Chapter 3

FIR Digital Filters

Chapter content:

3. 1 Introduction

We presented in the previous chapter the design of IIR filters. Although these filters have advantages, they also have disadvantages. For example, if we want to use the advantage of the speed of the FFT, this is not possible and therefore we must use a FIR filter. On the other hand, it is clear that with an IIR filter, it is not possible to obtain a linear phase. With a FIR filter, it is possible to obtain exactly a linear phase. The transfer function of a causal FIR filter is of the form

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \tag{3.1}$$

$H(z)$ is a polynomial in z^{-1} of order N-1. Thus, $H(z)$ has (N-1) zeros that are located in the Z plane and (N-1) poles of z = 0. The frequency response $H(e^{jw})$ is the trigonometric polynomial

$$H(e^{jw}) = \sum_{n=0}^{N-1} h(n) e^{-jwn} \tag{3.2}$$

Each finite sequence is completely specified by N samples of its Fourier transform. So the design of a FIR filter is accomplished by finding either the coefficients of the impulse response or N samples of its frequency response. If the impulse response meets the condition

$$h(n) = h(N-1-n) \tag{3.3}$$

Then the filter has a linear phase. Replacing (3.3) into (3.2), we have

$$H\left(e^{j\omega}\right) = \begin{cases} e^{-j\omega(N-1)/2}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{(N-3)/2} 2h(n)\cos\left(\omega(n-\frac{N-1}{2}\right)\right], & N \; impair \\ e^{-j\omega(N-1)/2}\left[\sum_{n=0}^{N/2-1} 2h(n)\cos\left(\omega(n-\frac{N-1}{2}\right)\right], & N \; pair \end{cases} \tag{3.4}$$

The condition of (3.3) implies a linear phase shift which corresponds to a delay of (N-1)/2 samples. If N is odd, the phase offset is a delay of an integer number of samples. If N is even, this delay corresponds to an integer plus half a sample. This distinction between N is of great importance. Some examples are shown in Fig. 3.1.

3. 2 FIR filter design by window method

A direct approach to obtaining a finite impulse response is to truncate an infinite impulse response. If one assumes that $H_d\left(e^{j\omega}\right)$ is an ideal frequency response desired, then

$$H_d\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} h_d(n) e^{-j\omega n} \tag{3.5a}$$

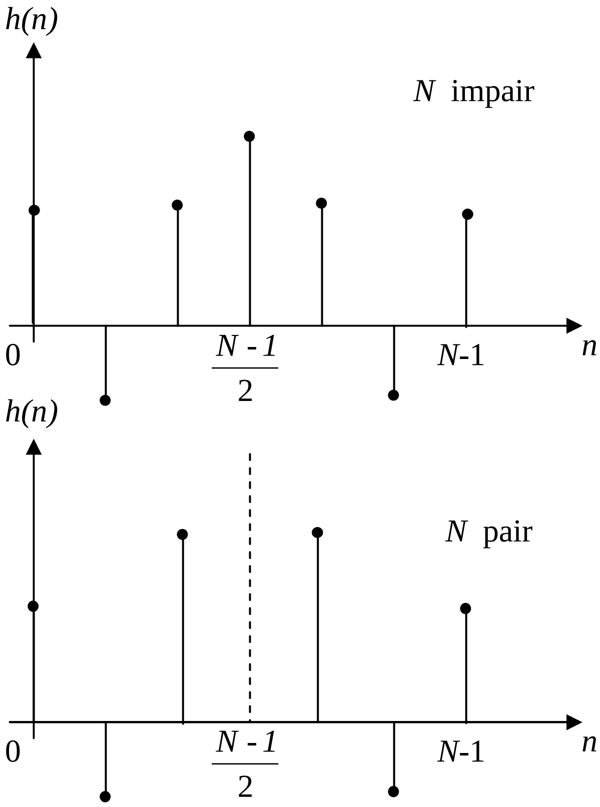


Fig. 3. 1 Typical impulse responses for linear phase FIR filters

where $h_d(n)$ is the corresponding impulse response, i.e.,

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi_c}^{\pi} H_d\left(e^{j\omega}\right) e^{j\omega n} d\omega \tag{3.5b}$$

