Ex01 : Soit la fonction (Champ) scalaire $F(x,y,z) = 3x^2y - y^3z^2$

1°/ Les dérivées partielles, sont des dérivées par rapport à la variable considérée, en considérant que les autres variables sont des constantes.

- $\frac{\partial F}{\partial x}$: est la dérivée partielle par rapport à la variable "x" alors que les variables "y" et "z" sont des constantes.

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(3x^2y - y^3z^2) = \frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial x}(y^3z^2) = 6xy - 0$$

$$\boxed{\frac{\partial F}{\partial x} = 6xy}$$

- $$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(3x^2 - y^3z^2) = \frac{\partial}{\partial y}(3x^2y) - \frac{\partial}{\partial y}(y^3z^2) = 3x^2 - 3y^2z^2$$

$$\boxed{\frac{\partial F}{\partial y} = 3(x^2 - y^2z^2)}$$

- $$\frac{\partial F}{\partial z} = \frac{\partial}{\partial z}(3x^2y - y^3z^2) = \frac{\partial}{\partial z}(3x^2y) - \frac{\partial}{\partial z}(y^3z^2) = 0 - 2y^3z$$

$$\boxed{\frac{\partial F}{\partial z} = -2y^3z}$$

2°/ Gradient de $F(x,y,z)$: Le gradient d'un champ (f^{ct}) scalaire est donné par : $\vec{\nabla}F = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k} = \overrightarrow{grad}(F)$

$$\Rightarrow \boxed{\vec{\nabla}F = 6xy\vec{i} + 3(x^2 - y^2z^2)\vec{j} - 2y^3z\vec{k}}$$

• au point $P(1,-2,1)$, $x=1$; $y=-2$, $z=1$

$$\vec{\nabla}F(1,-2,1) = 6\cdot(1)(-2)\vec{i} + 3\left[1^2-(-2)^2(1)^2\right]\vec{j} - 2(-2)^3(1)^2\vec{k}$$

$$\boxed{\vec{\nabla}F(1,-2,1) = -12\vec{i} - 9\vec{j} - 16\vec{k}}$$

3°/ Le gradient est toujours orienté le long de la normale de la fonction (surface) :

$\vec{\nabla}F = |\vec{\nabla}F|.\vec{n}$ $\quad$ $\vec{n}$: est la normale à la surface

au point $Q(1,1,1)$, $\vec{\nabla}F(1,1,1) = 6.1.1\,\vec{i} + 3(1^2-1^2.1^2)\vec{j} - 2.1^3.1$

$\vec{\nabla}F(1,1,1) = 6\vec{i} - 2\vec{k}$ $\Rightarrow$ $|\vec{\nabla}F| = \sqrt{6^2+2^2} = 2\sqrt{10}$

alors $\vec{n} = \dfrac{\vec{\nabla}F}{|\vec{\nabla}F|} = \dfrac{1}{2\sqrt{10}}(6\vec{i} - 2\vec{k})$: $\boxed{\vec{n} = \dfrac{1}{\sqrt{10}}(3\vec{i} - \vec{k})}$: $\vec{n}\begin{pmatrix} 3/ \\ 0 \\ -1/ \end{pmatrix}$

4°/ La dérivée directionnelle d'une fonction, est la projection de son gradient, le long de la direction considérée.

$\dfrac{dF}{du} = \vec{\nabla}F.\vec{u}_n$ au point $Q(1,1,1)$: $\vec{\nabla}F(1,1,1) = 6\vec{i} - 2\vec{k}$

et $\vec{u} = \vec{i} - \vec{j} - \vec{k}$

$\vec{u}_n = \vec{u}/|\vec{u}|$

$\Rightarrow \dfrac{dF}{du} = (6\vec{i} - 2\vec{k})\circ\left(\dfrac{\vec{i}-\vec{j}-\vec{k}}{\sqrt{3}}\right) = \dfrac{6+2}{\sqrt{3}} = 8/\sqrt{3}$

$$\boxed{\dfrac{dF}{du} = \dfrac{8}{\sqrt{3}}}$$

car $\vec{u} = \vec{i} - \vec{j} - \vec{k}$ est le vecteur unitaire dans cette direction est $\vec{u}_n = \vec{u}/|\vec{u}|$

