = 0.2$  for Euler's, the improved Euler's and the 4th order Runge–Kutta solutions are 1.222, 1.251531 and 1.253015 respectively. The actual value (correct to 6 d.p.)

is 1.253017 as evaluated in **Example 16**. The Runge–Kutta value might well be the same as, or closer to, the actual value if we had performed the calculations to more decimal places. However, notice how close the Runge–Kutta approximation is to the actual value compared to Euler’s and the improved Euler’s.

We can use MAPLE to plot solutions of Euler’s, the improved Euler’s and the 4th order Runge–Kutta methods, as the next example shows.

### Example 20


Consider the differential equation

$$\frac{dy}{dx} = x^2 + y^2 \text{ with } y(0) = 1$$

- i** Solve this differential equation.
- ii** Find  $y$  at  $x = 0.2, 0.4, 0.6$  and  $0.8$  by using Euler’s method with  $h = 0.1$ .
- iii** Repeat **ii** for the improved Euler’s method with  $h = 0.1$ .
- iv** Repeat **ii** for the 4th order Runge–Kutta method with  $h = 0.1$ .
- v** Plot, on the same axes, the graphs of Euler’s, the improved Euler’s and the 4th order Runge–Kutta solutions, and the solution of part **i**.  
What do you notice about your graphs?


Solution

Using Maple we have


- i** 

```
> de_20:=diff(y(x),x)=(x^2)+(y(x)^2);
```

$$de\_20: = \frac{\partial}{\partial x} y(x) = x^2 + y(x)^2$$

```
> soln:=dsolve({de_20,y(0)=1},y(x));
```
- ii** 

```
> e_soln:=dsolve({de_20,y(0)=1}, y(x), type=numeric, method=
classical, output=array([0.2, 0.4, 0.6, 0.8]), stepsize=0.1);
```

$$e\_soln: = \begin{bmatrix} [x, y(x)] \\ [.2 & 1.22200000000000020] \\ [.4 & 1.57348122078465646] \\ [.6 & 2.19954651435706426] \\ [.8 & 3.50783181255390319] \end{bmatrix}$$
- iii** 

```
> improve_soln:=dsolve({de_20, y(0)=1},y(x),type=numeric,
method=classical [heunform], output=array([.2,.4,.6,.8]),
stepsize=0.1);
```

$$improve\_soln: = \begin{bmatrix} [x, y(x)] \\ [.2 & 1.25153067368552052] \\ [.4 & 1.68800733199093366] \\ [.6 & 2.60002511535832781] \\ [.8 & 5.37146875547270320] \end{bmatrix}$$

Example 20 *continued*

**iv**

```
> rk_soln:=dsolve({de_20,y(0)=1},y(x),type=numeric,
method=classical[rk4], output=array([.2,.4,.6,.8]),
stepsize=0.1);
```



$$rk\_soln := \begin{bmatrix} [x, y(x)] \\ [.2 & 1.25301517460353452] \\ [.4 & 1.69609790372281698] \\ [.6 & 2.64386019707933162] \\ [.8 & 5.84201333527994748] \end{bmatrix}$$

**v**

```
> with(plots):p_1:=plots[odeplot](e_soln,[x,y(x)],
linestyle=4,color=black):
> p_2:=plots[odeplot](improve_soln,[x,y(x)],linestyle=2,
color=blue):
> p_3:=plots[odeplot](rk_soln,[x,y(x)],linestyle=6,color=coral):
> p_4:=plot(rhs(soln),x=0..0.8):
> display({p_1,p_2,p_3,p_4});
```

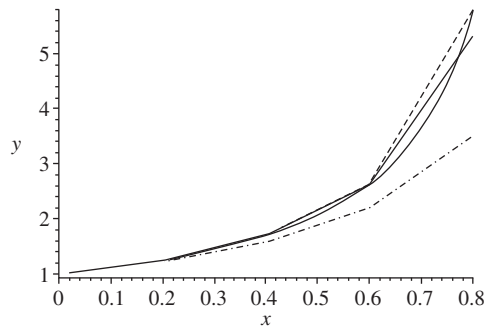


Fig. 34

Legend

- Euler solution
- Improved Euler solution
- Runge-Kutta solution
- Exact solution

The Runge–Kutta solution seems to be the closest to the actual solution. The improved Euler’s is a close second and Euler’s solution increases in error as you move away from 0. Remember that the numerical results are plotted as points. So when comparing exact and numerical solutions, we should ignore the portions between the kinks in the lines.