In general, $H_d\left(e^{j\omega}\right)$ for a selective frequency filter is a constant response with discontinuities at the boundaries between bands. In this case $h_d(n)$ is infinite and must be truncated to obtain a finite impulse response. Equations (3.5) can be considered as the representation of the Fourier series of the periodic frequency response $H_d\left(e^{j\omega}\right)$ with $h_d(n)$ as the Fourier coefficients. If $h_d(n)$ is infinite, one way to obtain a causal and finite $h(n)$ impulsive response is to simply truncate $h_d(n)$ as

$$h(n) = \begin{cases} h_d(n) & 0 \le n \le N-1 \\ 0 & ailleurs \end{cases} \tag{3.6}$$

En général, on peut représenter $h(n)$ comme le produit de la réponse impulsionnelle désirée $h_d(n)$ avec une « fenêtre » $w(n)$ i.e.,

In general, $h(n)$ can be represented as the product of the desired impulse response $h_d(n)$ with a "window" $w(n)$ i.e.,

$$h(n) = h_d(n)w(n) \tag{3.7}$$

where

$$w(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & ailleurs \end{cases} \tag{3.8}$$

Using complex convolution

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})d\theta \tag{3.9}$$

That is, $H(e^{j\omega})$ is the continuous periodic convolution of the desired frequency response with the window's T.F. From (3.9), we see that if $W(e^{j\omega})$ is narrow compared to the variations in $H_d(e^{j\omega})$, then $H(e^{j\omega})$looks more like $H_d(e^{j\omega})$. Therefore, the choice of the window is dictated (inspired) by the desire to have $w(n)$ as short as possible for reasons of calculation and $W(e^{j\omega})$ as narrow as possible in frequency to be as close as possible to the desired response. These are two contradictory conditions. In the case of a rectangular window, we have

$$\begin{aligned} W(e^{j\omega}) = \sum_{n=0}^{N-1} e^{-j\omega n} &= \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \\ &= e^{-j\omega(N-1)/2}\frac{\sin(\omega N/2)}{\sin(\omega/2)} \end{aligned} \tag{3.10}$$

(i.e., the result of the geometric series, $\sum_{n=0}^{N-1} q^n = \frac{1-q^N}{1-q}$ is used in (3.10))

Fig. 3. 2 sketches typical needed functions $H_d(e^{j\omega})$ and $W(e^{j\omega-\theta})$ in (3.9). The magnitude of $W(e^{j\omega})$ is shown in Fig. 3.3 for $N = 8$.

From (3.10), it is clear that the phase is linear. As N increases, the width of the main lobe decreases $(-2\pi/N, 2\pi/N)$. However, for a rectangular window, the secondary lobes are not insignificant and if N increases, the amplitude peaks of the primary and secondary lobes increase so that the area under each lobe is the same. By making the transition smoothly between 1 and 0. The heights of the secondary lobes can be decreased or depends on a larger main lobe and therefore wider transition band.

Some examples of the most used windows for $0 \le n \le N-1$ are given in Fig. 3. 4.

- Rectangular window : $w(n) = 1$

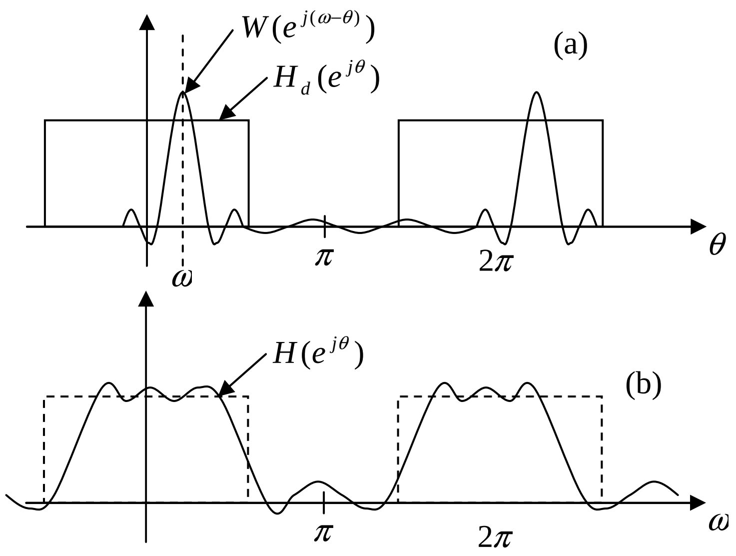


Fig. 3. 2 (a) Convolution process involved in truncating the desired impulse response **(b)** Typical Result approximation from the desired window impulse response.