Ex02: Soit la fonction (Champ) vectorielle $\vec{G}$ /

$$\vec{G} = xyz\,\vec{i} + 3x^2y\,\vec{j} + (xz^2 - y^2z)\,\vec{k}$$

1°/ La divergence d'une fonction vectorielle est donné par:

$$\vec{\nabla}\cdot\vec{G} = \operatorname{div}\vec{G} = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}$$

$$\vec{G} = G_x\vec{i} + G_y\vec{j} + G_z\vec{k} \Rightarrow \begin{cases} G_x = xyz \\ G_y = 3x^2y \\ G_z = xz^2 - y^2z \end{cases} \Rightarrow \begin{cases} \dfrac{\partial G_x}{\partial x} = yz \\ \dfrac{\partial G_y}{\partial y} = 3x^2 \\ \dfrac{\partial G_z}{\partial z} = 2xz - y^2 \end{cases}$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{G} = yz + 3x^2 + 2xz - y^2}$$

au point $P(2,-1,1)$: $\begin{pmatrix} x = 2 \\ y = -1 \\ z = 1 \end{pmatrix}$

$$\vec{\nabla}\cdot\vec{G}(2,-1,1) = (-1)(1) + 3(2)^2 + 2(2)(1) - (-1)^2 = 14$$

$$\boxed{\vec{\nabla}\cdot\vec{G}(2,-1,1) = 14}$$

2°/ Le rotationnel d'une fonction scalaire est donné par:

$$\vec{rot}\,\vec{G} = \vec{\nabla}\wedge\vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_x & G_y & G_z \end{vmatrix} = \vec{i}\left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}\right) - \vec{j}\left(\frac{\partial G_z}{\partial x} - \frac{\partial G_x}{\partial z}\right) + \vec{k}\left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right)$$

$$\Rightarrow \vec{\nabla}\wedge\vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix} = \vec{i}(-2yz - 0) - \vec{j}(z^2 - xy) + \vec{k}(6xy - xz$$

$$\boxed{\vec{\nabla}\wedge\vec{G} = -2yz\,\vec{i} + (xy - z^2)\,\vec{j} + (6xy - xz)\,\vec{k}}$$

au point $P(2,-1,1)$

$$\vec{\nabla}\wedge\vec{G}(2,-1,1) = -2(-1)(1)\vec{i} + \left[2(-1)-1^2\right]\vec{j} + \left[6.2(-1)-2.1\right]\vec{k}$$

$$\Rightarrow \boxed{\vec{\nabla}\wedge\vec{G}(2,-1,1) = 2\vec{i} - 3\vec{j} - 14\vec{k}}$$

3°/ $\vec{A} = (2x-y)\vec{i} + (z+y)\vec{j} + (1-a)z\,\vec{k}$

Pour que le champ $\vec{A}$ soit solénoïdal, il faut que sa divergence soit nulle. $\vec{\nabla}.\vec{A}=0$

$$\Rightarrow \vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 \qquad \begin{cases} A_x = 2x-y \\ A_y = z+y \\ A_z = (1-a)z \end{cases} \Rightarrow \begin{cases} \frac{\partial A_x}{\partial x} = 2 \\ \frac{\partial A_y}{\partial y} = 1 \\ \frac{\partial A_z}{\partial z} = 1-a \end{cases}$$

$$\vec{\nabla}.\vec{A} = 2+1+1-a = 0 \quad \Rightarrow \boxed{a=4}$$

Pour que $\vec{A}$ soit un champ solénoïdal, il faut que : $\underline{a=4}$

4°/ Soit $\vec{B} = (2z+ay)\vec{i} + (x-bz)\vec{j} + (y+cx)\vec{k} = B_x\vec{i} + B_y\vec{j} + B_z\vec{k}$

$$\begin{cases} B_x = 2z+ay \\ B_y = x-bz \\ B_z = y+cx \end{cases} \Rightarrow \vec{\nabla}\wedge\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2z+ay & x-bz & y+cx \end{vmatrix}$$

