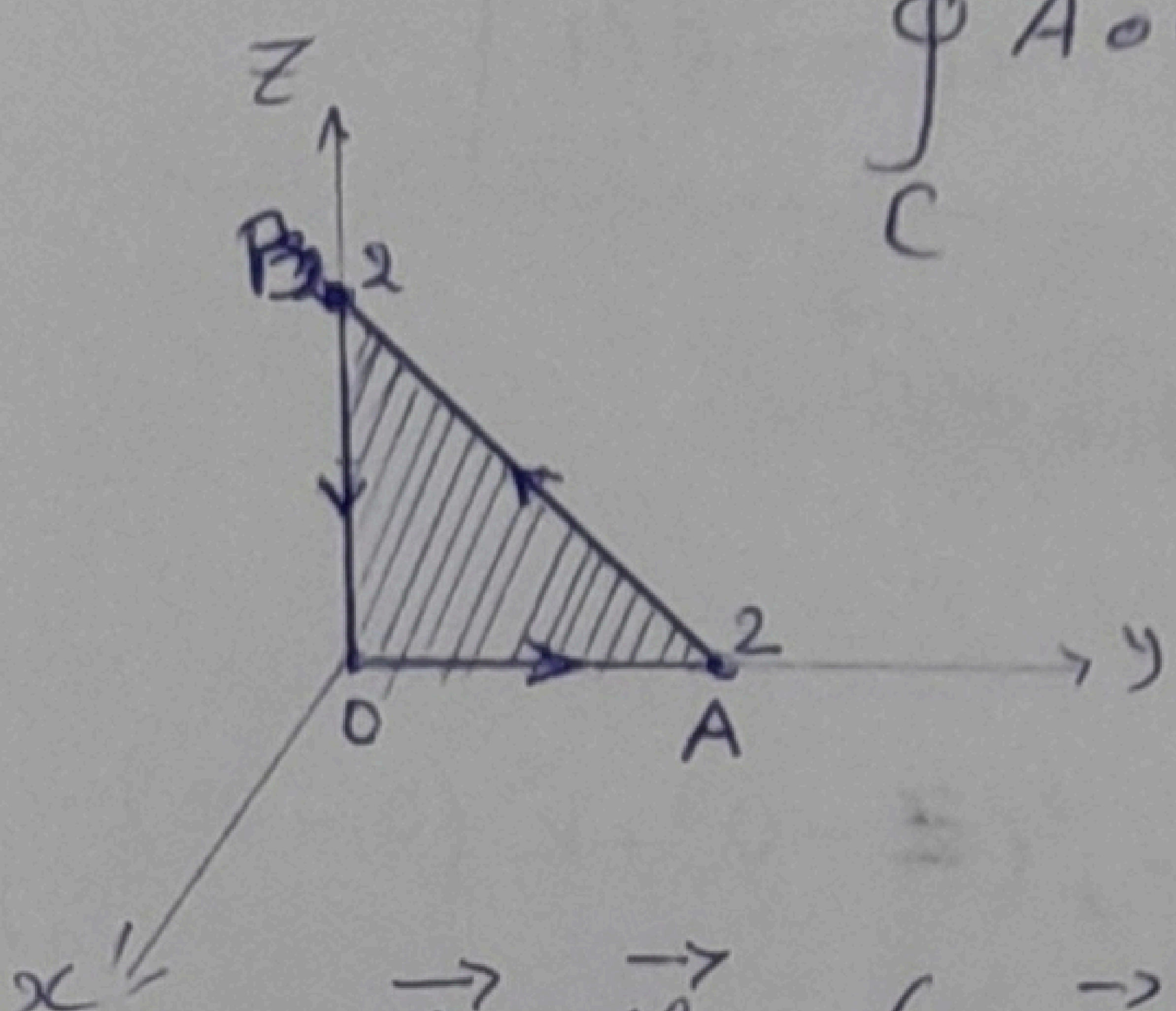
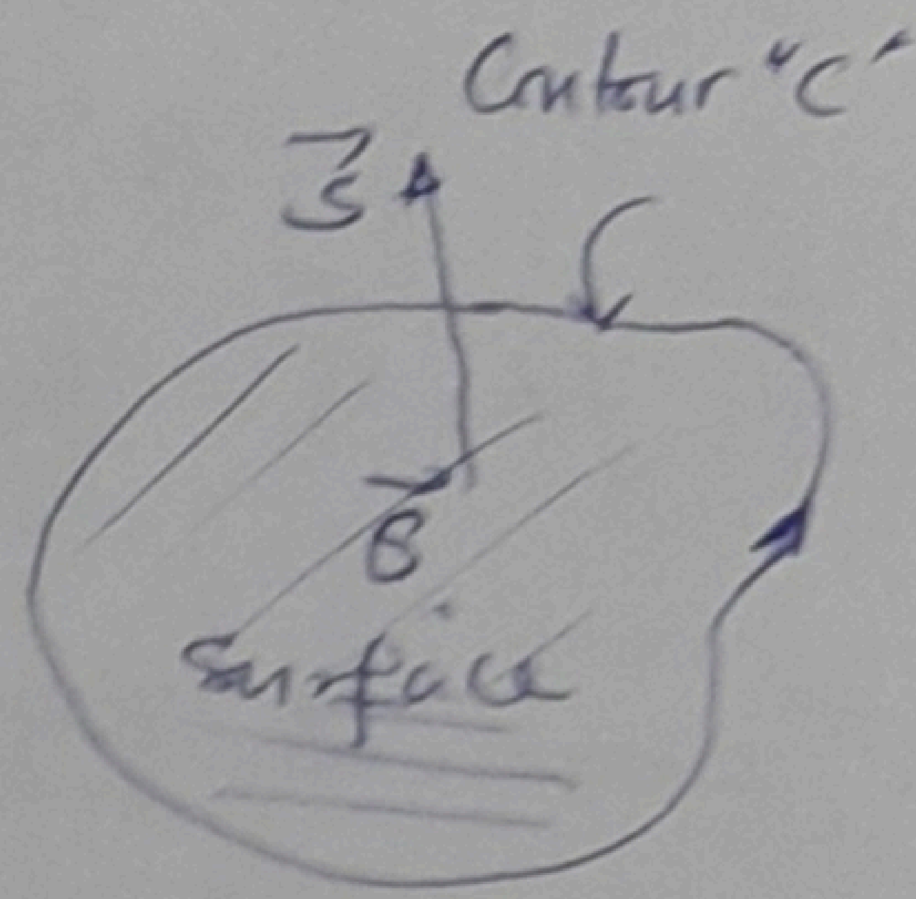
Ex04 : Soit le champ scalaire $\vec{A} = xy\vec{i} + 2yz\vec{j} + 3xz\vec{k}$