We can also find the Runge–Kutta solution of differential equations by using a spreadsheet. Try the above example on the type of spreadsheet available to you.

**SUMMARY**

The fourth order Runge–Kutta solution of the differential equation

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_n) = y_n$$

is given by

$$13.5 \quad y_{n+1} = y_n + \frac{h}{6} [K_1 + 2(K_2 + K_3) + K_4]$$

where

$$K_1 = f(x_n, y_n)$$

$$K_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hK_1\right)$$

$$K_3 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hK_2\right)$$

$$K_4 = f(x_n + h, y_n + hK_3)$$

**Exercise 13(g)**

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- 1 Consider the differential equation

$$\frac{dy}{dx} = x + y \text{ with } y(0) = 2$$

Use the 4th order Runge–Kutta method with  $h = 0.1$  to obtain approximate solutions of  $y$  at  $x = 0.1$  and  $x = 0.2$  (work to 6 d.p.).

- 2 For the initial value problem


$$\frac{dy}{dx} = \sqrt{x + y}, \quad y(0) = 0$$

find  $y(0.4)$  using the 4th order Runge–Kutta formula with  $h = 0.2$  (work to 4 d.p.).

- 3 Consider the differential equation

$$\frac{dy}{dx} = \ln(x + y) \text{ with } y(1.2) = 1$$

Approximate  $y(1.6)$  by employing the 4th order Runge–Kutta formula with  $h = 0.2$  (work to 4 d.p.).

- 4  [mechanics] The velocity,  $v$ , of an object dropped in air is given by

$$\frac{dv}{dt} = 9.8 - \frac{v^2}{10} \text{ with } v(0) = 0$$

Apply the 4th order Runge–Kutta method with  $h = 1$  to find an approximate  $v$  at  $t = 2$  s (work to 4 d.p.).

**Use a computer algebra system for the remaining questions.**

- 5 Consider the differential equation

$$\frac{dy}{dx} = x^4 + y^2 \text{ with } y(0) = 1$$

- i Solve this differential equation.
- ii Determine approximate values of  $y$  at  $x = 0.2, 0.4, 0.6$  and  $0.8$  by using Euler's method with  $h = 0.05$ .
- iii Determine approximate values of  $y$  at  $x = 0.2, 0.4, 0.6$  and  $0.8$  by using the 4th order Runge–Kutta method with  $h = 0.05$ .
- iv Plot, on the same axes, the graphs of Euler's, Runge–Kutta and the exact solutions.

- 6 Consider the differential equation

$$\frac{dy}{dx} = e^y \sin(x), \quad y(0) = -1$$

## Exercise 13(g) continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- i** Solve this differential equation.  
**ii** Determine approximate values of  $y$  at  $x = 1, 2, 3$  and  $4$  by using the improved Euler's method with  $h = 0.2$ .  
**iii** Determine the approximate values of  $y$  at  $x = 1, 2, 3$  and  $4$  by using the 4th order Runge–Kutta method with  $h = 0.2$ .  
**iv** Plot, on the same axes, the graphs of the improved Euler's, the Runge–Kutta and the exact solutions.