• Pour que le champ $\vec{B}$ soit irrotationnel il faut que : $\underline{\vec{\nabla}\wedge\vec{B}=\vec{0}}$

$$\Rightarrow \vec{\nabla}\wedge\vec{B} = (1+b)\vec{i} - (c-2)\vec{j} + (1-a)\vec{k} = 0$$

$$\Rightarrow \begin{cases} 1+b=0 \\ c-2=0 \\ 1-a=0 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=-1 \\ c=2 \end{cases} : \cancel{A}$$

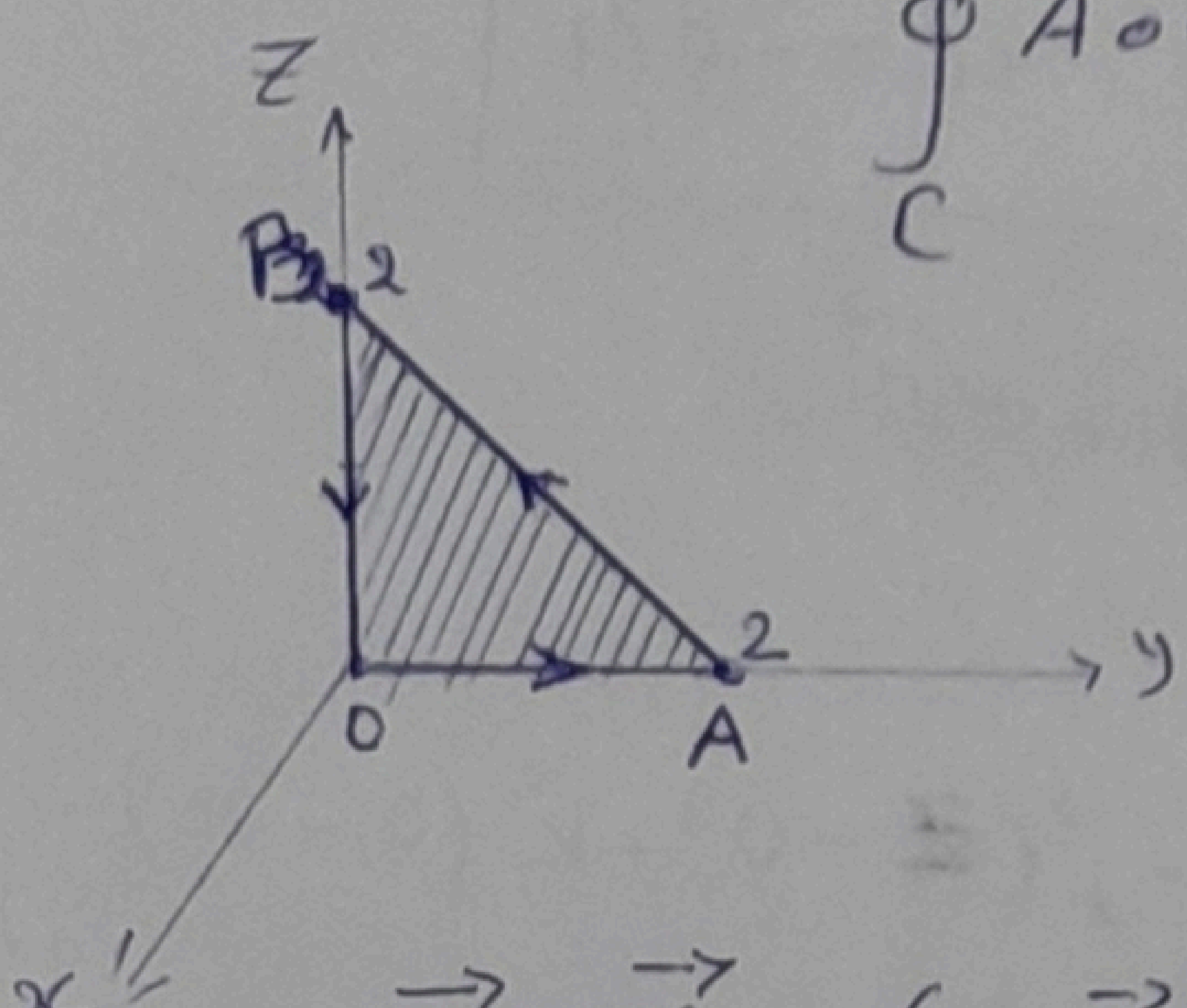
$\vec{A}$: irrotationnel

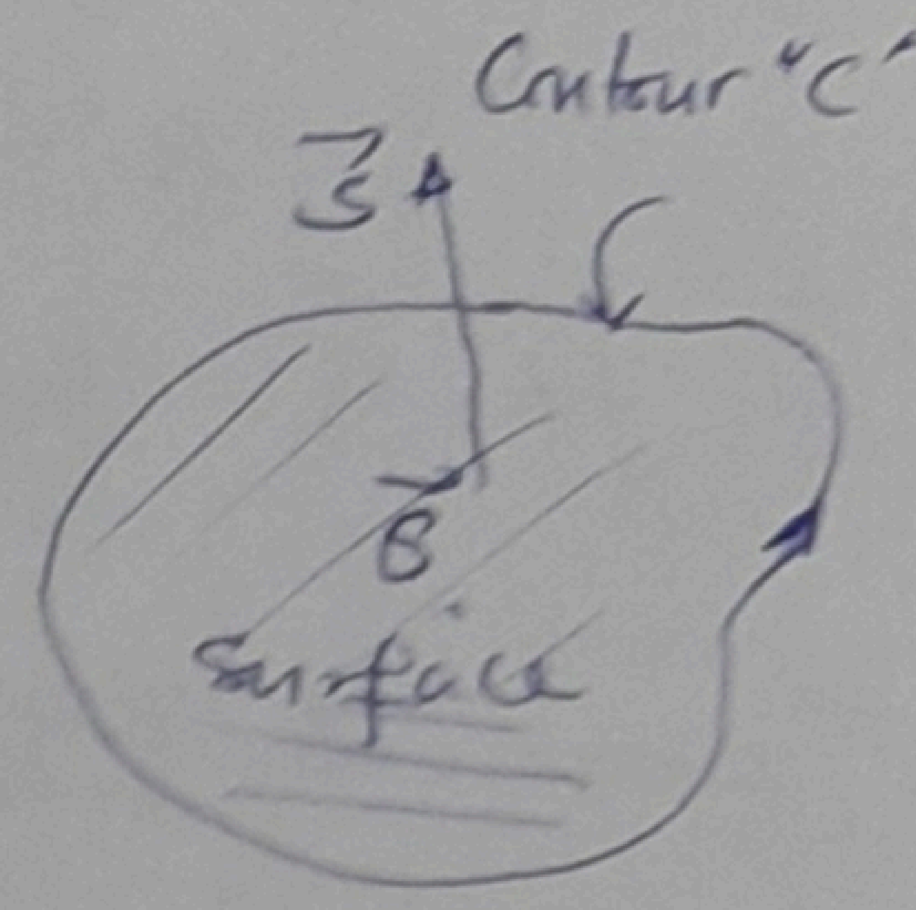
$a=1,\ b=-1,\ c=2$

Ex 04 : Soit le champ scalaire : $\vec{A} = xy\vec{i} + 2yz\vec{j} + 3xz\vec{k}$

Théorème de STOKES (théorème du rotationnel)

$$\oint_C \vec{A} \cdot \vec{dl} = \iint_S \vec{rot}\,\vec{A} \cdot \vec{ds}$$





contour C : OABO

surface du triangle $\overset{\Delta}{OAB}$

$$\vec{A} \cdot \vec{dl} = (xy\vec{i} + 2yz\vec{j} + 3xz\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$\vec{A} \cdot \vec{dl} = xy\,dx + 2yz\,dy + 3xz\,dz$$

on a : $\oint_C \vec{A} \cdot \vec{dl} = \int_{OABO} \vec{A} \cdot \vec{dl} = \int_{OA} \vec{A} \cdot \vec{dl} + \int_{AB} \vec{A} \cdot \vec{dl} + \int_{BO} \vec{A} \cdot \vec{dl}$