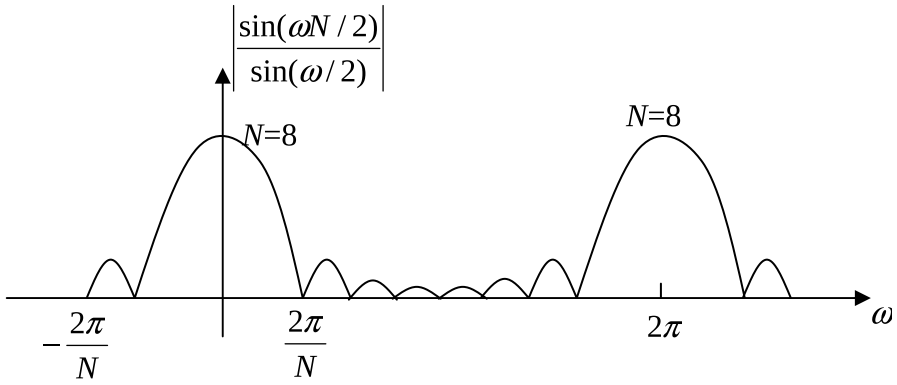


Fig. 3. 3 Magnitude of Module of rectangular window Fourier transform with $N = 8$.

- Bartlett window: $w(n) = \begin{cases} \frac{2n}{N-1} \\ 2 - \frac{2n}{N-1} \end{cases}$

- Hanning window: $w(n) = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right]$

- Hamming window: $w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)$

- Blackman window: $w(n) = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right)$

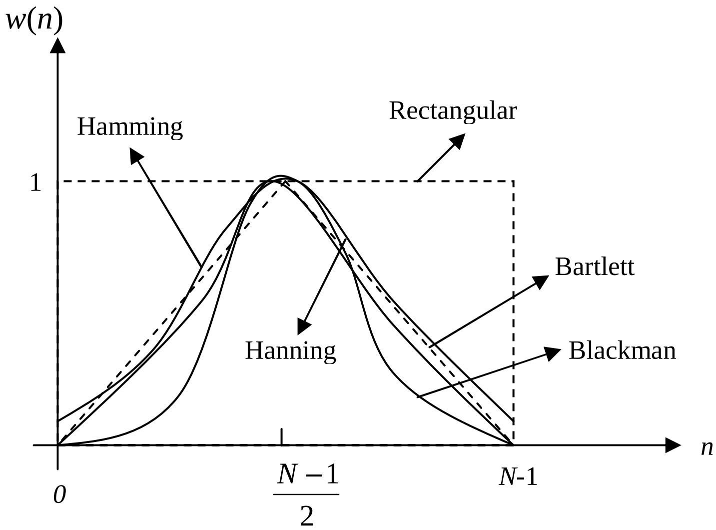


Fig. 3. 4 Usual windows for truncation of $h_d(n)$

For the previously mentioned truncation windows, the basic parameters for designing a low-pass filter are summarized in Table. 3. 1. It is well noted that the values in this table are approximate and depend only on N and the break frequency.

Table. 3. 1 Features of truncation windows.

Fenêtre	Pic d'amplitude du lobe secondaire (dB)	Largeur de transition du lobe principal	Atténuation minimale de la bande d'arrêt (dB)
Rectangulaire	-13	$4\pi/N$	-21
Bartlett	-25	$8\pi/N$	-25
Hanning	-31	$8\pi/N$	-44
Hamming	-41	$8\pi/N$	-53
Blackman	-57	$12\pi/N$	-74

Pulse response of a linear phase FIR filter:

A causal linear phase low-pass filter is considered. The desired frequency response is given by

$$H_d\left(e^{j\omega}\right)=\begin{cases} e^{-j\omega\alpha} & |\omega|\leq\omega_c \\ 0 & ailleurs \end{cases} \tag{3.11}$$

The corresponding impulse response is given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\alpha)}d\omega. = \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)} \qquad n \neq \alpha \tag{3.12}$$

Clearly, $h_d(n)$ has an infinite duration. To create a linear phase causal linear filter of finite duration N, we define

$$h(n) = h_d(n).w(n) \tag{3.13}$$

where $\alpha = \frac{N-1}{2}$ (linear phase condition).

