حلول السلسلة رقم 06

Exercice 01:

Rappel

Si f est integrable au sens de Remann alors

$$\int_a^b f(x)\,dx = \boxed{\lim_{n\to+\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)}$$

$$= \boxed{\lim_{n\to+\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a+k\frac{b-a}{n}\right)}$$

en particulier $\int_0^1 f(x)dx = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$

* Calculons $\int_0^1 x\,dx$. On a la fct définie par $f(x) = x$ est intégrable au sens de Riemann car elle est continue

Donc $\int_0^1 x\,dx = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{k}{n} = \lim \frac{1}{n^2} \sum_{k=0}^{n-1} k$

$$= \lim_{n\to+\infty} \frac{1}{n^2}\left(0+1+\cdots+n-1\right)$$

$$= \lim_{n\to+\infty} \frac{n(n-1)}{2n^2} = \boxed{\frac{1}{2}}$$

* $\int_0^1 x^2 dx$. On pose $f(x) = x^2$, On a f est cont

Donc $\int_0^1 x^2 dx = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k^2}{n^2} = \lim_{n\to+\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2$

$$= \lim_{n\to+\infty} \frac{1}{n^3}\left(1^2+2^2+\cdots+n^2\right)$$

$$= \lim \frac{1}{n^3}\left(\qquad\qquad\right) = \frac{1}{3}$$

* $\int_0^1 e^x dx$. On pose $f(x) = e^x$. On a f est continue donc

$$\int_0^1 e^x dx = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{\frac{k}{n}} = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(e^{\frac{1}{n}}\right)^k$$

02)

$$= \lim \frac{1}{n}\,\frac{1-(e^{1/n})^n}{1-e^{1/n}} \qquad \left(\lim_{x\to 0}\frac{e^x-1}{x}=1,\ x=1/n\right)$$

$$= \lim_{n\to+\infty}(1-e)\times \underbrace{\frac{1/n}{1-e^{1/n}}}_{\to -1} = \boxed{e-1}$$

Exercice 02 :

$$* \lim_{n\to+\infty}\sum_{k=0}^{n}\frac{1}{n+k} = \lim_{n\to+\infty}\left(\frac{1}{n}+\sum_{k=1}^{n}\frac{1}{n+k}\right)$$

$$= \lim_{n\to+\infty}\left(\underbrace{\frac{1}{n}}_{\to 0}+\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}\right)$$

$$= \int_0^1\frac{1}{1+x}\,dx = \Big[\ln(1+x)\Big]_0^1 = \boxed{\ln 2}$$

$$* \lim_{n\to+\infty}\frac{1}{\sqrt{n}}\sum_{k=0}^{n}\frac{1}{\sqrt{n+k}} = \lim\left(\frac{1}{n}+\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{1}{\sqrt{n+k}}\right)$$

$$= \lim_{n\to+\infty}\left(\frac{1}{n}+\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1+\frac{k}{n}}}\right)$$

$$= 0+\int_0^1\frac{1}{\sqrt{1+x}}\,dx = \Big[2\sqrt{1+x}\Big]_0^1 = \boxed{2\sqrt{2}-2}$$

$$* \lim_{n\to+\infty}\sum_{k=0}^{n}\frac{k}{n^2}\sin\frac{k\pi}{n} = \lim_{n\to+\infty}\sum_{k=0}^{n-1}\frac{k}{n^2}\sin\frac{k\pi}{n}$$

$$= \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\frac{k}{n}\sin\frac{k}{n}\pi$$

$$= \int_0^1 x\sin(\pi x)\,dx = \int_0^1 x\left(-\frac{1}{\pi}\cos\pi x\right)'dx$$

$$\overset{\text{Par parties}}{=} \left[-\frac{x}{\pi}\cos\pi x\right]_0^1+\frac{1}{\pi}\int_0^1\cos(\pi x)\,dx$$

$$= \frac{1}{\pi}-0+\frac{1}{\pi}\underbrace{\left[\frac{\sin\pi x}{\pi}\right]_0^1}_{0} = \boxed{\frac{1}{\pi}}$$

Exercice 03 :

(Par parties $\boxed{\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx}$)

* $I_1 = \int x \ln x \, dx$ on choisit $f'(x) = x$ et $g(x) = \ln x$.