Théorème de STOKES (théorème du rotationnel)

$$\oint_C \vec{A} \cdot \vec{dl} = \iint_S \vec{rot}\,\vec{A} \cdot \vec{ds}$$



contour C : OABO
surface du triangle $\overset{\Delta}{OAB}$



$$\vec{A} \cdot \vec{dl} = (xy\vec{i} + 2yz\vec{j} + 3xz\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$\vec{A} \cdot \vec{dl} = xy\,dx + 2yz\,dy + 3xz\,dz$$

on a : $\oint_C \vec{A} \cdot \vec{dl} = \int_{OABO} \vec{A} \cdot \vec{dl} = \int_{OA} \vec{A} \cdot \vec{dl} + \int_{AB} \vec{A} \cdot \vec{dl} + \int_{BO} \vec{A} \cdot \vec{dl}$

chemin OA : $(x=0,\ dx=0)$; $z=0$,

$$\Rightarrow \int_{OA} \vec{A} \cdot \vec{dl} = \int_{OA} (xy\,dx + 2yz\,dy + 3xz\,dz) = \cancel{\int_{OA} xy\,dx}^{\,x=0\ \ 0} + \cancel{\int_{OA} 2yz\,dy}^{\,z=0\ \ 0} + \cancel{\int_{OA} 3xz\,dz}^{\,x=z=0\ \ 0}$$