## Examination questions 13

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- 1** Determine the solution of the differential equations

**i**  $y \frac{dy}{dx} = 3x^2 \quad y(0) = 1$

**ii**  $\frac{dy}{dx} - y = e^{2x} \quad y(0) = 0$

Loughborough University, UK, 2009

- 2** The velocity  $v$  (m/s) of an object initially at rest (that is,  $v = 0$  at  $t = 0$ ) is given by

$$150 \frac{dv}{dt} = 350 - 15v$$

- a** Find an expression for  $v$  in terms of time  $t$ .  
**b** Find the value of  $t$  when  $v = 2$  m/s.

University of Portsmouth, UK, 2009

- 3** Dr Gourley likes to drink a cup of coffee before his 11 a.m. lecture to the engineering students. The coffee is at 200°F when poured at 10.15 a.m., and 15 minutes later it cools to 120°F in a room in which the temperature is 70°F. Dr Gourley will not drink the coffee until it has cooled to 90°F. At what time (to the nearest minute) will he start drinking it?

[Assume that the temperature  $T$  of the coffee satisfies  $dT/dt = -k(T - 70)$  for some number  $k$ .]

University of Surrey, UK, 2002

- 4 i** Use the integrating factor method to find the solution of the linear differential equation  $\frac{dy}{dx} + \frac{y}{x} = x$  for which  $y = 3$  at  $x = 1$ .

- \*ii** Use the substitution  $y = vx$  to find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x - y}$$

University of Manchester, UK, 2008

- 5** Solve the problem  $y' = e^{x+2y}$ .

University of California, Berkeley, USA 2005

- 6** Find  $f(x)$  if  $f'(x) = \frac{2x + \sqrt{x}}{x}$ ,  $f(1) = 3$ .

Arizona State University, USA (Review)

- 7** Use Euler's method with step size 0.5 to estimate  $y(3)$ , where  $y(x)$  is the solution of the initial value problem:  $y' = y - 2x$ ,  $y(1) = 0$ .

North Carolina State University, USA, 2010

- 8** Show that if  $y = \ln(\cos x)$ , then  $\frac{dy}{dx} = -\tan(x)$ . Use this result to find an integrating factor of the differential equation

$$\frac{dy}{dx} + 2y \tan x = \sin x$$

Hence solve the differential equation, given that  $y(\pi/3) = 0$ .

University of Sussex, UK, 2007

Examination questions **13 continued**Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

- 9**
- Solve the initial value problem

$$\frac{dy}{dt} - 2 \tan(t)y = \tan(t), y\left(\frac{\pi}{4}\right) = 5.$$

**Memorial University of  
Newfoundland, Canada, 2010**

- 10**
- Find all real solutions to the differential equation
- $x \frac{dy}{dx} + 2y = \sin x$
- .

**Princeton University, USA, 2006**

- 11**
- The population of fish in a lake is
- $m$
- million, where
- $m = m(t)$
- varies with time
- $t$
- (in years). The number of fish is currently 2 million.

- a**
- Suppose
- $m$
- satisfies the logistic-growth equation

$$\frac{dm}{dt} = 16m\left(1 - \frac{m}{4}\right)$$

When will the number of fish equal 3 million? You may use the fact that the general solution to the logistic-growth differential equation  $y' = ky[1 - (y/K)]$  is,  $y = K/(1 + Ae^{-kt})$  where  $A$  is a constant.

- b**
- Suppose instead that (because of fishing by humans)
- $m$
- satisfies

$$\frac{dm}{dt} = 16m\left(1 - \frac{m}{4}\right) - 12$$

Will the population ever equal 3 million? You must give justification for your answer.

**University of British Columbia,  
Canada, 2007**

- 12**
- Torricelli's law states that


$$A(y) \frac{dy}{dt} = -a\sqrt{2gy}$$

where  $y$  is the depth of a fluid in a tank at time  $t$ ,  $A(y)$  is the cross-sectional area of the tank at height  $y$  above the exit hole,  $a$  is the cross-sectional area of the exit hole, and  $g = 9.8 \text{ m/s}^2$  is the acceleration due to gravity.

