Contents

1 Real numbers

1.1 Preliminaries

Definition 1.1.
1. A set is a well-defined collection of distinct objects, called the elements or members of the set. Sets may be finite or infinite. They are typically denoted by curly braces { } and listing the elements separated by commas.
2. The empty set denoted by ϕ is a set that has no elements.
3. If x is an element of the set A, we write $x \in A$, if not we write $x \notin A$.
4. A set A is subset of B or A is included in B if every element of A belongs to B and we write $A \subset B$, that is,

$$x \in A \Longrightarrow x \in B.$$

5. Tow sets A and B are equals if its have the same elements and we write $A = B$. In other terms $A = B$ if $A \subset B$ and $B \subset A$, or

$$x \in A \Longleftrightarrow x \in B.$$

☞ **Example 1.1.**
- $A = \{1, 2, 3\}$ is a set containing the members 1, 2, and 3 (finite set).
- $A = \{0, 2, 4, 6, ...\}$ is a set of positive even integers (infinite set).
- $A = \{\frac{n^2+1}{n+1} \mid n \in \mathbb{N}\}$ is a set where the element are given by the expression $\frac{n^2+1}{n+1}$ for all $n \in \mathbb{N}$. We have $0 \notin A$, $1 \in A$ because $1 = \frac{1^2+1}{1+1}$, $2 \notin A$ because $2 \neq \frac{n^2+1}{n+1}$ for all $n \in \mathbb{N}$.
- $A = \{x \in \mathbb{R} : x^2 + 3x + 1 \leq 0\}$ is a set containing the solutions of the inequality $x^2 + x + 1 \leq 0$. For example, $0 \notin A$ because $0^2 + 3 \times 0 + 1 = 1 \not\leq 0$, $-1/2 \in A$ because $(-1/2)^2 + 3(-1/2) + 1 = -1/4 \leq 0$.

Definition 1.2.
1. The set of natural numbers denoted by $\mathbb{N}$ is defined by

$$\mathbb{N} := \{0, 1, 2, 3, \cdots\}$$

2. The set of integers denoted by $\mathbb{Z}$ is defined by

$$\mathbb{Z} := \{\cdots -2, -1, 0, 1, 2, \cdots\}$$

3. Endowed by the operation of addition " + ", the set of integers is an Abelien group. That is is
   - Closure: For all $x, y \in E$, $x + y \in \mathbb{Z}$.

- $+$ is commutative : $\forall x, y \in \mathbb{Z} : x + y = y + x$
- $+$ is associative : $\forall x, y, z \in \mathbb{Z} : (x + y) + z = x + (y + z)$
- **Identity Element**: There exists an element $0 \in \mathbb{Z}$ such that $\forall x \in \mathbb{Z} : x + 0 = x$.
- **symmetric Element**: For every $x \in E$, there exists an element $-x \in \mathbb{Z}$ such that $x + (-x) := x - y = 0$.

Definition 1.3 (Ordered sets). An ordered set is a set E endowed by a relation " $<$ " such that

- For all $x, y \in E$, exactly one of the following holds

$$x < y, \ x = y, \ \text{or } y < x$$

- For all $x, y, z \in E : x < y \wedge y < z \Longrightarrow x < z$ (transitivity)

We write $x \leq y$ if $x < y$ or $x = y$.

☞ **Example 1.2.** • The set of natural numbers $\mathbb{N} := \{0, 1, 2, 3, \cdots\}$ and the set of integers $\mathbb{Z} := \{\cdots - 2, -1, 0, 1, 2, \cdots\}$ are ordered sets with the relation (lower than) $<$ and we have

$$\cdots - 3 < -2 < -1 < 0 < 1 < 2 < 3 < \cdots$$

Definition 1.4. Let $(E, <)$ be an ordered set and let A be a subset of E.

- We say $a \in E$ is an lower-bound of A if

$$\forall x \in A : a \leq x$$

and if there exist an lower-bound of A, we say A is bounded below.

- We say $b \in E$ is an upper-bound of A if

$$\forall x \in A : x \leq b$$

and if there exist an upper-bound of A, we say A is bounded above.

- We say $a_0 \in E$ is the greatest lower-bound or the infimum of A if a_0 is an lower-bound of A and satisfies $a \leq a_0$ for every lower-bound $a \in E$. We write

$$a_0 := \inf A$$

- We say $b_0 \in E$ is the least upper-bound or the supremum of A if b_0 is an upper-bound of A and satisfies $b_0 \leq b$ for every upper-bound $b \in E$. We write

$$b_0 := \sup A$$

☞ **Example** **1.3.**

Definition **1.5.** The set of rational numbers is the set denoted by $\mathbb{Q}$ defined as follows

$$\boxed{\mathbb{Q} = \{\frac{p}{q} \mid (p,q) \in \mathbb{Z} \times \mathbb{Z}^*\}}$$

or

$$\boxed{\mathbb{Q} = \{\frac{p}{q} \mid (p,q) \in \mathbb{Z} \times \mathbb{N}^*\}}$$

or

$$\boxed{\mathbb{Q} = \{\frac{p}{q} \mid (p,q) \in \mathbb{Z} \times \mathbb{N}^*, \text{ with } p \wedge q = 1\}}$$

☞ **Remark** **1.1.** The set of rational numbers $\mathbb{Q}$ is an ordered set with the relation $<$ "lower than" defined as follow

$$\boxed{x < y \Longleftrightarrow y - x = p/q \text{ where } p, q \in \mathbb{N}}$$

and then we say that $y - x$ is negative, if it is not positive, we say that it is negative.

Definition **1.6.** The addition and multiplicative operations on $\mathbb{Q}$ are defined as follow

$$\boxed{\frac{p}{q} + \frac{p'}{q'} = \frac{pq' + qp'}{p'q'}, \quad \frac{p}{q} \cdot \frac{p'}{q'} = \frac{pp'}{qq'}, \quad \text{for all } p \in \mathbb{Z}, q \in \mathbb{Z}^*.}$$

Theorem **1.1.** The set of rational numbers $\mathbb{Q}$ endowed with the addition and multiplicative operations is an abilean field. That is

1. $(\mathbb{Q}, +)$ **is an abeliean group**
2. **Multiplicative Associativity**: For all $x, y, z \in \mathbb{Q}$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
3. **Multiplicative Identity Element**: There exists an element $1 \in \mathbb{Q}$ such that for all $x \in \mathbb{Q}$, $x \cdot 1 = 1 \cdot x = x$.
4. **Multiplicative Inverse Element (except for 0)**: For every non-zero $x \in \mathbb{Q}$, there exists an element $x^{-1} \in \mathbb{Q}$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$.
5. **Distributive Property**: For all $x, y, z \in \mathbb{Q}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

Definition **1.7 (least upper bound property).** Le E be an ordered set.