* $I_2 = \int \sin^2 x \, dx$ " " $f'(x) = \sin x$ et $g(x) = \sin x$

(on peut integrer directement en utilisant : $\sin^2 x = \frac{1-\cos 2x}{2}$)

* $I_3(x) = \int e^{-2x} \sin x \, dx$, on choisit $f'(x) = \sin x$ et $g(x) = e^{-2x}$
et on integre deux fois par parties

Exo4 :

$$F(x) = \int x^2 e^{-\frac{x}{3}} dx$$

* 1er methode : par parties (deux fois) on choisit $u' = e^{-\frac{x}{3}}$, $v = x^2$

* 2eme methode : On sait que $\int P_n(x) e^{ax} dx = Q_n(x) e^{ax} + c$
où Q_n est un polynôme de dég $=$ deg P_n

D'où $F(x) = (ax^2 + bx + c) e^{-\frac{x}{3}} + c^{te}$

On a donc $F'(x) = x^2 e^{-\frac{x}{3}}$. Pour déterminer a, b et c on identifie les coeficients

En effet : On a $F'(x) = (2ax+b) e^{-x/3} - \frac{1}{3}(ax^2+bx+c) e^{-x/3}$

$$= \left(-\frac{1}{3}ax^2 + \left(2a - \frac{b}{3}\right)x + b - \frac{c}{3}\right) e^{-\frac{x}{3}} = x^2 e^{-\frac{x}{3}}$$

Donc $\begin{cases} -\frac{1}{3}a = 1 \\ 2a - \frac{b}{3} = 0 \\ b - \frac{c}{3} = 0 \end{cases}$

Exercice 05 : $F(x) = \int_1^x \cos(\ln t) dt$, $G(x) = \int_1^x \sin(\ln t) dt$

1) Par parties

* $F(x) = \int_1^x \underbrace{(t)'}_{f'} \underbrace{\cos(\ln t)}_{g} dt = \left[t \cos(\ln t)\right]_1^x - \int_1^x \cancel{t} \left(\frac{1}{\cancel{t}}\right)(-\sin \ln t) dt$

$$= \boxed{x \cos \ln x - 1 + G(x)}$$

* $G(x) = \int_1^x (t)' \sin \ln t \, dt = \left[t \sin \ln t\right]_1^x - \int_1^x t\left(\frac{1}{t}\right)(\cos \ln t) dt$

$$= \boxed{x \sin \ln x - F(x)}$$

2) $F(x) = x \cos \ln x - 1 + (x \sin \ln x - F(x))$. D'où $F(x) = \frac{x}{2}(\cos \ln x + \sin \ln x)$

$G(x) = x \sin \ln x - \frac{x}{2}(\cos \ln x + \sin \ln x) + 1 = \frac{x}{2}(\sin \ln x - \cos \ln x) + 1$

Exercice 06 = $\int f'(x)[f(x)]^{\alpha} dx = \frac{f(x)^{\alpha+1}}{\alpha+1} + c \quad (\text{si } \alpha \neq -1)$

$$\int f'(x)\, G'(f(x))\, dx = G(f(x)) + c \,, \quad \int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

$* \int x(2x^2+1)^n dx = \frac{1}{4}\int \underbrace{4x}_{f'} (\underbrace{2x^2+1}_{f})^n dx = \frac{1}{4(n+1)} (2x^2+1)^{n+1} + c \quad (n \in \mathbb{N})$

$* \int \frac{1}{x(\ln x)^2} dx = \int (\ln x)'(\ln x)^{-2} dx = \frac{(\ln x)^{-1}}{-1} = -\frac{1}{\ln x} + c$

$\left(\text{ou } \int \frac{1}{x \ln^2 x} dx = -\int \frac{-(\ln x)'}{(\ln x)^2} dx = -\frac{1}{\ln x} + c \right)$

$* \int \sin x \cos^2 x \, dx = -\int (\cos x)' (\cos x)^2 dx = -\frac{\cos^3 x}{3} + c$

$* \int x (x^2+1)^{2022} \sqrt{x^2+1}\, dx = \frac{1}{2}\int 2x (x^2+1)^{2022+\frac{1}{2}} dx$

$$= \frac{1}{2} \frac{(x^2+1)^{2023+\frac{1}{2}}}{2023+\frac{1}{2}} + c = \frac{(x^2+1)^{\frac{4047}{2}}}{4047} + c^{te}$$