chemin OA : ($x=0$, $dx=0$) ; $z=0$,

$$\Rightarrow \int_{OA} \vec{A} \cdot \vec{dl} = \int_{OA} (xy\,dx + 2yz\,dy + 3xz\,dz) = \cancel{\int_{OA} xy\,dx}^{\,x=0 \;\; 0} + \cancel{\int_{OA} 2yz\,dy}^{\,z=0 \;\; 0} + \cancel{\int_{OA} 3xz\,dz}^{\,x=z=0 \;\; 0}$$

$$\Rightarrow \boxed{\int_{OA} \vec{A} \cdot \vec{dl} = 0}$$

chemin AB : ($x=0$, $dx=0$) , $y = 2 - z$, $y=0$, $0 \le z \le 2$, $dy = -dz$

$$\int_{AB} \vec{A} \cdot \vec{dl} = \cancel{\int_{AB} xy\,dx}^{\,x=0 \;\; 0} + \int_{AB} 2yz\,dy + 3\cancel{\int_{AB} xz\,dz}^{\,0 \;\; x=0} = \int_{AB} 2yz\,dy$$

$$\int_{AB} 2yz\,dy \Rightarrow \begin{cases} \int_0^2 2y(2-y)\,dy & y = 2 - z \Rightarrow z = 2 - y,\ \underline{2 \le y \le 0} \\ & dz = -dy \\ \int 2(2-z)z(-dz) & \underline{0 \le z \le 2} \end{cases}$$

$$\Rightarrow -2\int_0^2 (2z - z^2)\,dz = -2\left[z^2 \Big|_0^2 - \frac{1}{3} z^2 \Big|_0^2 \right] = -8/3$$

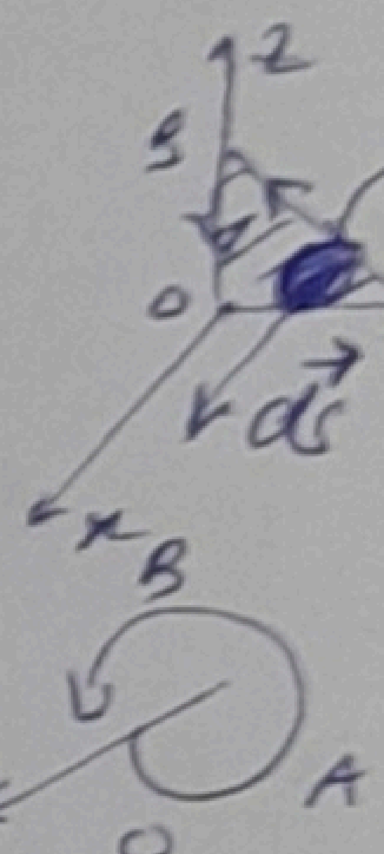
$$\Rightarrow \oint \vec{A}\cdot\vec{dl} = \int_{OA} \vec{A}\cdot\vec{dl} + \int_{AB}\vec{A}\cdot\vec{dl} + \int_{BO}\vec{A}\cdot\vec{dl}$$

Chemin BO : $x=0$, $y=0$, $2\leq z\leq 0$ $\Rightarrow \int_{BO}\vec{A}\cdot\vec{dl} = \int_{BO} xy\,dx + \int_{BO} 2yz\,dy + \int_{BO} 3xz\,dz = 0$

$$\Rightarrow \oint \vec{A}\cdot\vec{dl} = 0 - 8/3 + 0 \Rightarrow \boxed{\oint \vec{A}\cdot\vec{dl} = -8/3}$$

*2nd membre de l'égalité : $\iint_S \vec{rot}\,\vec{A}\cdot\vec{ds}$

$$\vec{rot}\,\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} = \vec{i}(0-2y) - \vec{j}(z-0) + \vec{k}(0-x)$$