1. We say that E satisfies the least upper bound property if every non empty subset A of E that is bounded from above has the least upper bound (i.e. $\sup A$ exists in E).
2. We say that E satisfies the greatest lower bound property if every non empty subset A of E that is bounded from below has the greatest lower bound (i.e. $\inf A$ exists in E)

☞ **Remark 1.2.** The ordered set $\mathbb{Q}$ does not satisfy the least upper bound property. Indeed consider the following subset of $\mathbb{Q}$:

$$A = \{x \in \mathbb{Q} : x^2 \leq 2\}.$$

This set is bounded above by 2 because for every $x \in A$ we have $x \leq 2$ (if not then $x^2 \geq 4$ and $x \notin A$). Suppose by absurd that A has a least upper bound denoted by b. Assume, for the sake of contradiction, that the set $A = \{x \in \mathbb{Q} : x^2 \leq 2\}$ has a least upper bound α in $\mathbb{Q}$. We divide the proof in two steps

- We claim that $\alpha^2 = 2$. Indeed, if $\alpha^2 > 2$, then for $h := \frac{\alpha^2-2}{2\alpha}$, we have $\alpha - h < \alpha$ and

  $$(\alpha - h)^2 = \alpha^2 - 2\alpha h + h^2 > \alpha^2 - 2\alpha h = 2.$$

  Thus, $\alpha - h$ is an upper bound of A, which contradicts the fact that $\alpha = \sup A$. If $\alpha^2 < 2$, then for $h := \min\{1, \frac{2-\alpha^2}{2\alpha+1}\} \in \mathbb{Q}$, we have $\alpha < \alpha + h$ and

  $$(\alpha + h)^2 = \alpha^2 + 2\alpha h + h^2 \leq \alpha^2 + 2\alpha h + h \leq 2.$$

  Thus $\alpha + h \in A$ and $\alpha < \alpha + h$. Then α is not an upper bound. Contradiction. Hence $\alpha^2 = 2.$

- Let us show that $\alpha \notin \mathbb{Q}$. If not then $\alpha = \frac{p}{q}$, where p and q are integers with no common factors other than 1. Hence $2 = \alpha^2 = \frac{p^2}{q^2}$ and $p^2 = 2q^2$. This implies that p^2 is an even number, and therefore, p is also be even (because the square of an odd number is odd). So we can write p as $p = 2k$ where k is an integer. Therefore $2q^2 = (2k)^2$. It follows that $q^2 = 2k^2$ is even and also is q However, this contradicts our initial assumption that p and q have no common factors other than 1, as both p and q are even. Consequently α is not rational number

☞ **Remark 1.3.** The ordered set $\mathbb{Z}$ has the least upper bound property and for every bounded set A of $\mathbb{Z}$, we have

$$\sup A \in A, \ \inf A \in A$$

1.2 The set of real numbers

We have seen in the previous remark that the set of rational numbers $\mathbb{Q}$ haven't the least upper bound property. So we need an other set larger than $\mathbb{Q}$, that satisfies this property. This set is the real number set $\mathbb{R}$ given by the following definition

Definition 1.8. The real number set $\mathbb{R}$ is an ordered field containing $\mathbb{Q}$ and satisfies the least upper bound property.

The following theorem guaranties the existence of $\mathbb{R}$.

Theorem 1.2. There is a unique ordered field which extends the field of rational numbers $\mathbb{Q}$ and satisfies the least upper bound property.

Proof. is accepted. □

1.3 Absolute value

Definition 1.9. The absolute value denoted by $|\cdot|$ is a function defined from $\mathbb{R}$ to $\mathbb{R}_+$ as follows

$$\forall x \in \mathbb{R} : |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

or

$$\forall x \in \mathbb{R} : |x| = \max\{x, -x\}$$

Proposition 1.3. for all $x, y \in \mathbb{R}$, we have

1. $|x| = |-x|$, $|xy| = |x||y|$
2. $|x| \leq y \Longleftrightarrow -|y| \leq x \leq |y|$, $|x| \geq y \geq 0 \Longleftrightarrow x \leq -y \vee x \geq y$
3. $-|x| \leq x \leq |x|$
4. $\left|x + y\right| \leq |x| + |y|$ (**Triangle inequality**)
5. $\left||x| - |y|\right| \leq |x - y|$

1.4 Archimedean property, density and integer part property

Definition 1.10. Let $x \in \mathbb{R}$

1. The integer part of x denoted as $[x]$ is the unique integer satisfying

$$[x] \leq x < [x] + 1$$

   or equivalently

$$x - 1 < [x] \leq x.$$

2. A set A is said to be dense in $\mathbb{R}$ if

$$\forall x, y \in \mathbb{R}, x < y, \ \exists z \in A : x < z < y.$$

☞ **Example 1.4.**

- $[0.5] = 0$ because $0 \leq 0.5 < 1$.
- $[-1.5] = -2$ because $-2 \leq -1.5 < -1$.
- If $x \in \mathbb{Z}$ then $[x] = x$ because $x \leq x < x + 1$.

Theorem 1.4 (Archimedean property). we have

$$\forall x \in \mathbb{R}_+^*, y \in \mathbb{R}, \exists n \in \mathbb{N} : nx \geq y.$$

Proof. Divide through by x. Then the Archimedean property says that for every real number $a = \frac{y}{x}$, we can find $n \in \mathbb{N}$ such that $n \geq a$. In other words, says that the set of natural numbers $\mathbb{N}$ is not bounded above. Suppose for contradiction that $\mathbb{N}$ is bounded above. Then due to the least upper bound axiom, there is $b = \sup \mathbb{N}$. Therefore number $b - 1$ cannot be an upper bound for $\mathbb{N}$ as it is strictly less than b (the least upper bound). Thus there exists an $m \in \mathbb{N}$ such that $m > b - 1$. it follows that $n := m + 1 > b$. This is contradiction since b being an upper bound. □

Theorem 1.5. The following properties are equivalent

1. **Archimedean property** $\forall x \in \mathbb{R}_+^*, y \in \mathbb{R}, \exists n \in \mathbb{N} : nx \geq y$.
2. **integer part property:** $\forall x \in \mathbb{R}, \exists n \in \mathbb{Z} : n \leq x < n + 1$
3. $\mathbb{Q}$ **is dense in** $\mathbb{R}$, that is $\forall x, y \in \mathbb{R}, x < y, \exists r \in \mathbb{Q} : x < r < y.$

Proof.

- 1) $\Longrightarrow$ 2) Let $x \in \mathbb{R}$ be given. We want to show that there exists an integer $n \in \mathbb{Z}$ such that $n \leq x < n + 1$. Consider the set

$$S = \{n \in \mathbb{Z} : n \leq x\}.$$

Due to the Archimedean property, the set S is non empty. Indeed. There is $n \in \mathbb{Z} : -n \geq -x$ then $n \leq x$ so $x \in S$. Since S is bounded above by x. By the well-ordering property of integers, there exists a greatest element in S denoted as n. Since n is the greatest integer less than x, we have $n \leq x < n+1$. Therefore, we have shown that for any real number x, there exists an integer n such that $n \leq x < n + 1$.

