Corrigé-type

Exercice 01(5 pts)
Le système caractéristique est

$$\begin{cases} \dfrac{\partial t}{\partial s} = 1 \\ \dfrac{\partial x}{\partial s} = 1 \\ \dfrac{\partial u}{\partial s} = u^2 \end{cases} \quad ...(0.75) \quad \Rightarrow \begin{cases} t = s + c_1 \\ x = s + c_2 \\ -\frac{1}{u} = s + c_3 \end{cases} \quad ...(0.75)$$

de la condition initiale

$$\begin{cases} t(0,\tau) = 0 = c_1 \\ x(0,\tau) = \tau = c_2 \\ u(0,\tau) = \ln(1+\tau^2) = -\frac{1}{c_3} \end{cases} \quad ...(0.75) \quad \Rightarrow \begin{cases} t(s,\tau) = s \\ x(s,\tau) = s + \tau \\ u(s,\tau) = -\frac{1}{s - \frac{1}{\ln(1+\tau^2)}} \end{cases} \quad ...(0.75)$$

puisque $J = \begin{vmatrix} \frac{dt}{dS} & \frac{dt}{d\tau} \\ \frac{dx}{dS} & \frac{dx}{d\tau} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0...(1)$
donc, on peut écrire s et τ en fonction de t et x

$$\begin{cases} s = t \\ \tau = x - t \end{cases} \quad ...(0.5)$$

d'où

$$u(x,y) = -\frac{1}{t - \frac{1}{\ln(1+(x-t)^2)}} ...(0.5)$$

Exercice 02(10 pts)
1)

$$x^2 u_{xx} - 2xyu_{xy} + y^2 u_{yy} + xu_x + yu_y = 0$$

$\Delta = (-2xy)^2 - 4(x^2)(y^2) = 0, ...(0.5)$
alors l'équation est parabolique sur tout $\mathbb{R} \times \mathbb{R}$...(0.5)
L'équation caractéristique est

$$\frac{dy}{dx} = \frac{B}{2A} = \frac{-2xy}{2(x^2)} = -\frac{y}{x} ...(0.5) \Rightarrow \frac{dy}{dx} = -\frac{y}{x} \Rightarrow xy = c...(0.5)$$

Les courbes caractéristiques associées est donc

$$\begin{cases} \zeta(x,y) = xy \\ \eta(x,y) = x \end{cases} \quad ...(0.5)$$

Par la règle de chaînes, on a

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = y\frac{\partial u}{\partial \zeta} + \frac{\partial u}{\partial \eta} ...(0.5)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u_x}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial x} = y(yu_{\zeta\zeta} + u_{\zeta\eta}) + yu_{\zeta\eta} + u_{\eta\eta} ...(0.75)$$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = x\frac{\partial u}{\partial \zeta}$...(0.5)

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial u_y}{\partial \zeta}\frac{\partial \zeta}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y} = x(xu_{\zeta\zeta} + u_{\zeta\eta}) + xu_{\zeta\eta} + u_{\eta\eta} = x^2 u_{\zeta\zeta}$...(0.75)

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u_y}{\partial x} = \frac{\partial}{\partial x}(x\frac{\partial u}{\partial \zeta}) = \frac{\partial u}{\partial \zeta} + x\frac{\partial}{\partial x}\frac{\partial u}{\partial \zeta} = u_\zeta + xy \ u_{\zeta\zeta} + xu_{\zeta\eta} =$ (0.5)

d'où la forme canonique de cette équation est donnée par

$$\frac{\partial^2 u}{\partial \zeta^2} = -\frac{1}{\zeta}\frac{\partial u}{\partial \zeta}...(1)$$

Si $x = y = 0$, l'équation est vérifiée en tout les points sauf l'origine

Si $x \neq 0$,et $y \neq 0$ on pose $w = \frac{\partial u}{\partial \zeta}$, d'où

$$\begin{aligned} w' &= -\frac{1}{\zeta}w \Rightarrow \ln w = -\ln(\zeta)w + f(\eta) \\ &\Rightarrow \quad w = \frac{f(\eta)}{\zeta} \end{aligned}$$

donc

$$u = \int w d\zeta = \int \frac{f(\eta)}{\zeta} d\zeta = f(\eta)\ln(\zeta) + g(\eta)$$

Par conséquent, la solution générale est

$$u(x, y) = f(x)\ln(xy) + g(x)$$

où $f, g \in C^2(R)$ sont des fonctions réelles arbitraires.

Exercice 03 (5 pts)

D'après le principe du maximum la solution u atteint le max et le min sur la frontière de Ω ; ce qui implique

$$\max_{(x,y)\in\bar{\Omega}} u(x, y) \leq u(x, y) \leq \min_{(x,y)\in\bar{\Omega}} u(x, y), \ \forall(x, y) \in \bar{\Omega}....(0.5)$$

$$\min_{x\in[0,1]} u(x, y) = \min_{x\in[0,1]}\{0, x - \ln(2x + 1), \cos(x)(x^2 + 6x - 1)\}...(0.25)$$

$$\max_{x\in[0,1]} u(x, y) = \max_{x\in[0,1]}\{0, x - \ln(2x + 1), \cos(x)(x^2 + 6x - 1)\}...(0.25)$$

Calcule de

$f1(x) = x - \ln(2x + 1)$

$f1'(x) = 1 - 2/(2x + 1) = 0 \Rightarrow x = \frac{1}{2}$...(0.5)

$0 < f1 < -0.1931$...(1)

$f2(x) = \cos(x)(x^2 + 6x - 1)$

$f2'(x) = \cos(x)(2x + 6) - \sin(x)(x^2 + 6x - 1) > 0 \Rightarrow$ ....(0.5)

$-1 < f2(x) < 3.241813835208839$...(1)

d'où

$$\max_{x\in[0,1]} u(x,y) = \max_{x\in[0,1]} \{0, -0.1931, -1, 3.2418\} = 3.2418...(0.5)$$

$$\min_{x\in[0,1]} u(x,y) = \min_{x\in[0,1]} \{0, -0.1931, -1, 3.2418\} = -1...(0.5)$$