A container in the form of an inverted right circular cone of radius 1 m and height 3 m is full of water. A circular plug of radius 1 cm is pulled open at the bottom of the container. How long will it take for the container to become completely empty of all water?

**University of Toronto, Canada, 2001**

Miscellaneous exercise **13**Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)


- 1**
- 
- [thermodynamics] In thermodynamics the relationship between mass,
- $m$
- , volume,
- $V$
- , and pressure,
- $P$
- , is given by

$$mCd(pV) = pVdm$$

where  $C$  is a constant. Show that

$$m = (pV)^C$$


(Assume the constant of integration to be zero.)

- 2**
- 
- [mechanics] The acceleration,
- $a$
- , of a vehicle is given by

$$a = v^2$$

where  $v$  is the velocity. Find an

expression for  $v$  given that  $a = \frac{dv}{dt}$  and when  $t = 0$ ,  $v = 10$ .

- 3**
- 
- [mechanics] The acceleration,
- $a = \frac{dv}{dt}$
- , of a vehicle is given by


$$a = -kv^2$$

where  $k$  is a positive constant and  $v$  is the velocity. Given that when  $t = 0$ ,  $v = 0.1$ , determine  $v$  as a function of time  $t$ .

## Miscellaneous exercise 13 continued


Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

By using a graphical calculator or a computer algebra system, plot graphs of  $v$  against  $t$  for  $k = 0.1, 1$  and  $10$  ( $0 \leq t \leq 10$ ). What effect does  $k$  have?

- 4  [mechanics] The velocity,  $v$ , of a vehicle during the application of brakes is given by

$$\frac{dv}{dt} = -kv$$


where  $t$  is time and  $k$  is a constant. If the velocity at time  $t = 0$  is  $V_0$ , show that  $v = V_0 e^{-kt}$ .

- 5  [thermodynamics] The  $PV$  relationship of an ideal gas is given by

$$\frac{dP}{dV} = -\frac{kP}{V}$$

where  $P$  is pressure,  $V$  is volume and  $k$  is a constant. Show that


$$PV^k = \text{constant}$$

- 6  [thermodynamics] The  $PV$  relationship of an ideal gas is given by

$$\frac{dP}{P} + k\frac{dV}{V} = 0$$

where  $P$  is pressure,  $V$  is volume and  $k$  is a constant. Show that


$$PV^k = \text{constant}$$

- 7  [electrical principles] The voltage,  $v$ , across an inductor of inductance  $L$ , is given by

$$v = L\frac{di}{dt}$$

where  $i$  is the current through the inductor. For  $v = 240\sin(100t)$ ,  $L = 1 \times 10^{-3}$  H and the initial condition  $t = 0, i = 0$ , show that


$$i = 2400[1 - \cos(100t)]$$

- 8  [electrical principles] A circuit consists of a resistor of resistance  $R$ , and a capacitor of capacitance  $C$ , in series, and is described by the differential equation

$$RC\frac{dV}{dt} + V = E$$

where  $E$  is the constant applied e.m.f. and  $V$  is the voltage across the capacitor. Show that, if  $V(0) = 0$ , then

$$V = E(1 - e^{-t/RC})$$

-  Questions 9 to 11 are in the field of [mechanics].

- 9 The velocity,  $v$ , of an object initially at rest ( $t = 0, v = 0$ ) is given by

$$100\frac{dv}{dt} = 200 - 10v$$

- Find an expression for  $v$  in terms of  $t$ .
- Find  $v$  for large  $t$  ( $t \rightarrow \infty$ ). (This  $v$  is called the terminal velocity.)
- Sketch the graph of  $v$  against  $t$  for  $t \geq 0$ .

- \*10 An object of mass,  $m$ , falls from rest ( $t = 0, v = 0$ ) and its velocity  $v$  is given by

$$m\frac{dv}{dt} = mg - kv \quad \left(g > \frac{k}{m} v\right)$$

where  $k$  is a positive constant. Show that

$$v = \frac{mg}{k} (1 - e^{-kt/m})$$

Determine the (terminal) velocity as  $t \rightarrow \infty$ .