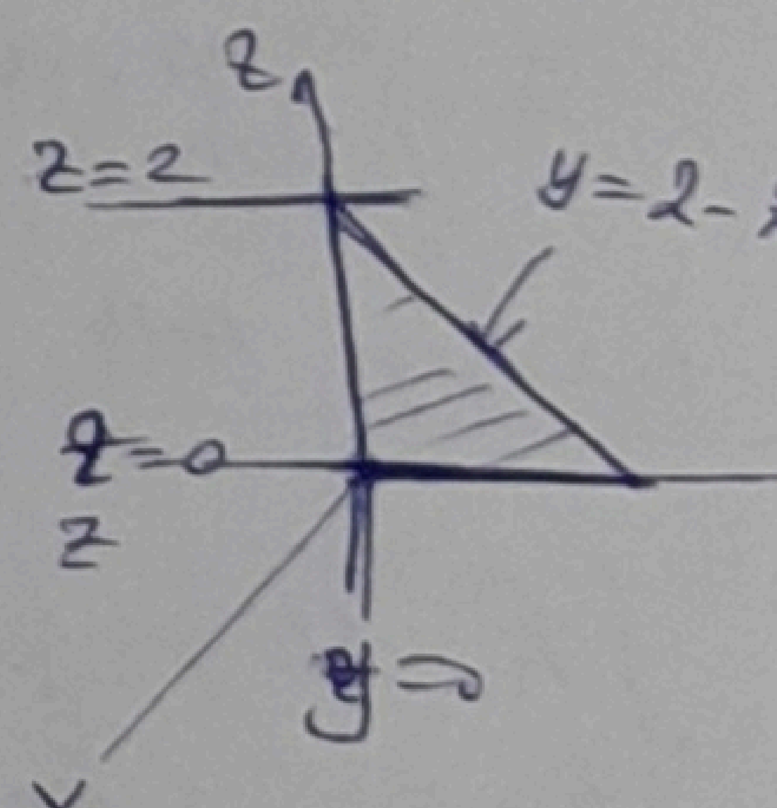
$\boxed{\vec{rot}\,\vec{A} = -2y\vec{i} - z\vec{j} - x\vec{k}}$, $\vec{ds} \perp$ au plan yoz

$\boxed{\vec{ds} = dy\,dz\,\vec{i}}$ $\quad \vec{ds} = ds\,\vec{i} = dy\,dz\,\vec{i}$

$$\iint_S \vec{rot}\,\vec{A}\cdot\vec{ds} = \iint -(2y\vec{i} + z\vec{j} + x\vec{k})\cdot(dy\,dz\,\vec{i}) = -2\iint y\,dy\,dz$$

$$\iint_S \vec{rot}(\vec{A})\cdot\vec{ds} = -2\iint y\,dy\,dz$$

les bornes d'intégration $0\leq z\leq 2$, $0\leq y\leq 2-z$



$$\Rightarrow -2\iint y\,dy\,dz = \begin{cases} -2\int_0^2 \left[\int_0^{2-z} y\,dy\right] dz \\ -2\int_0^2 y\left[\int_0^{2-y} dz\right] dy \end{cases}$$

$$\Rightarrow -2\int_0^2 y\int_0^{2-y} dz = -2\int_0^2 y\left[z\Big|_0^{2-y}\right] = -2\int_0^2 y(2-y)\,dy = -2\left[y^2 - y\right.$$

$$\Rightarrow \iint_S \vec{A}\cdot\vec{dl} = -2\iint y\,dy\,dz = -8/3 \quad \text{et} \quad \oint_C \vec{A}\cdot\vec{dl} = -8/3$$

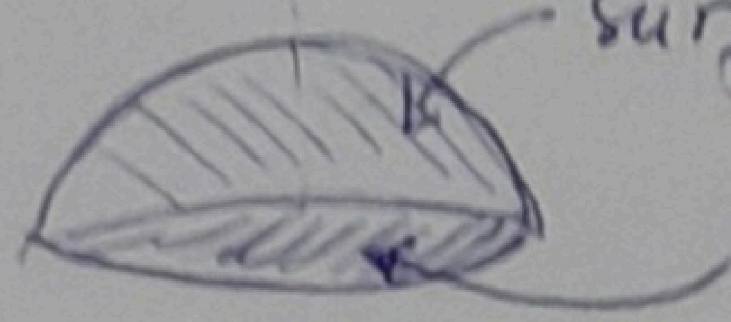
$$\Rightarrow \oint \vec{A}\cdot\vec{dl} = \iint_S \vec{rot}\,\vec{A}\cdot\vec{ds} \text{ est verifiée}$$