- 2) $\Longrightarrow$ 3). Given $x, y \in \mathbb{R} : x < y$. Due to 2) there exists $q \in \mathbb{Z}^*$ such that

$$q - 1 \leq \frac{1}{y-x} < q.$$

Which implies that

$$1 < q(y - x)$$

Then

$$qx + 1 < qy$$

By 2), there exists $p \in \mathbb{Z}$ such that $p - 1 \leq qx < p$. Hence

$$qx < p \leq qx + 1 < qy$$

Consequently, dividing by q, it follows $\boxed{x < \frac{p}{q} < y}$.

- 3) $\Longrightarrow$ 1). Given $x \in \mathbb{R}_+^*$, $y \in \mathbb{R}$. If $x \geq y$ it is enough to take $n = 1$. If not then $0 < x < y$. from 3), there are $p, q \in \mathbb{N}^*$ such that $\frac{p}{q} \geq \frac{y}{x}$ and then $px \geq qy \geq y$, $(q \geq 1)$.

□

Corollary 1.6. the irrational set $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Given $x, y \in \mathbb{R}$ such that $x < y$. form the density of $\mathbb{Q}$, there are $r_1, r_2 \in \mathbb{Q}$ such that $x < r_1 < r_2 < y$. We know that $\sqrt{2}$ is irrational and greater than 1. Then taking $z = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1) \notin \mathbb{Q}$ we obtain $r_1 < z < r_2$. □

1.5 Bounded subset in $\mathbb{R}$

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Theorem 1.7 (Characterisation of the supremum and infimum). Let A be a bounded subset of $\mathbb{R}$. Then

$$\alpha := \inf A \iff \begin{cases} \forall x \in A : x \geq \alpha \quad (\alpha \text{ is a lower bound of A}) \\ \forall \varepsilon > 0, \exists x_0 \in A : x_0 < \alpha + \varepsilon \quad (\alpha \text{ is greater than any lower bound}) \end{cases}$$

$$\beta := \sup A \iff \begin{cases} \forall x \in A : x \leq \beta \quad (\alpha \text{ is an upper bound of A}) \\ \forall \varepsilon > 0, \exists x_0 \in A : \alpha - \varepsilon < x_0 \quad (\beta \text{ is less than any upper bound}) \end{cases}$$

Definition 1.11 (Maximum and minimum). Let A be a subset of $\mathbb{R}$.

1. A maximum of A, denoted as $\max A$, is the greatest element of A. That is

$$\boxed{\max \in A \quad \text{and } \forall x \in A : x \leq \max A}$$

2. A minimum of A, denoted as $\min A$, is the least element of A. That is

$$\boxed{\min \in A \quad \text{and } \forall x \in A : x \geq \min A}$$

☞ **Remark 1.4.** Let A be a bounded subset.

- $\max A$ is an upper bound of A.
- If $\sup A \in A$, then $\max A = \sup A$.

- If $\max A$ exists then $\sup A = \max A$. Indeed, since $\max A$ is an upper bound of A, it suffices to show that

$$\forall \varepsilon > 0, \exists x_0 \in A : \max A - \varepsilon < x_0.$$

Given any $\varepsilon > 0$, we can take $x_0 = \max A$. Then we have $\max A - \varepsilon < \max A = x_0$.

- If $\sup A \notin A$, then $\max A$ does exists, because if not, $\sup A = \max A \in A$.

- Analogously for $\inf A$ and $\min A$.

☞ **Example 1.5.** Find $\sup A$, $\inf A$, $\max A$, $\min A$ if they exist, for the following cases.

1. Let $A := \{1, 2, 3\}$. We observe that $\min A = 1$, $\max A = 3$, leading to $\inf A = 1$ and $\sup A = 3$.

2. For $A =]0, 1]$, using the interval definition, we note that 0 is a lower bound, and 1 is an upper bound of A. Since $1 \in A$, we conclude that $\sup A = \max A = 1$. We now prove that $\inf A = 0$. Given $\varepsilon > 0$ (we can assume ε is arbitrarily small), if we choose $x_0 := \frac{\varepsilon}{2} \in A$, we have $x_0 < 0 + \varepsilon$. This shows that $\inf A = 0$. As $0 \notin A$, $\min A$ doesn't exist.

3. Let $A := \left\{\frac{n}{n^2+1} \mid n \in \mathbb{N}\right\}$. We observe that for all $n \in \mathbb{N}$, $0 < \frac{n}{n^2+1} \leq \frac{1}{2}$ (using $ab \leq \frac{1}{2}(a^2 + b^2)$). Thus, $\frac{1}{2}$ is an upper bound of A. Since $\frac{1}{2} = \frac{1}{1^2+1} \in A$, we deduce $\max A = \sup A = \frac{1}{2}$. Moreover, we can prove 0 is the infimum of A. For any $\varepsilon > 0$, we observe that

$$\frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n}, \quad \frac{1}{n} \leq \varepsilon \Longleftrightarrow n \geq \frac{1}{\varepsilon}.$$

Due to the Archimedean property, choose n such that $n \geq \frac{1}{\varepsilon}$ (e.g., $n = \left[\frac{1}{\varepsilon}\right] + 1$). This guarantees $\frac{n}{n^2+1} \leq \frac{1}{n} \leq \varepsilon$ Thus, 0 is indeed the infimum of A. As $0 \notin A$, $\min A$ doesn't exist.

1.6 Exercises

Exercise 1. Show that

1. $\forall x, y \in \mathbb{R} : \big|x + y\big| \leq |x| + |y|$ (Triangle inequality)
2. $\forall x, y \in \mathbb{R} : \big||x| - |y|\big| \leq |x - y|$
3. $\forall x_1, x_2, \dots, x_n \in \mathbb{R} : |x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$
4. $\forall x, y \in \mathbb{R} : 2|xy| \leq x^2 + y^2$ (For which values of x and y is equality achieved)
5. $\forall x, y \in \mathbb{R} : \max\{x, y\} = \dfrac{x + y + |x - y|}{2}, \ \min\{x, y\} = \dfrac{x + y - |x - y|}{2}$
6. $\forall x, y \in \mathbb{R} : |x| + |y| \leq |x + y| + |x - y|$

Solution:

1. It is enough to show that $-(|x| + |y|) \leq x + y \leq |x| + |y|$. By the definition of the absolute value, we have

$$-|x| \leq x \leq |x| \quad \text{and} \quad -|y| \leq y \leq |y|.$$

   Summing these inequalities, we obtain $-(|x| + |y|) \leq x + y \leq |x| + |y|$. This implies $|x + y| \leq |x| + |y|$.