- \*11 By applying Newton's law to a parachutist falling from rest, we obtain the first order differential equation

$$m\frac{dv}{dt} = mg - kv^2 \quad \left(v > \sqrt{\frac{mg}{k}}\right)$$

where  $m$  is mass,  $v$  is velocity,  $t$  is time,  $g$  is acceleration due to gravity and  $k$  is a constant. Show that




Miscellaneous exercise **13** continuedSolutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

$$v = \sqrt{\frac{mg}{k} \left( \frac{1 + Ae^{-2\beta t}}{1 - Ae^{-2\beta t}} \right)}$$

where  $\beta = \sqrt{\frac{gk}{m}}$  and  $A$  is a constant.

What is the value of the terminal velocity?

$$\left[ \text{Hint: Use } \int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| \right]$$

 **Questions 12 to 14 are in the field of [electrical principles].**

- 12** The current,  $i$ , through a series  $RL$  circuit is given by

$$L \frac{di}{dt} + Ri = t$$

Show that, if  $t = 0$ ,  $i = 0$ , then

$$i = \frac{t}{R} + \frac{L}{R^2} \left( e^{-\frac{R}{L}t} - 1 \right)$$

**13**

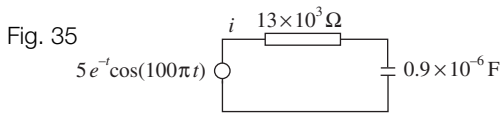


Figure 35 shows a voltage,  $5e^{-t}\cos(100\pi t)$ , applied to a circuit with a capacitance,  $(0.9 \times 10^{-6})$  F, in series with a resistance,  $(13 \times 10^3)$   $\Omega$ . Show that

$$\frac{dv}{dt} = 427.35e^{-t}\cos(100\pi t) - 85.47v$$

where  $v$  is the voltage across the capacitor.

- \*14** By applying Kirchhoff's law to a circuit we obtain the differential equation

$$L \frac{di}{dt} + Ri = E \cos(\omega t)$$

where  $L$  is inductance,  $R$  is resistance,  $E$  is e.m.f.,  $i$  is current,  $t$  is time and  $\omega$  is angular frequency.

For the initial condition, when  $t = 0$ ,  $i = 0$ , show that

$$i = \frac{E}{R^2 + \omega^2 L^2} \left[ R \cos(\omega t) + \omega L \sin(\omega t) - R e^{-Rt/L} \right]$$

**Use a computer algebra system for the remaining questions.**

- 15** Consider the differential equation

$$\frac{dy}{dx} = x^2 + y^2 \text{ with } y(0.5) = 0$$


- Solve this differential equation.
- Determine approximate values of  $y$  at  $x = 1, 1.2, 1.4, 1.6$  and  $1.8$  by using Euler's method with  $h = 0.1$ .
- Determine approximate values of  $y$  at  $x = 1, 1.2, 1.4, 1.6$  and  $1.8$  by using the 4th order Runge–Kutta method with  $h = 0.1$ .
- Plot, on the same axes, the graphs of Euler's, Runge–Kutta and the exact solutions.

What do you notice about your results?

- 16** Consider the differential equation

$$\frac{dy}{dx} = x^2 + ye^x, \quad y(0) = -1$$

- Solve this differential equation.
- Determine approximate values of  $y$  at  $x = 0.2, 0.4, 0.6$  and  $1.2$  by using the improved Euler's method with  $h = 0.2$ .
- Determine approximate values of  $y$  at  $x = 0.2, 0.4, 0.6$  and  $1.2$  by using the 4th order Runge–Kutta method with  $h = 0.2$ .
- Plot, on the same axes, the graphs of the improved Euler's, the Runge–Kutta and the exact solutions.