2/ Soit le champ vectoriel : $\vec{B} = r\sin\theta\,\vec{u}_r + r\sin\theta\,\vec{u}_\theta + r\sin\theta\cos\varphi\,\vec{u}_\varphi$ ④

Théorème de Gauss (théorème de la divergence)

$$\oiint_S \vec{B}\cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{B}\,dV = \iiint_V \mathrm{div}\vec{B}\,dV$$

Surface fermée : hémisphère nord + surface équatoriale



surface de l'hémisphère nord (S_1)

surface équatoriale (S_2)

$S = S_1 + S_2$

* hémisphère nord : S_1 : élément de surface de rayon R est
$dS_1 = R^2\sin\theta\,d\theta\,d\varphi \qquad 0 \le \theta \le \pi/2 \ , \ 0 \le \varphi \le 2\pi$
elle orientée ~~trag~~ vers l'extérieure suivant la direction radiale $\vec{u}_r$
$d\vec{S} = dS\,\vec{u}_r = R^2\sin\theta\,d\theta\,d\varphi\,\vec{u}_r$

* $\oiint \vec{B}\cdot d\vec{S}$; $\cancel{\iint_S \vec{B}\cdot d\vec{S}_1 = \iint (r\sin\theta\,\vec{u}_r + r\sin\theta\,\vec{u}_\theta + r\sin\theta\cos\varphi\,\vec{u}_\varphi)\cdot R^2\sin\theta\,d\theta\,d\varphi}$

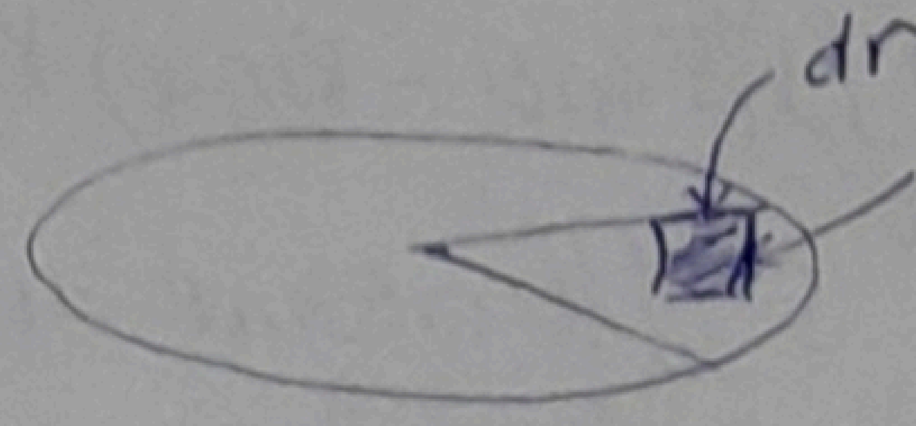
$$\iint_{S_1} \vec{B}\cdot d\vec{S}_1 = \iint (R\sin\theta\,\vec{u}_r + R\sin\theta\,\vec{u}_\theta + R\sin\theta\cos\varphi\,\vec{u}_\varphi)\cdot(R^2\sin\theta\,d\theta\,d\varphi\,\vec{u}_r)$$

$$= \iint R^3\sin\theta\sin\theta\,d\theta\,d\varphi \qquad 0 \le \varphi \le 2\pi,\ 0 \le \theta \le 2\pi$$

$$= R^3\int_0^{2\pi}\left[\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right]d\varphi = R^3\int_0^{2\pi}\left(\frac{1}{2}\sin^2\theta\Big|_0^{\pi/2}\right)d\varphi$$

$$\boxed{\iint_{S_1}\vec{B}\cdot d\vec{S}_1 = \pi R^3}$$

* Surface équatoriale :



$dS_2 = r\,dr\,d\varphi$

$dS_2 = r\,dr\,d\varphi$, orientée suivant $\theta = c^{te}$ c.a.d $\vec{u}_\theta$