It is easy to verify that if $w(n)$ is symmetrical, the choice of results in a sequence $h(n)$ that satisfies equation (3.3).

3. 3 FIR filter design by frequency sampling method

We have already seen that a finished sequence can be represented by its discrete Fourier transform. So FIR filters have a representation in terms of frequency samples.

$$\tilde{H}(k) = H(z)\Big|_{z=e^{j\frac{2\pi}{N}k}} = \sum_{n=0}^{N-1} h(n)e^{-j\frac{2\pi}{N}kn}, \quad k=0, 1, \ldots, N-1 \tag{3.14}$$

$H(z)$ can be represented by its samples according to the expression

$$H(z) = \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{\tilde{H}(k)}{1-e^{j\frac{2\pi}{N}k}z^{-1}} \tag{3.15}$$

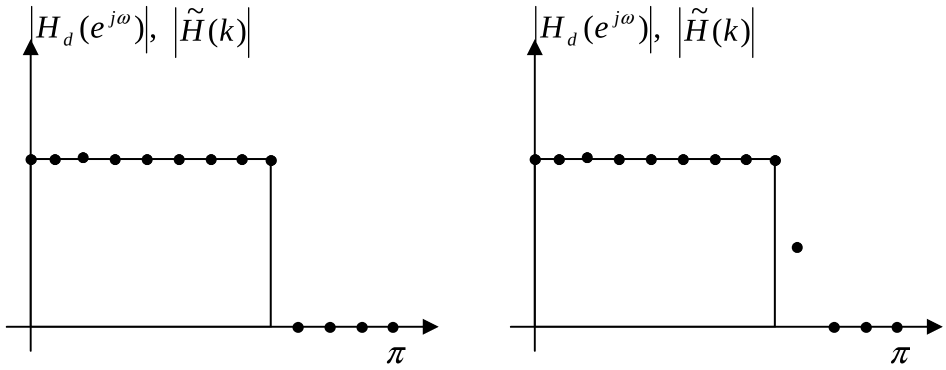
If $z = e^{j\omega}$, the frequency response is obtained

$$\begin{aligned} H(e^{j\omega}) &= \frac{1-e^{-j\omega N}}{N}\sum_{k=0}^{N-1}\frac{\tilde{H}(k)}{1-e^{j\frac{2\pi}{N}k}e^{-j\omega}} \\ &= \frac{e^{-j\omega\frac{N-1}{2}}}{N}\sum_{k=0}^{N-1}\tilde{H}(k)e^{j\pi k\left(1-\frac{1}{N}\right)}\frac{\sin\left[\frac{N}{2}\left(\omega-\frac{2\pi}{N}k\right)\right]}{\sin\left[\frac{1}{2}\left(\omega-\frac{2\pi}{N}k\right)\right]} \end{aligned} \tag{3.16}$$

Where

$$\tilde{H}(k) = H_d(e^{j\frac{2\pi}{N}k}), \ k=0, 1, \ldots, N-1 \tag{3.17}$$

The approach is therefore to specify the filter in terms of these samples from a single period of its desired response (Fig. 3. 5).

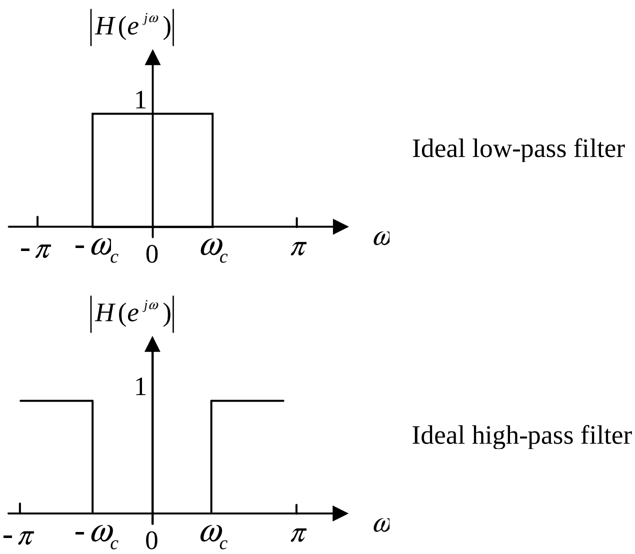


(a) **Fig. 3. 5.** Samples of the ideal low-pass filter frequency response
(a) Sampling without transition
(b) Sampling with transition