- 17**  [electrical principles] Applying Kirchhoff's law to a circuit we obtain the differential equation

## Miscellaneous exercise 13 continued

Solutions at end of book. Complete solutions available at [www.palgrave.com/engineering/singh](http://www.palgrave.com/engineering/singh)

$$(2 \times 10^{-3}) \frac{di}{dt} + (1 \times 10^3)i = 10$$

where  $i$  is the current through the circuit. Given the initial condition  $i(0) = 0$ :

- i determine the current  $i$
- ii plot the graph of  $i$  against  $t$  between  $t = 0$  and  $t = 10 \mu\text{s}$  ( $\mu = 10^{-6}$ ).

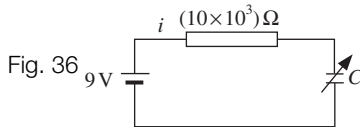
18  [electrical principles]

Figure 36 shows a variable capacitor of capacitance  $C$ , connected in series with a resistor of resistance  $10 \text{ k}\Omega$  and a  $9 \text{ V}$  supply. The differential equation for this circuit is

$$(10 \times 10^3)C \frac{dv}{dt} + v = 9$$

where  $v$  is the voltage across the capacitance,  $C$ .

- i Solve this differential equation for the initial condition,  $t = 0$ ,  $v = 0$ .
- ii Substitute the following values of  $C$  into your solution:  
 $C = 10 \times 10^{-6}$ ,  $20 \times 10^{-6}$ ,  $30 \times 10^{-6}$ ,  
 $40 \times 10^{-6}$  and  $50 \times 10^{-6}$
- iii Plot, on the same axes,  $v$  against  $t$  for the above values of  $C$  and  $0 \leq t \leq 1.5$ .
- iv What value does the voltage,  $v$ , tend to as  $t \rightarrow \infty$ ? (This is called the 'final value'.)

- v What effect does the change in capacitance,  $C$ , have on the voltage,  $v$ , across it?

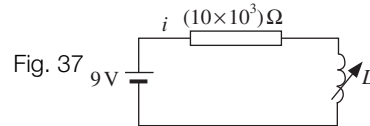
19  [electrical principles]

Figure 37 shows a variable inductor of inductance  $L$ , connected in series with a resistor of value  $10 \text{ k}\Omega$  ( $10 \times 10^3$ ) and an applied voltage of  $9 \text{ V}$ . The differential equation for this circuit is

$$(10 \times 10^3)i + L \frac{di}{dt} = 9$$

- i Solve this differential equation for the initial condition,  $t = 0$ ,  $i = 0$ .
- ii Substitute the following values of  $L$  into your solution:

$$L = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \text{ and } \frac{1}{32}$$

- iii Plot, on the same axes,  $i$  against  $t$  for the values of  $L$  in part ii and  $0 \leq t \leq 0.0001$ .
- iv What value does the current,  $i$ , tend to as  $t \rightarrow \infty$ ? (This is called the 'final value'.)
- v What effect does increasing the inductance,  $L$ , have on the current,  $i$ ?

# Second Order Linear Differential Equations

SECTION A **Homogeneous differential equations** 731

- ▶ State the relevant general solution to a homogeneous differential equation by examining the characteristic equation

SECTION B **Engineering applications** 738

- ▶ Solve engineering problems involving second order differential equations with initial conditions
- ▶ Solve boundary value problems

SECTION C **Non-homogeneous (inhomogeneous) differential equations** 746

- ▶ Understand the term non-homogeneous
- ▶ Use a trial function to find a particular integral
- ▶ Find general solutions of non-homogeneous second order differential equations

SECTION D **Particular solutions** 759

- ▶ Find particular solutions of non-homogeneous second order differential equations

SECTION A **Homogeneous differential equations**

By the end of this section you will be able to:

- understand the term *homogeneous second order differential equation*
- state the relevant solution to second order differential equations by examining solutions of the corresponding characteristic equation

A1 **Introduction**

Second order differential equations are frequently found in many areas of engineering and science, such as vibrations, deflection of beams and the electrical quantities within *RLC* circuits.