$d\vec{S}_2 = r\,dr\,d\varphi\,\vec{u}_\theta \qquad 0 \le r \le R \ ; \ 0 \le \varphi \le 2\pi$

$\Rightarrow$ ~~[illegible]~~

$$\Rightarrow \iint_{S_2} \vec{B}\cdot d\vec{S}_2 = \iint \left[r\cos\theta\,\vec{u}_r + r\sin\theta\,\vec{u}_\theta + r\sin\theta\cos\varphi\,\vec{u}_\varphi\right]\left[r\,dr\,d\varphi\,\vec{u}_\theta\right]$$

$$= \iint r^2 \sin\theta\, dr\, d\varphi \quad \text{or} \quad \theta = \pi/2$$

$$\Rightarrow \iint_{S_2} \vec{B}\cdot d\vec{S}_2 = \int_0^R\int_0^{2\pi} r^2\, dr\, d\varphi = \left(\int_0^R r^2\, dr\right)\left(\int_0^{2\pi} d\varphi\right) = \frac{2\pi R^3}{3}$$

$$\boxed{\iint \vec{B}\cdot d\vec{S}_2 = \frac{2\pi R^3}{3}}$$

$$\Rightarrow \oiint \vec{B}\cdot d\vec{S} = \iint_{S_1} \vec{B}\cdot d\vec{S}_1 + \iint_{S_2} \vec{B}\cdot d\vec{S}_2 = \frac{\pi R^3}{3} + \frac{2\pi R^3}{3} = \frac{5\pi R^3}{3}$$

$$\boxed{\oiint \vec{B}\cdot d\vec{S} = 5\pi R^3/3}$$

* $\text{Div}\,\vec{B}$: $B_r = r\cos\theta$, $B_\theta = r\sin\theta$, $B_\varphi = r\sin\theta\cos\varphi$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 B_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot r\cos\theta\right) = \frac{\cos\theta}{r^2}\frac{\partial (r^3)}{\partial r} = \underline{\underline{3\cos\theta}}$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, B_\theta\right) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, r\sin\theta\right) = \frac{r}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta\right) = \frac{2r\sin\theta\cos\theta}{r\sin\theta}$$

$$= \underline{\underline{2\cos\theta}}$$

$$\frac{1}{r\sin\theta}\frac{\partial B_\varphi}{\partial\varphi} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(r\sin\theta\cos\varphi\right) = \frac{r\sin\theta}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\cos\varphi\right) = -\underline{\underline{\sin\varphi}}$$

$$\boxed{\vec{\nabla}\cdot\vec{B} = 3\cos\theta + 2\cos\theta - \sin\varphi = 5\cos\theta - \sin\varphi}$$

élément de volume : $dv = r^2\sin\theta\, dr\, d\theta\, d\varphi$ $\quad 0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le \theta \le \frac{\pi}{2}$

$$\iiint_V \vec{\nabla}\cdot\vec{B}\, dv = \iiint (5\cos\theta - \sin\varphi)\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= 5\iiint r^2\cos\theta\sin\theta\, dr\, d\theta\, d\varphi - \iiint r^2 \sin\varphi\sin\theta\, dr\, d\theta\, d\varphi$$

$$\iiint \vec{\nabla}\cdot\vec{B} = 5\int_0^R r^2\left(\int_0^{2\pi}\left[\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta\right]d\varphi\right]dr - \int_0^R r^2\left[\int_0^{2\pi}\left[\int_0^{\pi/2}\sin\theta\sin\varphi\, d\theta\right]d\varphi\right]$$

$$= 5\int_0^R r^2\left(\int_0^{2\pi} \frac{\sin^2\theta}{2}\Big|_0^{\pi/2} d\varphi\right)dr - \int_0^R r^2\left[\int_0^{2\pi}\sin\varphi\left[\int_0^{\pi/2}\sin\theta\, d\theta\right]d\varphi\right]dr = \frac{5\pi R^3}{3}$$

$$\Rightarrow \oiint \vec{B}\cdot d\vec{S} = \iiint \nabla\cdot\vec{B}\, dv \xrightarrow{\quad} \text{vérifiée}$$