Chapter 1

Logical Concepts

At the intersection of philosophy and mathematics, logic is a fundamental branch that enables the determination of the truth value of propositions and the construction of mathematical reasoning.

This document serves as an introduction to this crucial branch of mathematics. We will define the concepts of proposition and operator, construct truth tables, explain implications, reciprocal implications, and equivalence, before delving into the various types of reasoning used in mathematics.

1.1 Definition

A logical proposition (or assertion) is a statement formed by combining symbols and words, concerning mathematical objects, to which a clear truth value, either true or false, can be assigned.

Let P be a proposition.

By definition, P satisfies the following three principles (or axioms):

- Principle of Identity: P is P

  In other words, if P is true, then P is true, and if P is false, then P is false.

- Principle of Non-contradiction: P cannot be both true and false simultaneously.

- Principle of the Excluded Middle: Either P is true, or P is false.

There is no other truth value in mathematical logic.

These three principles form the foundation of all mathematical reasoning. The last point deserves a moment of attention:

Let P be the proposition ”The square of a real number is strictly positive.”

So, is it true or false?

The initial intuition might be to say, ”It depends on the number.” This is true for most numbers, but it is false for zero (since 0^2 is not greater than 0).

The problem is that this response contradicts the Principle of the Excluded Middle. Therefore, it is necessary to unambiguously assign either the value of true or the value of false to this proposition.

Given that there is at least one number (in this case, zero) for which this proposition is false, we will say that proposition P is false.

1.1.1 Some Examples

$\boldsymbol{P}_1$: ”The number of letters in the French alphabet is 10.”

The proposition $\boldsymbol{P}_1$ is false.

$\boldsymbol{P}_2$**:** ”$2 + 2 = 4$”

The proposition $\mathbf{P}_2$ is true.

$\mathbf{P}_3$**:** ”x>1”

$\boldsymbol{P}_3$ is not a complete logical proposition because it contains a free variable x. We do not know what x represents (a point? an integer? a vector? a star in the universe?). Therefore, we cannot assign a truth value to the proposition $\mathbf{P}_3$.

$\boldsymbol{P'}_3$: ”Let x be a real number, then $x > 1$”

The proposition $\boldsymbol{P'}_3$ is false. Indeed, $\boldsymbol{P'}_3$ is a logical proposition because we have defined the variable x as a real number. However, it is false because, for example, 0 is a real number and $0 < 1$.

Here, a counterexample is used to prove that the proposition $\boldsymbol{P'}_3$ is false.

(This type of reasoning will be further explored later).

Key Takeaways

Logical propositions can only take two values: TRUE or FALSE (hence the name bivalent logic).

It is important to distinguish between a proposition (which is a sentence) and its truth value (which is either TRUE or FALSE). We say that proposition p is false.

1.2 Basic Operators

Operators allow us to construct new propositions from one or more initial propositions.

Let's start with the first (and simplest!) one, the "NOT" operator.

1.2.1 Negation (not): Learning to Say No!

Let **P** be a proposition. We define a proposition "not **P**" which is denoted as "$\neg$**A**" (with a sort of small L elongated downwards) or simply as $\overline{\boldsymbol{P}}$.

If $\boldsymbol{P}$ is true, then $\overline{\boldsymbol{P}}$ is false.

If $\boldsymbol{P}$ is false, then $\overline{\boldsymbol{P}}$ is true.

For those who do programming, the "NOT" operator (denoted as in math) is often written as "!" in computer science.

We can establish the truth table for the negation operator based on its definition.

Definition. A truth table is a table that defines the value of a logical function for each possible combination of inputs.

Explanation. In the first column, we list all the possible values of A (i.e., True or False). In the second column, we place the corresponding truth value of $\bar{A}$.

By convention, and to facilitate the reading of large tables, we write $\boldsymbol{F}$ for the value $FALSE$ and $\boldsymbol{V}$ for the value $TRUE$.

$\boldsymbol{P}$	$\overline{\boldsymbol{P}}$
$\boldsymbol{V}$	$\boldsymbol{F}$
$\boldsymbol{F}$	$\boldsymbol{V}$

It is important to understand how to construct a truth table as we will use it many times in this course.

This connector is quite intuitive as we use it in our daily lives.