$$\Rightarrow \boxed{\int_{OA} \vec{A} \cdot \vec{dl} = 0}$$

chemin AB : $(x=0,\ dx=0)$, $y = 2-z$, $y=0$, $0 \leq z \leq 2$, $dy = -dz$

$$\int_{AB} \vec{A} \cdot \vec{dl} = \cancel{\int_{AB} xy\,dx}^{\,x=0\ \ 0} + \int_{AB} 2yz\,dy + 3\cancel{\int_{AB} xz\,dz}^{\,0\ \ x=0} = \int_{AB} 2yz\,dy$$

$$\int_{AB} 2yz\,dy \Rightarrow \begin{cases} \int_0^2 2y(2-y)\,dy & y = 2-z \Rightarrow z = 2-y,\ 2 \leq y \leq 0 \\ & dz = -dy \\ \int 2(2-z)z(-dz) & 0 \leq z \leq 2 \end{cases}$$

$$\Rightarrow -2\int_0^2 (2z - z^2)\,dz = -2\left[z^2\Big|_0^2 - \frac{1}{3}z^2\Big|_0^2 \right] = -8/3$$

$$\Rightarrow \oint \vec{A}\cdot\vec{dl} = \int_{OA} \vec{A}\cdot\vec{dl} + \int_{AB} \vec{A}\cdot\vec{dl} + \int_{BO} \vec{A}\cdot\vec{dl}$$

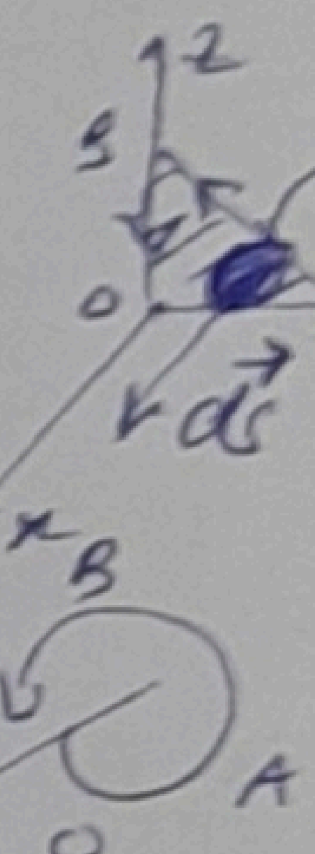
Chemin BO : $x=0$, $y=0$, $2 \leq z \leq 0$

$$\Rightarrow \int_{BO} \vec{A}\cdot\vec{dl} = \int_{BO} \overset{x=0}{\cancel{xy}}\,dx + \int_{BO} \overset{y=0}{\cancel{2yz}}\,dy + \int_{BO} \overset{z=0}{\cancel{3xz}}\,dz = 0$$

$$\Rightarrow \oint \vec{A}\cdot\vec{dl} = 0 - 8/3 + 0 \quad \Rightarrow \quad \boxed{\oint \vec{A}\cdot\vec{dl} = -8/3}$$

*2nd membre de l'égalité : $\iint_S \vec{rot}\,\vec{A}\cdot\vec{ds}$

$$\vec{rot}\,\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3xz \end{vmatrix} = \vec{i}(0-2y) - \vec{j}(z-0) + \vec{k}(0-x)$$