2. We have from the triangle inequality

$$|x| = |x - y + y| \leq |x - y| + |y|$$

   and then $|x| - |y| \leq |x - y|$. By the same way we obtain $|y| - |x| \leq |y - x|$, that is $|x| - |y| \geq -|x - y|$. This implies that

$$\big||x| - |y|\big| \leq |x - y|.$$

3. By induction on $n \in \mathbb{N}^*$. For $n = 1$, the inequality is obvious. Let us suppose that the inequality is satisfied for $n \in \mathbb{N}$ and show that it is satisfied for $n + 1$, that is,

$$|x_1 + x_2 + \cdots + x_n + x_{n+1}| \leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|.$$

   From the triangle inequality, we have

$$|x_1+x_2+\cdots+x_n+x_{n+1}| = |(x_1+x_2+\cdots+x_n)+x_{n+1}| \leq |x_1+x_2+\cdots+x_n|+|x_{n+1}|.$$

   By the hypothesis of induction, we have $|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$. Hence,

$$|x_1 + x_2 + \cdots + x_n + x_{n+1}| \leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|.$$

4. We have
$$x^2 + y^2 - 2|xy| = |x|^2 + |y|^2 - 2|x||y| = (|x| - |y|)^2 \geq 0.$$
Then $2|xy| \leq x^2 + y^2$.

5. We distinguish two cases

 - If $x \leq y$ then
$$\max\{x, y\} = y = \frac{(x+y) - (x-y)}{2} = \frac{x + y + |x-y|}{2}.$$
$$\min\{x, y\} = x = \frac{(x+y) + (x-y)}{2} = \frac{x + y - |x-y|}{2}.$$
 - If $x \geq y$ then
$$\max\{x, y\} = x = \frac{(x+y) + (x-y)}{2} = \frac{x + y - |x-y|}{2}.$$
$$\min\{x, y\} = y = \frac{(x+y) - (x-y)}{2} = \frac{x + y + |x-y|}{2}.$$

6. We have
$$|x| = \frac{|(x+y) + (x-y)|}{2} \leq \frac{|x+y| + |x-y|}{2}.$$
$$|y| = \frac{|(x+y) + (y-x)|}{2} \leq \frac{|x+y| + |x-y|}{2}.$$
By addition, we get $|x| + |y| \leq |x+y| + |x-y|$.

Exercise 2. Let $S_n := \sum_{i=0}^{n} 3^i$, $P_n := \prod_{i=2}^{n} 4^i$. Calculate:

$$S_0,\ S_3,\ P_2,\ P_4,\ \sum_{i=2}^{3} \prod_{j=1}^{i} j3^j.$$

Solution: $S_n := \sum_{i=0}^{n} 3^i$, $P_n := \prod_{i=2}^{n} 4^i$. We have
$$S_0 = 3^0 = 1,\ S_3 = 3^0 + 3^1 + 3^2 + 3^3 =$$
$$P_2 = 4^2 = 16,\ P_4 = 4^2 + 4^3 + 4^4 =$$
$$\begin{aligned}\sum_{i=2}^{3} \prod_{j=1}^{i} j3^j &= \prod_{j=1}^{2} j3^j + \prod_{j=1}^{3} j3^j \\ &= (1 \times 3^1) \times (2 \times 3^2) + (1 \times 3^1)(2 \times 3^2)(3 \times 3^3) = \end{aligned}$$

Exercise 3. Let $[x]$ be the integer part of x. Show that

1. If $n \in \mathbb{Z}$ such that $n \leq x$, then $n \leq [x]$
2. $\forall x, y \in \mathbb{R} : x \leq y \Longrightarrow [x] \leq [y]$ (Does $x < y \Longrightarrow [x] < [y]$?)

3. $\forall x \in \mathbb{R},\ n \in \mathbb{Z} : [x+n] = [x] + n$ (Does $[x+y] = [x] + [y], \quad \forall x, y \in \mathbb{R}$?)

4. $[x] + [-x] = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Z} \\ -1 & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$

Solution: Recall that the integer part of x denoted as $[x]$ is the unique integer satisfying $[x] \leq x < [x] + 1$ (or $x - 1 < [x] \leq x$).

1. By contradiction, if $n > [x]$, then $n \geq [x] + 1$ (because n and $[x]$ are integers). Hence, we have
$$[x] \leq x < [x] + 1 \leq n.$$
This implies that $x < n$, which contradicts the hypothesis.

2. If $x \leq y$ then $[x] \leq x \leq y$. Since $[x] \in \mathbb{Z}$ then according to the previous question $[x] \leq [y]$. The strict inequality does not always hold; for example, take $x = \frac{1}{4}$ and $y = \frac{1}{2}$. We have $x < y$, but $[x] = [y] = 0$.

3. It is sufficient to show the following: $[x] + n \leq x + n < [x] + n + 1$. By definition, we have $[x] \leq x < [x] + 1$. Therefore, $[x] + n \leq x + n < [x] + n + 1$. Since $[x] + n \in \mathbb{Z}$, then $[x+n] = [x] + n$.

4. We have two cases:
   - If $x \in \mathbb{Z}$, then $[x] = x$ and $[-x] = -x$. Hence, $[x] + [-x] = 0$.
   - If $x \notin \mathbb{Z}$, then $-x \notin \mathbb{Z}$. In this case, we have
   $$x - 1 < [x] < x \quad \text{and} \quad -x - 1 < [-x] < -x.$$
   By summing these inequalities, we obtain: $-2 < [x] + [-x] < 0$. Since $[x] + [-x] \in \mathbb{Z}$, then $[x] + [-x] = -1$.

Exercise 4. Show that

$$\forall n \in \mathbb{N} : \left[(\sqrt{n} + \sqrt{n+1})^2\right] = 4n + 1$$

Solution: The goal is to demonstrate the following inequality: $4n+1 \leq (\sqrt{n}+\sqrt{n+1})^2 < 4n+2$. We have
$$(\sqrt{n} + \sqrt{n+1})^2 = 2n + 1 + 2\sqrt{n^2 + n}.$$
Furthermore, we can observe that
$$n = \sqrt{n^2} \leq \sqrt{n^2 + n} < \sqrt{n^2 + 2n + 1} = n + 1.$$
Therefore,
$$4n + 1 \leq (\sqrt{n} + \sqrt{n+1})^2 < 4n + 2.$$

Exercise 5. 1. Show that if $n \in \mathbb{N}$ is not the square of a natural number, then $\sqrt{n}$ is irrational.

2. Deduce that $\sqrt{2}+\sqrt{3}$ is irrational.

Solution:

1. By contradiction: Suppose that $\sqrt{n}$ is a rational number. Then there exist integers p and q such that $\sqrt{n}=\frac{p}{q}$. It follows that $p^2=nq^2$. It is well-known that any integer can be expressed as a unique product of prime factors. Therefore, the exponents in the prime factorization of nq^2 must be even because $nq^2=p^2$, which implies that the exponents in the prime factorization of n are even. This, in turn, means that n is the square of a natural number, leading to a contradiction.

2. We can observe that
$$(\sqrt{2}+\sqrt{3})^2=5+2\sqrt{6}.$$
Hence, $\sqrt{6}=\frac{1}{2}\left((\sqrt{2}+\sqrt{3})^2-5\right)$. It follows that if $\sqrt{2}+\sqrt{3}$ is rational, then $\sqrt{6}$ is also rational. However, as shown in the previous question, $\sqrt{6}$ is irrational.