Exo7 = $ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$

On a $x^2+2x+3 = (x+1)^2 - 1 + 3 = \boxed{(x+1)^2 + 2}$

1) $* \int \frac{dx}{x^2+2x+3} dx = \int \frac{dx}{2\left[\left(\frac{x+1}{\sqrt{2}}\right)^2+1\right]}$. On pose $u = \frac{x+1}{\sqrt{2}}$, $du = \frac{dx}{\sqrt{2}}$

$$= \frac{1}{2}\int \frac{\sqrt{2}\, du}{u^2+1} = \frac{\sqrt{2}}{2} \operatorname{Arctg} u + C$$

$$= \frac{1}{\sqrt{2}} \operatorname{Arctg}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

$* \int \frac{1}{\sqrt{x^2+2x+3}} dx = \int \frac{dx}{\sqrt{2}\sqrt{\left(\frac{x+1}{\sqrt{2}}\right)^2+2}} = \frac{1}{\sqrt{2}} \int \frac{\sqrt{2}\, du}{\sqrt{u^2+1}} = \operatorname{Argsh} u + C$

$$= \operatorname{Argsh} \frac{x+1}{\sqrt{2}} + C$$

$* \int \sqrt{x^2+2x+3}\, dx = \sqrt{2} \int \sqrt{\left(\frac{x+1}{\sqrt{2}}\right)^2 + 1}\, dx = 2\int \sqrt{u^2+1}\, du$

On pose ~~t = sh u, D'où dt = ch u du~~ $u = \operatorname{sh} t$, D'où $du = \operatorname{ch} t\, dt = \sqrt{u^2+1}\, dt$

$\int \sqrt{u^2+1}\, du = \int u^2+1\, dt = \int (\operatorname{sh}^2 t + 1) dt = \int \frac{\operatorname{ch} 2t + 1}{2} dt = \boxed{\frac{\operatorname{sh} 2t}{4\cancel{t}} + \frac{t}{2} + C}$

Exercice 07

Forme Canonique d'un polynôme de degré 2 :

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right]$$

$$x^2+2x+3 = (x+1)^2 + 2 = 2\left[\left(\frac{x+1}{\sqrt{2}}\right)^2 + 1\right]$$

$$* \; I_1 = \int \frac{dx}{x^2+2x+3} = \frac{1}{2} \int \frac{dx}{\left(\frac{x+1}{\sqrt{2}}\right)^2+1}$$

On pose $u = \frac{x+1}{\sqrt{2}}$. D'où $\boxed{du = \frac{dx}{\sqrt{2}}}$

$$I_1 = \frac{1}{2} \int \frac{\sqrt{2}\,du}{u^2+1} = \frac{1}{\sqrt{2}} \operatorname{Arctg} u + C$$

$$= \boxed{\frac{1}{\sqrt{2}} \operatorname{Arctg}\left(\frac{x+1}{\sqrt{2}}\right) + C}$$

$$* \; I_2 = \int \frac{dx}{\sqrt{x^2+2x+3}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\frac{x+1}{2}\right)^2+1}} = \frac{1}{\cancel{\sqrt{2}}} \int \frac{\cancel{\sqrt{2}}\,du}{\sqrt{u^2+1}}$$

$$= \operatorname{Argsh} u + C = \boxed{\operatorname{Argsh}\left(\frac{x+1}{\sqrt{2}}\right) + C}$$

$$* \; I_3 = \int \sqrt{x^2+2x+3}\,dx = 2 \int \sqrt{u^2+1}\,(\sqrt{2})\,du = 2\sqrt{2} \int \sqrt{u^2+1}\,du$$

On pose ~~t≠~~ $u = \operatorname{sh} t$. D'où $du = \operatorname{ch} t\,dt = \sqrt{\operatorname{sh}^2 t+1}\,dt =$

$$= \sqrt{u^2+1}\,dt$$

$$I_3 = 2\sqrt{2}\int (u^2+1)\,dt = 2\sqrt{2}\int (\operatorname{sh}^2 t+1)\,dt = 2\sqrt{2}\int \operatorname{ch}^2 t\,dt \quad \left(\operatorname{ch}^2 - \operatorname{sh}^2 = 1\right)$$