A commonplace application is that of calculating an object's acceleration  $a$  in terms of its displacement  $x$ :

$$a = \frac{d^2x}{dt^2} \text{ where } t \text{ is time}$$

A demonstration of this is given in the following example.

Example 1 *mechanics*

Consider a deep well as shown in Fig. 1. A stone is dropped down the well and a splash is heard after **three** seconds. The acceleration  $a = \frac{d^2x}{dt^2}$  of the stone is  $9.8 \text{ m/s}^2$ .

Determine the depth of the well. Assume at  $t = 0$ , the velocity  $\frac{dx}{dt} = 0$  and depth  $x = 0$ .

Solution

We are given that  $\frac{d^2x}{dt^2} = 9.8$  and we need to find a formula for depth  $x$ . **How?**

Integrate  $\frac{d^2x}{dt^2} = 9.8$  twice. Integrating this once gives

$$\dagger \quad \frac{dx}{dt} = \int 9.8 \, dt = 9.8t + C$$

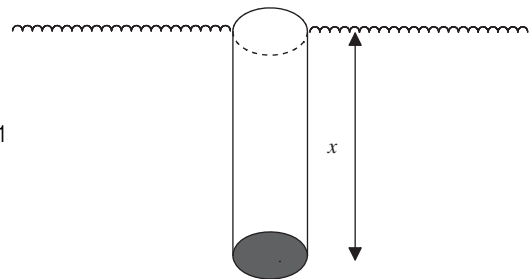


Fig. 1



Example 1 *continued*

We are given that at  $t=0$  the velocity  $\frac{dx}{dt}=0$  therefore we can find the constant  $C$ . **How?**

By substituting this condition  $\frac{dx}{dt}=0$  at  $t=0$  into  $\text{†}$  :

$$(9.8 \times 0) + C = 0 \text{ implies that } C = 0$$

We have  $\frac{dx}{dt} = 9.8t$ . Integrating this again gives

$$\begin{aligned} x &= \int 9.8t \, dt = \frac{9.8t^2}{2} + D \\ &= 4.9t^2 + D \end{aligned}$$



$D$  is the constant of integration. **How can we find  $D$ ?**

By substituting the other given condition, that at  $t = 0$ ,  $x = 0$ :

$$(4.9 \times 0^2) + D = 0 \text{ gives } D = 0$$

Hence  $x = 4.9t^2$ . The splash is heard after 3 seconds therefore we substitute  $t = 3$  into  $x = 4.9t^2$  to find the depth of the well:

$$x = 4.9t^2 = 4.9 \times 3^2 = 44.1$$

Thus the depth of the well is 44.1 m.

$\frac{d^2x}{dt^2} = 9.8$  is an example of a **second order** differential equation. Second order differential equations may also contain the variables  $\frac{dx}{dt}$  and  $x$ , as we discuss in the next subsection.

In solving a second order differential equation we obtain two arbitrary constants such as  $C$  and  $D$  as seen in **Example 1** above. We can calculate values for these constants if we have initial conditions.

## A2 Homogeneous equations

Generally an equation of the form:

$$\text{14.1} \quad a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \quad (a \neq 0)$$

is called a second order differential equation (2nd order DE), where  $a$ ,  $b$  and  $c$  are constants and  $f(x)$  is a function of  $x$ . Note the presence of  $\frac{dy}{dx}$  and  $y$  in the general case. Also  $y$  is a



function of the independent variable,  $x$ . **What does the term 'second order' mean?**

The highest order of the derivative in the differential equation is 2. If  $f(x) = 0$  then **14.1** becomes

$$\text{14.2} \quad a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

This, **14.2**, is called a **homogeneous** second order differential equation because the Right-Hand Side  $f(x)$  is zero.

**?** If we are asked to solve such an equation then what are we trying to find?

We need to find  $y$ , a **function** of  $x$ , which satisfies **14.2**. Note that the solution is a function of  $x$ , that is  $y = y(x)$ .

