LOGIC AND MATHEMATICAL REASONING

1.1 LOGIC

1.1.1 Assertion

In mathematical logic a ststement (assertion) or proposition is a setence that is either true or false, but not both.

Examples

* Every rectangle is a square.

* The sun rises in the Easte and sets in the West.

* 2+2=5.

*$2 > 1$.

* "The square root of 9 is 3." is a true statement.

Remark: Some sentences that do not have a truth value or may have more than one truth value are not proposition.

For example

* What time is it?.......(Question).

* Go out and play.......(command).

* 7+16........................(this is not even a sentence).

* $x^2 > 0$.......(this depends on the value of x).

1.1.2 Logical Connectives

Logical connectives are the operators used to compine one or more propositions. In propositionall logic there are 5 basic connectives.

Let P and Q be propositions:

•$\overline{P}$ means not P, called a ***negation.***

•$P \wedge Q$ means P and Q, called a ***conjunction***.

•$P \vee Q$ means P or Q, called a ***disjunction***.

•$P \Longrightarrow Q$ means if P, then Q or (P implies Q), called an ***implication*** or ***conditional***.

•$P \Longleftrightarrow Q$ means P if and only if Q or (P is equivalent to Q), called a ***equivalent*** or ***biconditional***.

Truth Conditions for Connectives

•$\overline{P}$ is true when P is false.

•$P \wedge Q$ is true when both P and Q are true.

•$P \vee Q$ is true when P or Q or both are true.

•$P \Longrightarrow Q$ is true when P is false or Q is true.

•$P \Longleftrightarrow Q$ is true when P and Q are both true, or both false.

The truth table is as follow:

P	Q	$\overline{\text{P}}$	P∧Q	P∨Q	P⟹Q	P⟺Q
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Propositions: Let P and Q be two assertions then:

1)* $P \Longrightarrow Q \Longleftrightarrow \overline{P} \vee Q$.

*$\overline{P \Longrightarrow Q} \Longleftrightarrow P \wedge \overline{Q}$.

*$\overline{\overline{P}} \Longleftrightarrow P$.

2) De Morgan's laws:

* $(\overline{P \wedge Q}) \Longleftrightarrow (\overline{P} \vee \overline{Q})$.

* $(\overline{P \vee Q}) \Longleftrightarrow (\overline{P} \wedge \overline{Q})$.

Exercise

Consider the following statements, wich ones are true, wich ones are false? Provide their negation.

*$(1 < 2) \wedge (4 < 3)$. (false)

*$(\sqrt{30} > 6) \vee (\sqrt{30} > 5)$. (true)

*$b^2 - 4ac > 0 \Longrightarrow ax^2 + bx + c = 0$, has two distinct real solutions. (true)

*$2+3=5 \Longrightarrow 1=0$. (false)

*$x^2=4 \Longleftrightarrow x=2$. (false)

*$x>y \Longleftrightarrow x-y>0$.(true)

inverse, Converse, and Contrapositive

Let P and Q be two statements :

1) The **inverse** of $P \Longrightarrow Q$ is the proposition $\overline{P} \Longrightarrow \overline{Q}$.

2) The **converse** of $P \Longrightarrow Q$ is the proposition $Q \Longrightarrow P$.

3) The **contrapositive** of $P \Longrightarrow Q$ is the proposition $\overline{Q} \Longrightarrow \overline{P}$.

1.1.3 Quantifiers

As mentioned before, the phrase "$x^2>0$" is not a mathematical statement (unless we assign a value to x.

However, we can turn it into a statement as follows.

(i) "For all real numbers x, we have $x^2>0$." or (ii) "There exists a real number x such that $x^2>0$."

Each of the phrases (i) and (ii) are mathematical statements. Statement (i) is false, as not every real number satisfies $x^2>0$ (if $x=0$, then $x^2=0$). On the other hand, statement (ii) is true, as $x^2>0$ does hold for some real numbers.

Words such as "**for all**", "**for every**", "**for each**", "**there is**", "**there exists**", "**for some**", etc. are called **quantifiers**, and are used to turn **predicates** (sentences with variables) into statements. Quantifiers play an essential role in building mathematical statements. As we have just seen, replacing one quantifier by another can change the meaning of the statement.

1)The universal quantifier is "$\forall$". The sentence "$\forall x : P(x)$" is read "**for all"** or "**for every"**or "**for each"** $x, P(x)$.