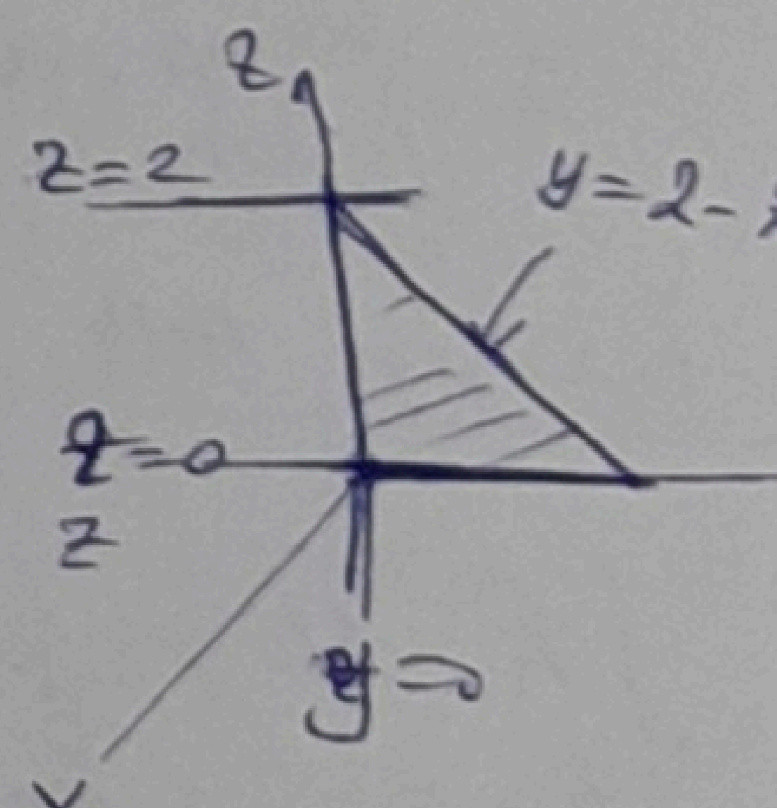
$\boxed{\vec{rot}\,\vec{A} = -2y\vec{i} - z\vec{j} - x\vec{k}}$, $\vec{ds} \perp$ au plan yoz

$\boxed{\vec{ds} = dydz\,\vec{i}}$ $\qquad \vec{ds} = ds\,\vec{i} = dydz\,\vec{i}$

$$\iint_S \vec{rot}\,\vec{A}\cdot\vec{ds} = \iint -(2y\vec{i} + z\vec{j} + x\vec{k})\cdot(dydz\,\vec{i}) = -2\iint y\,dydz$$

$$\iint_S \vec{rot}(\vec{A})\cdot\vec{ds} = -2\iint y\,dydz$$

les bornes d'intégration $0 \leq z \leq 2$, $0 \leq y \leq 2-z$



$$\Rightarrow -2\iint y\,dydz = \begin{cases} -2\int_0^2 \left[\int_0^{2-z} y\,dy\right] dz \\ -2\int_0^2 y\left[\int_0^{2-y} dz\right] dy \end{cases}$$

$$\Rightarrow -2\int_0^2 y \int_0^{2-y} dz = -2\int_0^2 y\left[z\Big|_0^{2-y}\right] = -2\int_0^2 y(2-y)\,dy = -2\left[y^2 - y\right.$$

$$\Rightarrow \iint_S \vec{A}\cdot\vec{dl} = -2\iint y\,dydz = -8/3 \quad \text{et} \quad \oint_C \vec{A}\cdot\vec{dl} = -8/3$$

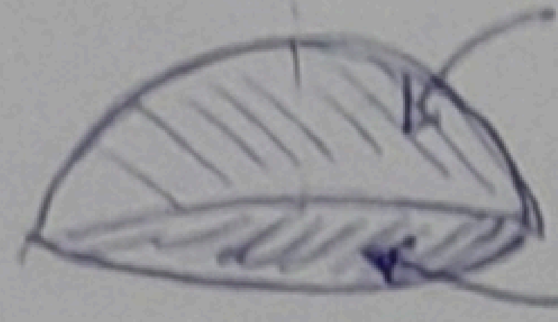
$\Rightarrow \oint \vec{A}\cdot\vec{dl} = \iint_S \vec{rot}\,\vec{A}\cdot\vec{ds}$ est vérifié

2/ Soit le champ vectoriel : $\vec{B} = r\cos\theta\,\vec{u}_r + r\sin\theta\,\vec{u}_\theta + r\sin\theta\cos\varphi\,\vec{u}_\varphi$ ④