Exercise 6. 1. Find $n\in\mathbb{N}$ such that: $\frac{1}{n}<\frac{1}{2023}$, $\frac{n}{n^2+1}<\frac{1}{2023}$

2. Find $N\in\mathbb{N}$ such that: $n\geq N\Longrightarrow\frac{1}{n}\leq\frac{1}{2023}$, $n\geq N\Longrightarrow\frac{n}{n^2+1}\leq\frac{1}{2023}$

3. Show that $\forall\varepsilon>0,\exists n\in\mathbb{N}^*:\frac{1}{n}<\varepsilon$

4. Show that $(\forall\varepsilon>0:0\leq x\leq\varepsilon)\Longrightarrow x=0$

5. Show that $(\forall n\in\mathbb{N}^*:0\leq x\leq\frac{1}{n})\Longrightarrow x=0$

Solution:

1. Let $n\in\mathbb{N}$. We have $\frac{1}{n}<\frac{1}{2023}\iff n>2023$. Therefore, we can choose $n=2024$ (or any natural number greater than 2023). We observe that $\frac{n}{n^2+1}<\frac{n}{n^2}=\frac{1}{n}$. Consequently,
$$\frac{1}{n}<\frac{1}{2023}\Rightarrow\frac{n}{n^2+1}<\frac{1}{2023}.$$
So we can select $n=2024$.

2. According to the previous question, we can take $N=2024$. Therefore
$$n\geq N\Longrightarrow\begin{cases}1/n & \leq & \frac{1}{2023}\\ & \wedge & \\ \frac{n}{n^2+1} & < & 1/n\leq\frac{1}{2023}\end{cases}$$

3. Given any $\varepsilon>0$. We have $1/n\leq\varepsilon\iff n\geq1/\varepsilon$. Therefore, we can select $n=[1/\varepsilon]+1$

4. By contradiction, suppose that $x>0$. Then there exists $\varepsilon=x/2>0$ such that $0<\varepsilon<x$. This contradicts the initial assumption.

5. By contradiction, suppose that $x > 0$. Then there exists $n := [\frac{1}{x}] + 1 > \frac{1}{x}$. Therefore, $\frac{1}{n} < x$. This contradicts the initial assumption.

Exercise 7. Let A and B be two non-empty and bounded sets in $\mathbb{R}$. Prove that:

1. If $A \subseteq B$, then $(\sup A \leq \sup B \wedge \inf A \geq \inf B)$
2. $\sup(A \cup B) = \max\{\sup A, \sup B\}$, $\inf(A \cup B) = \min\{\inf A, \inf B\}$
3. If $A \cap B \neq \emptyset$, then: $\sup(A \cap B) \leq \min\{\sup A, \sup B\}$, $\inf(A \cap B) \geq \max\{\inf A, \inf B\}$ (Give an example where strict inequalities hold).

Solution: Note that

$$\inf A \geq \alpha \iff \alpha \text{ is an lower bound of } A$$

$$\sup A \leq \beta \iff \beta \text{ is an upper bound of } A$$

1. Supose that $A \subset B$.
   - Prove that $\sup A \leq \sup B$. It is enough to show that $\sup B$ is an upper bound of A (i.e. $\forall x \in A : x \leq \sup B$). Let $x \in A$. Since $A \subset B$ then$x \in B$. Therefore $x \leq \sup B$.
   - Prove that $\inf A \geq \inf B$. It is enough to show that $\inf B$ is an lower bound of A (i.e. $\forall x \in A : x \geq \inf B$). Let $x \in A$. Since $A \subset B$ then$x \in B$. Therefore $x \geq \inf B$.
2. Prove that $\sup(A \cup B) = \max\{\sup A, \sup B\}$. We set $\alpha = \max\{\sup A, \sup B\}$.
   - Prove that α is an upper bound of $A \cup B$ (i.e. $\forall x \in A \cup B : x \leq \alpha$). Let $x \in A \cup B$. hence $x \in A$ or $x \in B$. It follows that $x \leq \sup A$ or $x \leq \sup B$. This implies that $x \leq \alpha = \max\{\sup A, \sup B\}$.
   - Prove that α is the least upper bound of $A \cup B$. Let M be an upper bound of $A \cup B$. Then M is an upper bound of A and B. It follows that $\sup A \leq M$ and $\sup B \leq M$. Then $\alpha \leq M$. In a similar manner, we can establish the equality of the infimum (inf).
3. We have $A \cap B \subset A$ and $A \cap B \subset B$. It follows that

$$\begin{cases} \sup(A \cap B) \leq \sup A & \wedge \quad \sup(A \cap B) \leq \sup B \\ \inf(A \cap B) \geq \sup A & \wedge \quad \inf(A \cap B) \geq \sup B \end{cases}$$

   It follows that

$$\begin{cases} \sup(A \cap B) & \leq & \min\{\sup A, \sup B\} \\ \inf(A \cap B) & \geq & \max\{\inf A, \inf B\} \end{cases}$$

Exercise 8. Let A, B be non-empty and bounded subsets of $\mathbb{R}$. Define:

$$-A := \{-x \ / \ x \in A\}, \quad A + B := \{x + y \ / \ x \in A, y \in B\}.$$

Prove that

1. $\sup(-A) = -\inf A, \quad \inf(-A) = -\sup A$
2. $\sup(A + B) = \sup A + \sup B, \quad \inf(A + B) = \inf A + \inf B$

Solution:

1. Prove that $\sup(-A) = -\inf A$.

    - We prove that $-\inf A$ is an upper bound of $-A$. Let $x \in -A$, then $-x \in A$. it follows that $-x \geq \inf A$. Hence $x \leq -\inf A$.
    - Prove that $-\inf A$ is the least upper bound of $-A$. That is

    $$\forall \varepsilon > 0, \exists y_0 \in -A : y_0 > -\inf A - \varepsilon.$$

    By the definition of $\inf A$, we have

    $$\forall \varepsilon > 0, \exists x_0 \in A : x_0 < \inf A + \varepsilon.$$

    Therefore

    $$\forall \varepsilon > 0, \exists y_0 := -x_0 \in -A : y_0 > -\inf A - \varepsilon.$$

   In a similar manner, we can establish the equality of the infimum (inf).

2. Prove that $\sup(A + B) = \sup A + \sup B$.

    - We prove that $\sup A + \sup B$ is an upper bound of $A + B$. Let $z \in A + B$, then $z = x + y$ where $x \in A$ and $y \in B$. it follows that $x \geq \sup A$ and $y \geq \sup B$. Hence $z = x + y \leq \sup A + \sup B$.
    - Prove that $\sup A + \sup B$ is the least upper bound of $A + B$. That is

    $$\forall \varepsilon > 0, \exists z_0 \in -A : z_0 > \sup A + \sup B - \varepsilon.$$

    By the definition of $\sup A$ and $\sup B$, we have

    $$\forall \varepsilon > 0, \exists x_0 \in A : x_0 < \sup A - \varepsilon/2.$$

    $$\forall \varepsilon > 0, \exists y_0 \in B : x_0 < \sup B - \varepsilon/2.$$

    Therefore, by addition

    $$\forall \varepsilon > 0, \exists z_0 := x_0 + y_0 \in A + B : z_0 > \sup A + \sup B - \varepsilon.$$

   In a similar manner, we can establish the equality of the infimum (inf).