**?** We could begin by substituting the function  $y = e^{mx}$  where  $m$  is unknown. Why?

Because if you differentiate  $e^{mx}$  you end up with another  $e^{mx}$  term. Thus

$$\text{if } y = e^{mx} \text{ then } \frac{dy}{dx} = me^{mx}$$

$$\text{Differentiating again: } \frac{d^2y}{dx^2} = m^2e^{mx}$$

Substituting these into

$$\mathbf{14.2} \quad a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

gives

$$\begin{aligned} am^2 e^{mx} + bme^{mx} + ce^{mx} &= 0 \\ e^{mx}(am^2 + bm + c) &= 0 \quad [\text{Factorizing out } e^{mx}] \end{aligned}$$

**?** So we have  $e^{mx} = 0$  or  $am^2 + bm + c = 0$ . What about  $e^{mx} = 0$ ?

For any real values of  $mx$ ,  $e^{mx} \neq 0$ . (One of the properties of the exponential function is that it cannot equal zero.) Therefore we must have

$$am^2 + bm + c = 0$$

which is a quadratic equation. So if we solve this quadratic equation for  $m$ , we have a value for the ' $m$ ' in  $y = e^{mx}$ . All we need is a value for ' $m$ ', which we can find by solving  $am^2 + bm + c = 0$ . Since every quadratic equation is solvable,  $y = e^{mx}$  is a solution of **14.2** provided that  $m$  satisfies

$$\mathbf{14.3} \quad am^2 + bm + c = 0$$

This quadratic equation is called the **characteristic** or **auxiliary equation** of the differential equation

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

**?** What is the characteristic equation of  $2 \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + 7y = 0$ ?

$$2m^2 + 3m + 7 = 0$$

The solution of **14.3** is given by the quadratic equation formula:

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or by factorizing the quadratic where appropriate.

**?** How many different types of roots can result from a quadratic equation?

Three:

- i** Real and different,  $m_1$  and  $m_2$
- ii** Real and equal,  $m_1 = m_2 = m$
- iii** Complex,  $m = \alpha \pm j\beta$

The theory of differential equations says that, if  $y_1(x)$  and  $y_2(x)$  are two independent solutions of the differential equation

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

then  $Ay_1(x) + By_2(x)$  is also a solution ( $A$  and  $B$  are any real constants). As demonstrated in **Example 1** above there are two arbitrary constants in the solution of a second order differential equation. This is only true if the differential equation is linear, that is  $y$  has index 1 in the differential equation.

For the case of **i** real and different roots,  $m_1$  and  $m_2$ , we have  $y_1(x) = e^{m_1x}$  and  $y_2(x) = e^{m_2x}$  as solutions of the differential equation. Hence

$$y = Ae^{m_1x} + Be^{m_2x}$$

is also a solution. This is called the **general solution** because we do not have specific values for the constants  $A$  and  $B$ .

Hence the solution,  $y$ , of the differential equation

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

depends on the type of root of the characteristic equation

$$am^2 + bm + c = 0$$

If the roots are real and different,  $m_1$  and  $m_2$ , then

$$\text{14.4} \quad y = Ae^{m_1x} + Be^{m_2x}$$

If the roots are real and equal,  $m_1$  (or  $m_2$ ), then

$$\text{14.5} \quad y = (A + Bx)e^{m_1x}$$

If the roots are complex,  $m = \alpha \pm j\beta$ , then

$$\text{14.6} \quad y = e^{\alpha x} [A \cos(\beta x) + B \sin(\beta x)]$$

( $A$  and  $B$  are constants in **14.4** to **14.6**.)

We will not show results **14.5** and **14.6** but just use them whenever appropriate.

Every homogeneous differential equation has a **trivial solution**,  $y = 0$ , because differentiating  $y = 0$  once and twice still gives 0, therefore it satisfies the homogeneous differential equation. So if the question says solve a differential equation it means find the **non-trivial** solution.