Théorème de Gauss (théorème ou la divergence)

$$\oiint_S \vec{B}\cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{B}\,dv = \iiint_V \mathrm{div}\vec{B}\,dv$$

Surface fermée : hémisphère nord + surface équatoriale



surface de l'hémisphère nord (S_1)

surface équatoriale (S_2)

$S = S_1 + S_2$

* hémisphère nord, S_1, élément de surface de rayon R et

$dS_1 = R^2\sin\theta\,d\theta\,d\varphi \qquad 0 \le \theta \le \pi/2\ ,\ 0 \le \varphi \le 2\pi$

elle orientée ~~trag~~ vers l'extérieure suivant la direction radiale $\vec{u}_r$

$d\vec{S} = dS\,\vec{u}_r = R^2\sin\theta\,d\theta\,d\varphi\,\vec{u}_r$

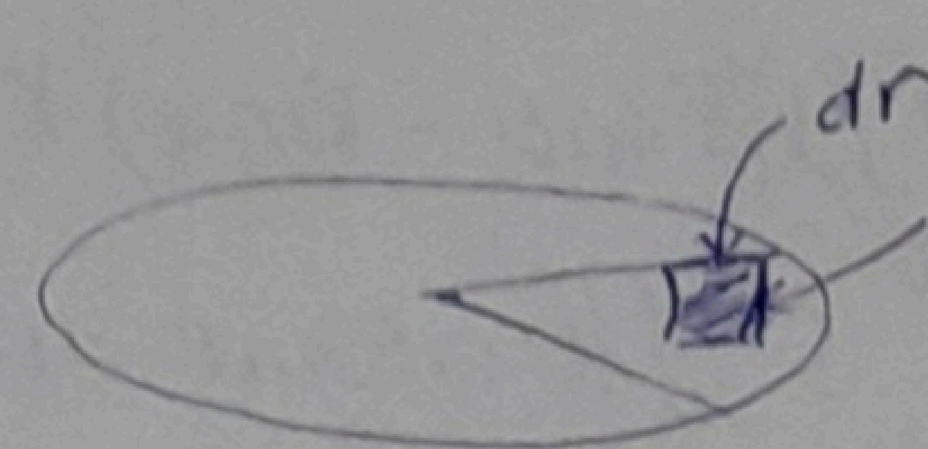
* $\oiint \vec{B}\cdot d\vec{S}$; $\iint_S \vec{B}\cdot d\vec{S}_1 = \iint$ ~~(r cosθ u_r + r sinθ u_θ + r sinθ cosφ u_φ) R² sinθ dθ dφ u_r~~

$$\iint_{S_1} \vec{B}\cdot d\vec{S}_1 = \iint \left(R\cos\theta\,\vec{u}_r + R\sin\theta\,\vec{u}_\theta + R\sin\theta\cos\varphi\,\vec{u}_\varphi\right)\cdot\left(R^2\sin\theta\,d\theta\,d\varphi\,\vec{u}_r\right)$$

$$= \iint R^3\cos\theta\sin\theta\,d\theta\,d\varphi \qquad \begin{matrix} 0 \le \varphi \le 2\pi \\ 0 \le \theta \le 2\pi \end{matrix}$$

$$= R^3\int_0^{2\pi}\left[\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right]d\varphi = R^3\int_0^{2\pi}\left(\frac{1}{2}\sin^2\theta\Big|_0^{\pi/2}\right)d\varphi$$

$$\boxed{\iint_{S_1} \vec{B}\cdot d\vec{S}_1 = \pi R^3}$$



$dS_2 = r\,dr\,d\varphi$

* Surface équatoriale :

$dS_2 = r\,dr\,d\varphi$, orientée suivant $\theta = c^{te}$ c.a.d $\vec{u}_\theta$

$d\vec{S}_2 = r\,dr\,d\varphi\,\vec{u}_\theta \qquad 0 \le r \le R\ ;\ 0 \le \varphi \le 2\pi$