Exercise 9. Determine the supremum, infimum, maximum, and minimum (if they exist) of the following sets:

$$A = [0, 1] \cup [2, 3[, \;\; B = \{x \in \mathbb{R} : x^2 - x - 6 < 0\}, \;\; C = \{x \in \mathbb{Z} : x^2 - x - 6 < 0\}$$

Solution:

- $\sup A = \max\{\sup[0,1], \sup[2,3[\} = \max\{1,3\} = 3.$, $\max A$ doesn't exist $\inf A = \min\{\inf[0,1], \inf[2,3[\} = \min\{0,2\} = 0 = \min A$.
- $B = \{x \in \mathbb{R} : x^2 - x - 6 < 0\} =]-2,3[$. $\sup B = 3$, $\inf B = -2$. $\max B$ and $\min B$ don't exist.

Exercise 10. Determine the supremum, infimum, maximum, and minimum (if they exist) of the following sets:

- $A_1 = \{1 + \frac{1}{n} \ / \ n \in \mathbb{N}^*\}$
- $A_2 = \{1 - \frac{1}{n} \ / \ n \in \mathbb{N}^*\}$
- $A_3 = \{(-1)^n + \frac{1}{n} \ / \ n \in \mathbb{N}^*\}$
- $A_4 = \{\frac{1}{n} + \frac{1}{n^2} \ / \ n \in \mathbb{N}^*\}$
- $A_5 = \{\cos(n\pi) \ / \ n \in \mathbb{N}\}$
- $A_6 = \{\cos(\frac{n\pi}{2}) \ / \ n \in \mathbb{N}\}$
- $A_7 = \{\frac{1}{x} / \ 1 < x < 2\}$
- $A_8 = \{-\frac{1}{x} \ / \ 1 < x < 2\}$.

Solution:

1. $A_1 = \{1 + \frac{1}{n} \ / \ n \in \mathbb{N}^*\}$. We have $\forall n \in \mathbb{N}^* : 1 + 1/n \leq 1 + 1 = 2 \in A_1$. Hence 2 is an upper bound of A_1 and since it belongs to A_1, then

$$\sup A_1 = \max A_1 = 2.$$

For each $n \in \mathbb{N}^*$, $1 + \frac{1}{n} \geq 1$. Thus 1 is a lower bound of A_1. We observe that $1 + \frac{1}{n}$ decreases, approaching 1 as $n \to +\infty$. Hence we claim that $\inf A_1 = 1$. Indeed, for any $\varepsilon > 0$, there exists an n such that $1 + \frac{1}{n} < 1 + \varepsilon$ (e.g. $n = [1/\varepsilon] + 1$). Finally, since $\inf A_1 = 1 \notin A_1$, then $\min A_1$ doesn't exist.

2. $A_2 = \{1 - \frac{1}{n} \ / \ n \in \mathbb{N}^*\}$. Similarly, we obtain

$$\inf A_2 = \min A_2 = 0, \quad \sup A_2 = 1, \ \max A_2 \text{ don't exist.}$$

3. $A_3 = \{(-1)^n + \frac{1}{n} \ / \ n \in \mathbb{N}^*\}$. We have

$$A_3 = \underbrace{\{-1 + \frac{1}{2n+1} \ / \ n \in \mathbb{N}\}}_{A} \cup \underbrace{\{1 + \frac{1}{2n} \ / \ n \in \mathbb{N}^*\}}_{B}$$

As in the set A_1, we have $\sup A = 0$, $\sup B = 1 + 1/2 = 3/2$, $\inf A = -1$, $\inf B = 1$. Therefore

$$\sup A_3 = \max\{\sup A, \sup B\} = 3/2, \quad \inf A_3 = \min\{\inf A, \inf B\} = -1.$$

We Observe that $\sup A_3 = 3/2 \in A_3$. Thus $\max A_3 = 3/2$. $\inf A_3 = -1 \notin A_3$ because $-1 < (-1)^n + 1/n$ for all n. Thus $\min A_3$ don't exist.

4. $A_4 = \{\frac{1}{n} - \frac{1}{n^2} \ / \ n \in \mathbb{N}^*\}$ We have

$$\forall n \in \mathbb{N}^* : 0 < \frac{1}{n} - \frac{1}{n^2} \leq \frac{1}{n} \leq 1.$$

Hence A_4 is bounded and 0 is a lower bound and 1 is an upper bound of A_4.

- For the infimum, we observe $\frac{1}{n} - \frac{1}{n^2} \longrightarrow 0$ as $n \to +\infty$. So we claim that $\inf A_4 = 0$. Indeed, given any $\varepsilon > 0$. Observing that $\frac{1}{n} - \frac{1}{n^2} < \frac{1}{n}$, there exists n such that $1/n \leq \varepsilon$ and consequently $\frac{1}{n} - \frac{1}{n^2} \leq \varepsilon$. Thus $\inf A_4 = 0$. Since $0 \notin A_4$ because $0 < \frac{1}{n} - \frac{1}{n^2}$, then $\min A_4$ doesn't exist.
- For supremum, we have

$$\frac{1}{n} - \frac{1}{n^2} = \frac{1}{4} - \left(\frac{1}{n} - \frac{1}{2}\right)^2 \leq \frac{1}{4}$$

and we observe that for $n = 2$, $\frac{1}{n} - \frac{1}{n^2} = 1/4 \in A_4$. Therefore $\sup A_4 = \max A_4 = 1/4$.

5. $A_5 = \{\cos(n\pi) \ / \ n \in \mathbb{N}\}$. We observe that $\cos(n\pi) = +1$ if n is even and $\cos(n\pi) = -1$ if n is odd. Then $A_5 = \{-1, +1\}$. Therefore $\sup A_5 = \max A_5 = +1$ and $\inf A_5 = \min A_5 = -1$.

6. $A_6 = \{\cos(\frac{n\pi}{2}) \ / \ n \in \mathbb{N}\}$. We have

$$\cos(\frac{n\pi}{2}) = \begin{cases} +1 & \text{if} \quad n = 4k \\ 0 & \text{if} \quad n = 4k+1 \\ -1 & \text{if} \quad n = 4k+2 \\ 0 & \text{if} \quad n = 4k+3 \end{cases}$$

Hence $A_6 = \{-1, 0, 1\}$.Therefore $\sup A_6 = \max A_6 = +1$ and $\inf A_6 = \min A_6 = -1$.