Example: "$\forall x \in \mathbb{R},\ x^2 \geq 0$ ". Would read as "**for all** real numbers x, x squared is greater than or equal to 0".

2)**The existential quantifier** is "$\exists$". Is read "**there is**" or "**there exists**".

Exemple: "$\exists x \in \mathbb{R}, x \leq 2$" is read as "**there exists** a real number x that is less than or equal to 2."

Remark

$*$ The negation of "$\forall x : P(x)$" is "$\exists x : \overline{P(x)}$".

$*$The negation of "$\exists x : P(x)$" is "$\forall x : \overline{P(x)}$".

$*$We sometimes encounter the symbol $\exists!$, which means "there exists a unique." For example: $\exists! x \in \mathbb{R}$: $x^3 = 1$.

This symbol is used to indicate that there is one and only one element satisfying a particular condition. In the given example, it means "there exists a unique real number x such that $x^3 = 1$".

The unique real number satisfying this equation is 1, since $1^3 = 1$.

Lecture exercises 6+7

Exercise 06

Consider the following three statements:

a) For every natural number n. if n is even, then n^2 is also even.

b) there exists a real number x, less than or equal -2.

c) There is one and only one natural number such that: $n + 1 = 3$.

1-Write the following statements using quantifiers?.

2-which ones are true, which ones are false.

3-Give the negation of (a) and (b).

Exercise 07

P, Q and R are three propositions.

Write the converse, negation and contrapositive of:

$P \Longrightarrow (Q \wedge R)$.

1.2 Modes of reasoning

1.2.1 Direct proof

A direct proof is one of the most familiar forms of proof. We use it to prove statements of the form "if P then Q" or "P implies Q", which we can write as $P \implies Q$.

To prove $P \implies Q$ is true by direct proof:

Assume P is true. Explain, explain, ... , explain. Therefore Q is true.

Example: prove that all integers n, if n is odd, then n^2 is also odd.

Proof: Let P be the statement that n is an odd integer, and Q be the statement that n^2 is an odd integer.

Assume that n is an odd integer, then by definition, $n = 2k+1$ for some integer k. Then: $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

since $2k^2 + 2k$ is an integer and $n^2 = 2(2k^2 + 2k) + 1$,we see that n^2 is odd. Completing the proof.

1.2.2 Proof by Cases

Proof by cases is a method of mathematical proof where you consider different cases to establishe the validity of a statement. Here's an example:

Lets prove by cases that, for all $n \in \mathbb{N}$, $n^2 + 3n + 7$ is odd.

Proof: If $n \in \mathbb{N}$, then either n is even or n is odd.

Case 1: If n is even, then $n = 2k$ for some $k \in \mathbb{N}$. Therefore, $n^2+3n+7 = (2k)^2+3(2k)+7$ $= 2(2k^2 + 3k + 3) + 1 = 2l + 1$.

Where $l = 2k^2 + 3k + 3$. So since $2k^2 + 3k + 3 \in \mathbb{N}$, we have $n^2 + 3n + 7$ is odd.

case 2: similary, if n is odd, then $n = 2k + 1$ for some integer k. Thus $n^2 + 3n + 7 = (2k + 1)^2 + 3(2k + 1) + 7 = 2(2k^2 + 5k + 5) + 1 = 2t + 1$, where $t = 2k^2 + 5k + 5$. Since $2k^2 + 5k + 5 \in \mathbb{N}$,we have $n^2 + 3n + 7$ is odd.

We can conclude that for any integer n, if n is even or odd, then $n^2 + 3n + 7$ is odd.

This is an example of a proof by cases where we consider two cases (even and odd) to find the desired result.

1.2.3 Proof by contrapositive

The reasoning by contrapositive is based on the following equivalence:

The statement "$P \Longrightarrow Q$ "is equivalent to "$\overline{Q} \Longrightarrow \overline{P}$".

To prove $P \Longrightarrow Q$ is true by contrapositive:

assume $\overline{Q}$ is true. Explain, explain,.....explain. Therfore $\overline{P}$ is true.

Example: prove for all integers a and b, if $a+b$ is odd, then a is odd or b is odd.

$\forall a, b \in \mathbb{Z}$, ($a+b$ is odd $\Longrightarrow$ a is odd **or** b is odd)$\Longleftrightarrow$ (a is ever **and** b is even $\Longrightarrow a+b$ is even)(contrapositive).

proof: Let a and b be integers. Assume that a and b are even. Then $a = 2k$ and $b = 2l$ for some integers k and l. Now, let's find the sum of a and b : $a + b = 2k + 2l = 2(k + l)$. Since $k + l$ is an integer, we see that $a + b$ is even. Completing the proof.