Some Examples

P : "Algiers is the capital of Algeria"	(its value is $\boldsymbol{V}$)
$\bar{P}$: "Algiers is not the capital of Algeria"	(its value is $\boldsymbol{F}$)
Q : "π is an integer"	(F)
$\bar{Q}$: "π is not an integer"	(V)
R : "5 is an odd number"	(V)
$\bar{R}$: "5 is an even number"	(F)

This first operator should now seem quite simple to you. In order to construct logical reasoning, we need to use operators that link two logical propositions together (these are called **binary operators**).

1.2.2 Conjunction "and", denoted $\wedge$

Let P and Q be two propositions.

We define a new proposition "P AND Q" which is denoted as "$P \wedge Q$". This new proposition is:

True when both P and Q are true.

False in all other cases.

From this definition, we can derive the truth table for the proposition "$P \wedge Q$":

P	Q	$P \wedge Q$
V	V	V
V	F	F
F	V	F
F	F	F

The first two columns list all possible cases for the truth values of P and Q. The last column corresponds to the truth value of the proposition "$P \wedge Q$".

It is important to understand the truth table of the "AND" operator as it is used in many logical reasoning.

Some Examples

Example 1: "5 is a number less than 10 **and** 5 is even."

Let $\boldsymbol{P}$: "5 is a number less than 10." $\boldsymbol{P}$ is true.

Let $\boldsymbol{Q}$: "5 is even." $\boldsymbol{Q}$ is false.

The proposition A is the proposition "$P \wedge Q$".

According to the truth table of the "AND" operator, we conclude that proposition A is false.

Example 2: "The letter A is a vowel and T is a consonant."

By reasoning in the same way, we conclude that proposition **B** is true.

1.2.3 Disjunction "or", denoted $\vee$

The second binary operator we are going to study is the "OR" operator.

Let P and Q be two propositions.

We define a new proposition "P or Q" which is denoted as "$P \vee Q$".

This proposition is:

False when both P and Q are false.

True otherwise.

The truth table for the proposition "$P \vee Q$" is as follows:

P	Q	$P \vee Q$
V	V	V
V	F	V
F	V	V
F	F	F

In other words, the proposition "$P \vee Q$" is true only if either P or Q is true (or both!).

Example: "5 is a number less than 10 OR 5 is even"

What is the truth value of this proposition?

Solution: Let $\boldsymbol{P}$: "5 is a number less than 10". It is true.

Let $\boldsymbol{Q}$: "5 is even". It is false.

The proposition "$\boldsymbol{P} \vee \boldsymbol{Q}$"

According to the truth table of the "OR" operator, the proposition in the example is true.

The binary operators "NOT," "AND," and "OR" are the most important in mathematics because they allow us to define all other operators.

We are now at the heart of the matter! Indeed, implications and equivalences are used in the majority of mathematical proofs. Understanding them well allows us to avoid reasoning errors in exams... and in life too!

1.2.4 Notion of Implication "$\Rightarrow$"

Implication is a binary operator (i.e., it connects two propositions).

Let P and Q be two propositions.

We write $P \Rightarrow Q$ (and read "P implies Q").

Multiple Wordings for the Same Concept

$P \Rightarrow Q$ can also be read as:

✓ If P, then Q

✓ It is sufficient for P to have Q

✓ It is necessary for Q to have P

✓ Q is required for P

Hence, every time you hear one of these wordings in everyday language, it is actually an implication.

Example:

"I am joyful if he is here" corresponds to "He is here" $\Rightarrow$ "I am joyful"

"It is raining" $\Rightarrow$ "The ground is wet". If it is raining, then the ground is wet. It means that it is impossible for it to rain and the ground not to be wet.

"If I am tired, I will rest." This means that "I am tired" $\Rightarrow$ "I will rest."

Let P and Q be two propositions.

We define a new proposition "$P \Rightarrow Q$" (read as "P implies Q").

This proposition is:

False when P is true and Q is false.

True otherwise.

The truth table for the proposition "$P \Rightarrow Q$" is as follows:

P	Q	$P \Rightarrow Q$
V	V	V
V	F	F
F	V	V
F	F	V

In other words, the proposition "$P \Rightarrow Q$" is false only when P is true and Q is false.

1.2.5 Reciprocal Implication "$\rightleftharpoons$"

Here's one more thing. P and Q are still two propositions. The proposition $Q \Rightarrow P$ is called the reciprocal implication of the proposition $P \Rightarrow Q$. Remember this expression, we will use it again shortly!

1.2.6 Equivalence "$\Leftrightarrow$"

The symbol for equivalence is $\Leftrightarrow$, a double arrow that resembles the implication arrow discussed earlier.