7. $A_7 = \{\frac{1}{x} / \ 1 < x < 2\}$. We have $1 < x < 2 \iff 1/2 < 1/x < 1$. Then $A_7 =]1/2, 1[$. Hence $\sup A_7 = 1$, $\inf A_7 = 1/2$. Since $\sup A_7$ and $\inf A_7$ don't belong to A_7, then $\max A_7$, $\min A_7$ don't exit

8. $A_8 = \{-\frac{1}{x} \ / \ 1 < x < 2\}$. We observe that $A_8 = -A_7$. Then

$$\sup A_8 = -\inf A_7 = -1/2, \ \inf A_8 = -\sup A_7 = -1.$$

$\max A_8$, $\min A_8$ don't exit

Exercise 11. Let $A = \{x^2 + y^2 \ / \ x, y \in \mathbb{R}, xy = 1\}$

1. Prove that A is bounded below and calculate $\inf A$.
2. Is A bounded above?

Solution:

1. We have $\forall x, y \in \mathbb{R} : x^2 + y^2 \geq 0$. Hence A is bounded below. We observe that if $xy = 1$ then $x^2 + y^2 \geq 2xy = 2$. Therefore 2 is a lower bound of A. We have $2 \in A$ because $2 = 1^2 + 1^2$ and $1 \times 1 = 1$. Then $\inf A = 2 = \min A$.

2. Let $M \in \mathbb{R}$. We have for $x = \sqrt{|M| + 1}$, $y = 1/x$: $z = x^2 + y^2 \in A$ and $z > M$. Therefore A is not bounded above.

Exercise 12. Calculate the supremum and (if it exists, the maximum) of the following sets:

$$A = \{x \in \mathbb{Q} : x^2 \leq 2\}, \quad B = \{x \in \mathbb{R} \setminus \mathbb{Q} : x^2 < 2\}.$$

Solution: We observe that $\sqrt{2}$ is an upper bound for sets A and B because if not, there exists an $x \in A$ (or B) such that $x > \sqrt{2}$. This implies that $x^2 > 2$, but $x^2 \leq 2$. We claim that $\sqrt{2}$ is the least upper bound of set A. Indeed, if not, there must exist an upper bound M such that $M < \sqrt{2}$. Due to the density of rational numbers $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational number $x \in \mathbb{Q}$ such that $M < x < \sqrt{2}$. This means $M^2 < x^2 < 2$. Hence, x belongs to A, and this implies that M is not an upper bound. This leads to a contradiction. Since $\sqrt{2}$ is irrational number, then $\sqrt{2} \notin A$. Hence $\max A$ doesn't exist.

2 Complex numbers

The set of complex numbers is essentially an algebraic representation of the plane. If we represent the set of points on a line using the set of real numbers, then the set of points in the plane is represented by the set of complex numbers. In this context, each point in the plane is represented by the following number: $x + iy$, where x represents the abscissa of the point, and y represents its ordinate. The imaginary unit i is introduced to distinguish between the abscissa and ordinate of the points represented by complex numbers during mathematical operations.

Complex numbers are a fundamental concept in mathematics that extends the real numbers to include the imaginary unit. In this course, we will explore the properties, operations, and applications of complex numbers.

Definition 2.1.
1. A complex number is an element of the form $z = x + iy$, where x and y are real numbers, and i is the imaginary unit ($i^2 = -1$).
2. x is called the real part of z ($\mathrm{Re}(z)$) and y is the imaginary part of z ($\mathrm{Im}(z)$).
3. The set of complex numbers is denoted by $\mathbb{C}$, that is

$$\boxed{\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}.}$$

2.1 Definition of the Field of Complex Numbers

Theorem 2.1. The set of complex numbers $\mathbb{C}$ is an Abiliene field with the following two operations:

1. Addition: For complex numbers $z = x + ix'$ and $z' = x' + iy'$, the sum $z + z'$ is defined as
$$z + z' = (x + x') + i(y + y')$$
2. Multiplication: The product $z \cdot z'$ is defined as
$$z \cdot z' = (xx' - yy') + i(xy' + x'y)$$
and in particular $i^2 = -1$.

☞ **Remark 2.1.** The field $\mathbb{C}$ extends the real numbers $\mathbb{R}$ by introducing the imaginary unit i, which satisfies $\boxed{i^2 = -1}$.

Proposition 2.2. The set $\mathbb{C}$ has the following properties:

- Multiplicative Inverse: Every non-zero complex number $z = x+iy$ has a multiplicative inverse z^{-1}, given by

$$z^{-1} = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

- Identity Elements: The additive identity is $x + iy = 0 \Longleftrightarrow x = y = 0$, and$0 \times i = 0$.

The field of complex numbers $\mathbb{C}$ is a fundamental mathematical structure that plays a significant role in various mathematical, scientific, and engineering applications.

Definition 2.2 (Conjugate and Modulus). The complex conjugate of $z = x + iy$ is $\bar{z} = x - iy$. The modulus (magnitude) of z is $|z| = \sqrt{x^2 + y^2}$.

☞ **Remark 2.2 (Subtraction and Division).** Since $\mathbb{C}$ is a field, then it follows

$$\boxed{z - z' = (x - x') + i(y - y')}$$

$$\boxed{\frac{z}{z'} = \frac{x + iy}{x' + iy'} = \frac{(x + iy)(x' - iy')}{x'^2 + y'^2}}$$

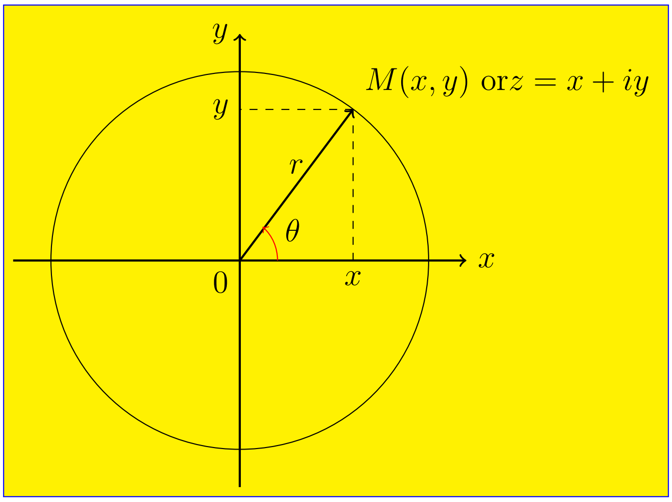
☞ **Remark 2.3.** Both the real part and the imaginary part of the complex number can represent a specific point in the plane. This representation aims to differentiate between the points representing real values and their complex counterparts during mathematical operations.

Proposition 2.3. We have the following useful properties.

- **Equality to Zero**:
$$|z| = 0 \iff z = 0$$
- **Triangle Inequality**:
$$|z + z'| \leq |z| + |z'|$$
- **Multiplicative Property**:
$$|zz'| = |z| \cdot |z'|$$
- **Modulus of the Conjugate**:
$$|z| = |\overline{z}|$$
- **Modulus of Quotient**:
$$\left|\frac{z}{z'}\right| = \frac{|z|}{|z'|}, \text{ if } z' \neq 0$$
- **Modulus of a Complex Conjugate Product**:
$$z \cdot \overline{z} = |z|^2$$

2.2 Geometric Interpretation of Complex Numbers

In the complex number system, each complex number $z = x + iy$ can be associated with a point (x, y) in the complex plane. The complex plane is a two-dimensional plane where the horizontal axis represents the real part of the complex number (x), and the vertical axis represents the imaginary part (y).