On sait que $\operatorname{ch}^2 t = \frac{\operatorname{ch} 2t + 1}{2}$

$$I_3 = \cancel{2}\sqrt{2} \int \frac{\operatorname{ch} 2t+1}{\cancel{2}}\,dt = \sqrt{2}\left(\frac{1}{2}\operatorname{sh} 2t + t\right) + C$$

$$= \frac{\sqrt{2}}{2} \times 2 \operatorname{sh} t \operatorname{ch} t + \sqrt{2}\,t + C$$

$$= \frac{\sqrt{2}}{\cancel{2}} \times \cancel{2} \times u\sqrt{1+u^2} + \sqrt{2} \operatorname{Argsh} u + C$$

$$= \boxed{\sqrt{2}\left(\frac{x+1}{\sqrt{2}}\right)\sqrt{1+\frac{(x+1)^2}{2}} + \sqrt{2} \operatorname{Argsh}\frac{x+1}{\sqrt{2}} + C}$$

canonique d'un poly

Exercice 08

$* \int \frac{dx}{x(x+1)(x+2)}$ $D = \mathbb{R} \setminus \{-2, -1, 0\}$

* On a $\frac{1}{x(x+1)(x+2)} = \frac{a}{x} + \frac{b}{x+1} + \frac{c}{x+2}$ ...... (*)

Pour trouver a, On multiplie (*) par x, on obtient

$$\frac{1}{(x+1)(x+2)} = a + x\left(\frac{b}{x+1} + \frac{C}{x+2}\right)$$

$$x = 0 \Longrightarrow \boxed{\frac{1}{2} = a}$$

Pour trouver b, on multiplie (*) par x+1 :

$$\frac{1}{x(x+2)} = (x+1)\frac{a}{x} + b + \frac{c(x+1)}{x+2}$$

$$x = -1 \Longrightarrow \boxed{-1 = b}$$

Pour trouver c, on multiplie par x+2 :

$$\frac{1}{x(x+1)} = (x+2)\frac{a}{x} + \frac{b(x+2)}{x+1} + C$$

$$x = -2 \Longrightarrow \boxed{+\frac{1}{2} = C}$$

D'où : $\int \frac{dx}{x(x+1)(x+2)} = \int \left(\frac{1}{2x} + \frac{-1}{x+1} + \frac{1}{2(x+2)}\right) dx$

$$= \frac{1}{2}\ln|x| - \ln|x+1| + \frac{1}{2}\ln|x+2| + C(x)$$

où $C(x) = \begin{cases} c_1 & \text{sur }]-\infty, -2[\\ c_2 & \text{sur }]-2, -1[\\ c_3 & \text{sur }]-1, 0[\\ c_4 & \text{sur }]0, +\infty[\end{cases}$

~~Remarque: Pour Calculer a, b et c on peut calcu~~

* $\int \frac{1}{x(x+1)^3}\,dx$, $D = \mathbb{R} \setminus \{0,-1\}$

On d'après la règle de décomposition :

$$\frac{1}{x(x+1)^3} = \frac{a}{x} + \frac{b}{x+1} + \frac{c}{(x+1)^2} + \frac{d}{(x+1)^3} \text{ - - - } ⊛$$

- Pour trouver a, on multiplie par x et puis on remplace x par 0 :

On obtient $\boxed{1 = a}$

- Pour trouver d, on multiplie ⊛ par $(x+1)^3$:

$$\frac{1}{x} = (x+1)^3\frac{a}{x} + b(x+1)^2 + c(x+1) + d$$

$$x = -1 \Longrightarrow \boxed{-1 = d}$$

- Pour trouver **b**, on multiplie par $x+1$:

$$\frac{1}{x(x+1)} = a\frac{x+1}{x} + b + \frac{c}{x+1} + \frac{d}{(x+1)^2}$$

$$x \longrightarrow +\infty \Longrightarrow 0 = a + b + 0 + 0$$

$$\Longrightarrow \boxed{b = -a = -1}$$

- Pour trouve c, On substitue x par -2 (valeur simple à calculer)

$$+\frac{1}{2} = -\frac{a}{2} - b + c - \frac{d}{[illegible]}$$

D'où $c = \frac{1}{2} + \frac{a}{2} + b + d = \boxed{-1}$

Alors : $\int \frac{dx}{x(x+1)^3} = a\ln|x| + b\ln|x+1|$ [illegible] $+ \frac{-c}{x+1} - \frac{d}{2(x+1)^2} + C(x)$