The phase is assumed to be linear with a duration of (N-1)/2 samples. The impulse response can be obtained using the discrete reverse Fourier transform as

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{H}(k)e^{j\frac{2\pi}{N}kn}\ , \quad n=0,\ 1,\ \ldots,\ N\text{-}1 \tag{3.17}$$

The desired answers of the ideal filters considered in the field of signal processing are shown in Fig. 3. 6.



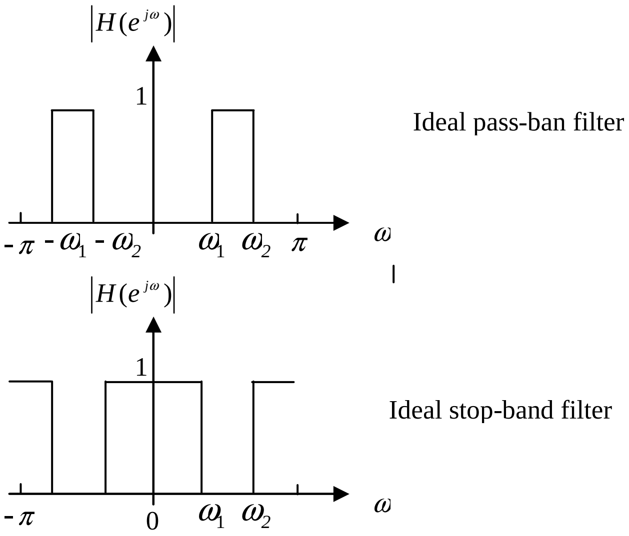


Fig. 3. 6. Frequency responses of ideal filters.

3. 4 Performances comparison of IIR and FIR filters

The Table. 3. 2 shows comparisons between SIR filters and FIR filters that may be encountered in practice.

Table. 3. 2 IIR and FIR filters comparisons.

Criterion	**FIR**	**IIR**
Phase control	yes	no
Complexity	very low possible calculation by DFT	low
Stability	always	problem risk in the case of insufficient calculation accuracy
Necessary coefficients number	medium	low
Necessary calculation accuracy	medium	large enough
Adapted to multi-cadence	yes	no

3. 5 Recitation

Exercise 1 :

Calculate the impulse response, $h(n)$ of a linear phase FIR low-pass filter (Fig. 1) with the following assumptions:

$N = 8$, $\omega_c = 0.25\pi$ and utilization of the following Hamming window

$$w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \le n < N-1$$

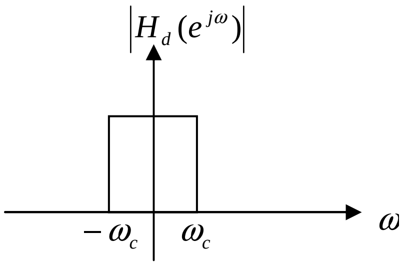


Fig. 1: FIR low-pass filter

Exercise 2 :

We also want to design a linear phase RIF low pass filter (Fig. 1) but with the use of the Hanning window with, $N = 7$ and $\omega_c = 0.2\pi$.

1) Calculate the impulse response coefficients.

2) Calculate the frequency response phase and amplitude.

Exercise 3 :

We want to design a linear phase FIR band pass filter (see Fig. 2) using the Hamming window with:

$N = 7$, $\omega_1 = 0.2\pi$ and $\omega_2 = 0.4\pi$.

1) Calculate the impulse response coefficients.

2) Calculate the frequency response phase and amplitude.

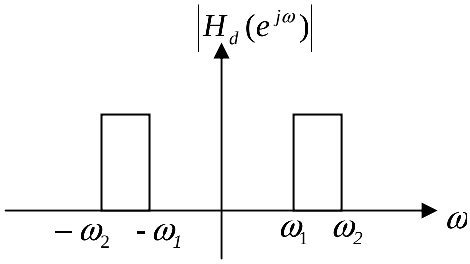


Fig. 2 : Pass-band FIR filter.