$\Rightarrow$ ~~[illegible]~~

$$\Rightarrow \iint_{S_2} \vec{B}\cdot d\vec{S_2} = \iint \left[r\cos\theta\, \vec{u_r} + r\sin\theta\, \vec{u_\theta} + r\sin\theta\cos\varphi\, u_\varphi\right]\left[r\, dr\, d\varphi\, \vec{u_\theta}\right]$$

$$= \iint r^2 \sin\theta\, dr\, d\varphi \quad \text{or} \quad \theta = \pi/2$$

$$\Rightarrow \iint_{S_2} \vec{B}\cdot d\vec{S_2} = \int_0^R\int_0^{2\pi} r^2\, dr\, d\varphi = \left(\int_0^R r^2 dr\right)\left(\int_0^{2\pi} d\varphi\right) = \frac{2\pi R^3}{3}$$

$$\boxed{\iint \vec{B}\cdot d\vec{S_2} = \frac{2\pi R^3}{3}}$$

$$\Rightarrow \oiint \vec{B}\cdot d\vec{S} = \iint_{S_1} \vec{B}\cdot d\vec{S_1} + \iint_{S_2} \vec{B}\cdot d\vec{S_2} = \frac{\pi R^3}{3} + \frac{2\pi R^3}{3} = \frac{5\pi R^3}{3}$$

$$\boxed{\oiint \vec{B}\cdot d\vec{S} = 5\pi R^3/3}$$

* $\underline{\text{Div}\,\vec{B}}$: $B_r = r\cos\theta$, $B_\theta = r\sin\theta$, $B_\varphi = r\sin\theta\cos\varphi$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot r\cos\theta) = \frac{\cos\theta}{r^2}\frac{\partial (r^3)}{\partial r} = \underline{\underline{3\cos\theta}}$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, B_\theta) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, r\sin\theta) = \frac{r}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta) = \frac{2r\sin\theta\cos\theta}{r\sin\theta}$$

$$= \underline{\underline{2\cos\theta}}$$

$$\frac{1}{r\sin\theta}\frac{\partial B_\varphi}{\partial\varphi} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(r\sin\theta\cos\varphi) = \frac{r\sin\theta}{r\sin\theta}\frac{\partial}{\partial\varphi}(\cos\varphi) = -\underline{\underline{\sin\varphi}}$$

$$\boxed{\vec{\nabla}\cdot\vec{B} = 3\cos\theta + 2\cos\theta - \sin\varphi = 5\cos\theta - \sin\varphi}$$

élément de volume : $dv = r^2\sin\theta\, dr\, d\theta\, d\varphi$ $\quad 0 \le r \le R$, $0 \le \varphi \le 2\pi$, $0 \le \theta \le \frac{\pi}{2}$

$$\iiint_V \vec{\nabla}\cdot\vec{B}\, dv = \iiint (5\cos\theta - \sin\varphi)\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= 5\iiint r^2\cos\theta\sin\theta\, dr\, d\theta\, d\varphi - \iiint r^2\sin\varphi\sin\theta\, dr\, d\theta\, d\varphi$$

$$\iiint \vec{\nabla}\cdot\vec{B} = 5\int_0^R r^2\left(\int_0^{2\pi}\left[\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta\right] d\varphi\right] dr - \int_0^R r^2\left[\int_0^{2\pi}\left[\int_0^{\pi/2}\sin\theta\sin\varphi\, d\theta\right] d\varphi\right]$$

$$= 5\int_0^R r^2\left(\int_0^{2\pi} \frac{\sin^2\theta}{2}\Big|_0^{\pi/2} d\varphi\right) dr - \int_0^R r^2\left[\int_0^{2\pi}\sin\varphi\int_0^{\pi/2}\sin\theta\, d\theta\right] d\varphi\right] dr = \frac{5\pi R^3}{3}$$

$$\Rightarrow \oiint \vec{B}\cdot d\vec{s} = \iiint \nabla\cdot\vec{B}\, dv \xrightarrow{2} \text{vérifiée}$$