où $C(x) = \begin{cases} C_1 \text{ sur }]-\infty,-1[\\ C_2 \text{ sur }]-1,0[\\ C_3 \text{ sur }]0,+\infty[\end{cases}$

$$* \quad \frac{1}{x(x^2+1)} = \frac{a}{x} + \frac{bx+c}{x^2+1} \quad \ldots \ldots \circledast$$

$$\circledast \Rightarrow \frac{1}{x^2+1} = a + x\left(\frac{bx+c}{x^2+1}\right) = a + \frac{bx^2+cx}{x^2+1}$$

$$\xrightarrow{x=0} \boxed{1 = a}$$

$$\xrightarrow{x \to +\infty} 0 = a + b \Rightarrow \boxed{b = -a = -1}$$

$$\xrightarrow{x=1} \frac{1}{2} = a + \frac{b+c}{2} \Rightarrow c = 0$$

$$\int \frac{dx}{x(x^2+1)} = \int \frac{dx}{x} + \int \frac{-x\,dx}{x^2+1}$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + C(x)$$

$$\text{avec} \quad C(x) = \begin{cases} c_1 & \text{sur }]-\infty, 0[\\ c_2 & \text{sur }]0, +\infty[\end{cases}$$

$$* \int \frac{e^{2x}}{1+e^{x}}\,dx = \int \frac{t}{1+t}\,dt \qquad \left(t = e^{x},\ dt = e^{x}dx\right)$$

$$= \int \left(1 - \frac{1}{1+t}\right) dt$$

$$= t - \ln|1+t| + C, \quad C \in \mathbb{R}$$

$$= \boxed{e^{x} - \ln|1+e^{x}| + C}$$

$$* \int \frac{\sqrt{x}}{\sqrt[3]{x}+1}\,dx = \int \frac{\sqrt[2]{t^6}}{\sqrt[3]{t^6}+1}\,6t^5\,dt \qquad \left(\begin{array}{l} x = t^6,\ dx = 6t^5dt \\ 6 = \text{ppcm}(2,3) \end{array}\right)$$

$$= \int \frac{6t^8}{t^2+1}\,dt, \qquad \begin{array}{r|l} t^8 & t^2+1 \\ \hline -\ \ t^8+t^6 & t^6 - t^4 + t^2 - 1 \\ \hline -t^6 & \\ -t^6 - t^4 & \\ \hline +t^4 & \\ t^4 + t^2 & \\ \hline -t^2 & \\ -t^2 - 1 & \\ \hline +1 & \end{array}$$

$$= 6\int \left[\left(t^6 - t^4 + t^2 - 1\right) + \frac{1}{t^2+1}\right] dt$$

$$= \frac{6}{7}t^7 - \frac{6}{5}t^5 + \frac{6}{3}t^3 - \frac{6}{1}t + \text{Arctg}\,t + C$$

$$= \boxed{\sqrt[6]{x}\left(\frac{6}{7}x - \frac{6}{5}x^{\frac{4}{6}} + 2\sqrt[3]{x} - 6\right) + \text{Arctg}\sqrt[6]{x} + C}$$

$$* \int \frac{3+\ln x}{(4+\ln x)^2}\,dx = \int \frac{3+t}{(4+t)^2}\,e^{t}\,dt \qquad \left(\begin{array}{l} x = e^{t},\ dx = e^{t}dt \\ t = \ln x \end{array}\right) \quad |x|$$

$$= \int \left(\frac{e^{t}}{4+t}\right)' dt = \frac{e^{t}}{4+t} + C(x)$$

$$= \frac{x}{4+\ln x} + C(x)$$

$$\text{où } C(x) = \begin{cases} c_1 & \text{sur }]-\infty, e^{-4}[\\ c_2 & \text{sur }]e^{-4}, +\infty[\end{cases}$$

Exercice 09

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\, dx, \quad J = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\, dx$$

On a $I + J = \int_0^{\frac{\pi}{2}} 1\, dx = \frac{\pi}{2}$

$$I - J = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\sin x + \cos x}\, dx = \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)'}{\sin x + \cos x}\, dx$$

$$= \Big[\ln(\sin x + \cos x) \Big]_0^{\frac{\pi}{2}} = 0$$

$$\begin{cases} I + J = \frac{\pi}{2} \\ I - J = 0 \end{cases} \Rightarrow \begin{cases} I = \frac{\pi}{4} \\ J = \frac{\pi}{4} \end{cases}$$