1.2.4 Proof by contradiction (indirect proof)

In this method of proof, we assume a proposition is not true, then through that premise and logic find a contradiction that shows our original premise must have been incorrect, and therefore, the proposition was true.

*For one proposition P, assume $\overline{P}$ is true, then find a contradiction that shows $\overline{P}$ is false, so P is true.

*For an implication $P \Longrightarrow Q$, assume P and $\overline{Q}$ are true ($\overline{P \Longrightarrow Q} \Longleftrightarrow P \wedge \overline{Q}$), then find a contradiction that shows either $P \Longrightarrow Q$ or $\overline{Q} \Longrightarrow \overline{P}$.

Example 01: prove by contradiction $\sqrt{2}$ is irrational.(P)

Assume $\sqrt{2}$ is rational ($\overline{P}$). Then there exists 2 integers a and b such that $\sqrt{2} = \dfrac{a}{b}$, $b \neq 0$ and a and b have no common factors.

$$\sqrt{2} = \frac{a}{b}$$

$$2 = \frac{a^2}{b^2}$$
$$2b^2 = a^2$$

Therefore a^2 must be even. If $a^2$is even then a must be even. Since a is even, $a = 2c$, for some $c \in \mathbb{Z}$. Then we have :

$$2b^2 = (2c)^2.$$
$$2b^2 = 4c^2.$$
$$b^2 = 2c^2.$$

Therefore b must be even. Since a and b are both even, they have a common factor. (contradiction).

Therefore, $\sqrt{2}$ is irrational.

Example 02: prove if $3n + 2$ is even ($n \in \mathbb{Z}$), then n is even.

Assume $3n + 2$ is even, and n is odd ($n \in \mathbb{Z}$),Since $3n + 2$ is even, then $3n$ is even by subtraction of two even integers. Then $3n - n$ must be odd by subtractionof an even and odd integer. However $3n - n = 2n$, which is even by definition which is a contradiction our original assumptions.

Therefore: if $n \in \mathbb{Z}$ and $3n + 2$ is even, then n is even.

1.2.5 Proof by counterexample

A counterexample is an example that shows that a statement is not always true. It is sufficient to just give one example.

Examples: show by counter-example that the following statements are not always true:

1) For any prime number p, $2p + 1$ is also prime.

When $p = 7, 2p + 1 = 15$.

Therefore, the statement (1) is untrue.

2)For all values of x and y, $(x + y)^2 \geq x^2 + y^2$.

Whene $x = 1$ and $y = -1$: $(1 + -1)^2 = 0$, and $(1)^2 + (-1)^2 = 2$.

$0 \geq 2$ is wrong. Therefore the statement (2) is untrue.

1.2.6 Proof by induction

A proof by induction of $P(n)$, a mathematical statement involving a value n, involves these main steps:

•Prove directly that P is correct for the initial value of n (for most examples you will see this is zero or one). This is called the **Basis Step**.

•Assume for some value k that $P(k)$ is correct. This is called the **Induction Hypothesis**.

• We will now prove directly that $P(k) \implies P(k+1)$. That means prove directly that $P(k+1)$ is correct by using the fact that $P(k)$ is correct. This is called the **Induction Step**.

The compination of these steps shows that $P(n)$ is true for all values of n.

Example

Prove by induction that : $1+2+3+...+n = \dfrac{n(n+1)}{2}, \forall n \in \mathbb{Z}_{+} ......P(n)$

Solution

1) **Basis:** when $n = 1$ the left hand side of the equation is 1 and the right hand side is $\dfrac{1(1+1)}{2} = 1$. So $P(1)$ is correct.

Induction Hypothesis: Assume that $P(k) : 1+2+3+...+k = \dfrac{k(k+1)}{2}$. is correct for some positive integer k.

Induction Step: We want to show that

$P(k+1) : 1+2+3+...+k+(k+1) = \dfrac{(k+1)(k+2)}{2}$ is correct.

So starting withe the left hand side we have:

$$1+2+3+...+k+(k+1) = (1+2+3+...+k)+(k+1)$$
$$= \frac{k(k+1)}{2} + (k+1) \text{ by the induction hypothesis.}$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$ since the numerator has a common factor of $(k+1)$, and this is the right hand side.

So $P(k+1)$ is correct. Hence by mathematical induction $P(n)$ is correct for all positive integers n.