Let P and Q be two propositions.

We define a new proposition "$P \Leftrightarrow Q$" which is read as "P is equivalent to Q".

Alternatively, it can be read as "if and only if Q".

It is also understood as "the implication $P \Rightarrow Q$ and the reciprocal implication $Q \Rightarrow P$".

This proposition has the following truth conditions:

True when P and Q have the same truth value (both true or both false).

False otherwise.

P	Q	$P \Leftrightarrow Q$
V	V	V
V	F	F
F	V	F
F	F	V

When proving an equivalence, the double implication rule is often employed:

✓ First, we establish one direction of implication,

✓ then we prove the reciprocal implication.

Avoid Confusion!

Do not confuse implications and equivalences.

Whenever determining the truth value of an equivalence, be sure to check the truth value of the double implication.

1.2.7 Properties

1. $(P_1 \Leftrightarrow P_2) \Leftrightarrow (P_1 \Rightarrow P_2) \wedge (P_2 \Rightarrow P_1)$

2. $\overline{\overline{P}}_1 \Leftrightarrow P_1$

3. $P_1 \vee P_1 \Leftrightarrow P_1$

4. $P_1 \vee P_1 \Leftrightarrow P_1$

5. $\overline{P_1 \vee P_2} \Leftrightarrow \overline{P_1} \wedge \overline{P_2}$

6. $\overline{P_1 \wedge P_2} \Leftrightarrow \overline{P_1} \vee \overline{P_2}$

7. $P_1 \wedge (P_2 \wedge P_3) \Leftrightarrow (P_1 \wedge P_2) \wedge P_3$

8. $P_1 \vee (P_2 \vee P_3) \Leftrightarrow (P_1 \vee P_2) \vee P_3$

9. $P_1 \wedge (P_2 \vee P_3) \Leftrightarrow (P_1 \wedge P_2) \vee (P_1 \wedge P_3)$

10. $P_1 \vee (P_2 \wedge P_3) \Leftrightarrow (P_1 \vee P_2) \wedge (P_1 \vee P_3)$

11. $\overline{(P_1 \Rightarrow P_2)} \Leftrightarrow P_1 \wedge \overline{P_2}$

12. $P_1 \Rightarrow P_2 \Leftrightarrow \overline{P_2} \Rightarrow \overline{P_1}$ "Law of contrapositive"

Proof

✓ Let's prove property 1 : $\overbrace{(P_1 \Leftrightarrow P_2)}^{(1)}\overbrace{(P_1 \Rightarrow P_2) \wedge (P_2 \Rightarrow P_1)}^{(2)}$

We use the truth table.

P_1	P_2	$P_1 \Rightarrow P_2$	$P_2 \Rightarrow P_1$	(1): $P_1 \Leftrightarrow P_2$	$(2) : (P_1 \Rightarrow P_2) \wedge (P_2 \Rightarrow P_1)$	$(1) \Leftrightarrow (2)$
V	V	V	V	V	V	$\boldsymbol{V}$
V	F	F	V	F	F	$\boldsymbol{V}$
F	V	V	F	F	F	$\boldsymbol{V}$
F	F	V	V	V	V	$\boldsymbol{V}$

1.3 Quantifiers

Let P be the proposition ”8 is an even number”. We can replace the number 8 with any other number to form new propositions. For example, we can write the proposition $P(6)$ as ”6 is an even number”, which is true, or the proposition $P(3)$ as ”3 is an even number”, which is false.

We can then write the general form of this proposition as

$P(x)$: ”x is an even number”, where x is called the argument of the proposition P. The truth value of the proposition $P(x)$ depends on x.

The problem is that I don’t know what x is in the proposition $P(x)$. In our example, x is a number, but it needs to be specified because otherwise our proposition doesn’t make sense (for example, $P(ABC)$: ”Triangle ABC is an even number” doesn’t make sense).

Therefore, we have invented quantifiers to indicate that we take our x from a determined set.

1.3.1 The Universal Quantifier ”$\forall$”

We write ”for all x element of E, the proposition $P(x)$ is true” as ”$\forall x$ in E, $P(x)$.”

Hold on! What are all these symbols?!

Stay calm, stay calm. You quickly get used to reading these mathematical symbols.

✓ The symbol $\forall$ (a reversed A) is read as ”for all” or ”for every.” It is a quantifier that indicates that the property is true for all objects satisfying the given condition.

✓ x is a mathematical object (a number, a point, a vector...).