The complex number $z = x + iy$ corresponds to the point (x, y) on the complex plane. The distance from the origin to this point is the modulus of the complex number, given by $r = \sqrt{x^2 + y^2}$. The angle θ that the line connecting the origin and the point makes with the positive real axis is the argument of the complex number.

Using this geometric interpretation, addition and subtraction of complex numbers correspond to vector addition and subtraction in the complex plane. Multiplication by a complex number corresponds to scaling and rotation, where multiplication by i results in a counterclockwise rotation by 90 degrees.

The geometric interpretation of complex numbers provides an intuitive way to understand their behavior and operations in terms of points and vectors on the complex plane.

2.3 Polar Form

Definition 2.3. A complex number $z = x + iy$ can also be represented in polar form as

$$z = r(\cos\theta + i\sin\theta),$$

where:

- r is the modulus (magnitude) of the complex number, given by $r = |z| = \sqrt{a^2 + b^2}$.
- θ is the argument (angle) of the complex number in the complex plane. It is characterized by :

$$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}.$$

The following properties apply to the argument $\arg(z)$:

Proposition 2.4. 1. **Argument of a Real Positive Number:** For a positive real number x,

$$\arg(x) = 0$$

.

2. **Argument of a Non-Positive Real Number:** For a non-positive real number x,

$$\arg(x) = \pi$$

.

3. **Argument of a Pure Imaginary Number:** For a pure imaginary number yi, where y is a real number, $\arg(yi) = \frac{\pi}{2}$ if $y > 0$ and $\arg(yi) = -\frac{\pi}{2}$ if $y < 0$.

4. **Argument of a Product:** For two complex numbers z_1 and z_2,

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

.

5. **Argument of a Quotient:** For two complex numbers z_1 and z_2,

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

.

6. **Argument of a Complex Conjugate:** For a complex number $z = a + bi$,

$$\arg(\overline{z}) = -\arg(z)$$

.

7. **Argument of the Reciprocal:** For a non-zero complex number z,

$$\arg\left(\frac{1}{z}\right) = -\arg(z)$$

.

8. **Argument of Powers:** For a non-zero complex number z and a positive integer n,

$$\arg(z^n) = n \arg(z)$$

.

2.4 Euler's Formula

Consider the function $f : \mathbb{R} \to \mathbb{C}$ defined by

$$f(\theta) = \cos\theta + i\sin\theta.$$

We observe that $f(0) = 1$, and its derivative is $f'(\theta) = if(\theta)$. This observation prompts us to define $f(\theta) = e^{i\theta}$, and let us to the following definition.

Definition 2.4 (Euler's Formula). Euler's formula for complex numbers is as follows:

$$e^{i\theta} := \cos\theta + i\sin\theta.$$

Consequently, for any complex number z :

$$z = |z|e^{i\theta} = re^{i\theta}.$$

2.5 Exercises

Exercise 13. Express the following complex numbers in algebraic form $(a + ib)$:

$$\frac{1}{5+3i}, \quad \frac{1}{(1+i)(1+i\sqrt{3})}$$

Solution: We set $z = \frac{1}{(1+i)(1+i\sqrt{3})}$. First, we'll multiply both the numerator and denominator by the complex conjugate of the denominator to rationalize the expression:

$$\begin{aligned} z &= \frac{1}{(1+i)(1+i\sqrt{3})} \cdot \frac{(1-i)(1-i\sqrt{3})}{(1-i)(1-i\sqrt{3})} \\ &= \frac{(1-i)(1-i\sqrt{3})}{4\times 2} = \frac{(1-i)(1-i\sqrt{3})}{8} = \frac{1+\sqrt{3}}{8} - i\frac{1+\sqrt{3}}{8}. \end{aligned}$$

Exercise 14. Calculate the cube roots of 1.

Solution: Let $z := re^{i\theta} \in \mathbb{C}$ be a cube root of 1. That is $z^3 = r^3e^{3i\theta} = 1$. Hence

$$\begin{cases} r^3 = 1 \\ 3\theta = 2n\pi \end{cases} \text{ and then } \begin{cases} r^3 = 1 \\ \theta = \frac{2n}{3}\pi \end{cases}$$

Therefore

- if $n = 3k$, then $\theta = 2k\pi$. Hence $z = e^{2ik\pi} = 1$.
- if $n = 3k + 1$, then $\theta = \frac{2\pi}{3} + 2k\pi$. Hence $z = e^{2i\pi/3+2ik\pi} = -1/2 + i\sqrt{3}/2$
- if $n = 3k + 2$, then $\theta = \frac{4\pi}{3} + 2k\pi$. Hence $z = e^{4i\pi/3+2ik\pi} = -1/2 - i\sqrt{3}/2$.

Exercise 15. 1. Give the exponential form of the complex numbers: $1 + i$, $1 + i\sqrt{3}$.

2. Calculate the real and imaginary parts of $\left(\frac{1+i\sqrt{3}}{1+i}\right)^{2022}$.

Solution:

1. Let $z_1 = 1+i,\ z_2 = 1+i\sqrt{3}$ and $\theta_1,\ \theta_2$ its argument respectively. We have $|z_1| = \sqrt{2}$ and $|z_2| = 2$. Therefore

$$\begin{cases} \cos\theta_1 = 1/\sqrt{2} \\ \sin\theta_1 = 1/\sqrt{2} \end{cases} \text{and} \begin{cases} \cos\theta_2 = 1/2 \\ \sin\theta_2 = \sqrt{3}/2 \end{cases}$$

Then $\theta_1 = \pi/4 + 2n\pi$, $\theta_2 = \pi/3 + 2n\pi$. Hence

$$z_1 = \sqrt{2}e^{i\pi/4}, \quad z_2 = 2e^{i\pi/3}.$$

2. Let $z = \left(\frac{1+i\sqrt{3}}{1+i}\right)^{2022} = \left(\frac{z_2}{z_1}\right)^{2022}$. Then

$$|z| = \left|\frac{z_2}{z_1}\right|^{2022} = \left(\sqrt{2}\right)^{2022} \text{ and } arg(z) = 2022(\frac{\pi}{3} - \frac{\pi}{4}) = \frac{337\pi}{2} = \frac{\pi}{2} + 168\pi.$$

Hence $z = (\sqrt{2})^{2022} i$.

Exercise 16. 1. Prove that $\forall z \in \mathbb{C} \setminus \{1\} : \frac{1+z}{1-z} \in i\mathbb{R} \iff |z| = 1$

2. Solve the equation: $z^3 = \bar{z}$

Solution:

1. Note that $z \in i\mathbb{R} \iff z = -\bar{z}$. Then

$$\begin{aligned} \frac{1+z}{1-z} = i\mathbb{R} &\iff \frac{1+z}{1-z} = -\frac{1+\bar{z}}{1-\bar{z}} \\ &\iff (1+z)(1-\bar{z}) = -(1-z)(1+\bar{z}) \\ &\iff 1 + z - \bar{z} - |z|^2 = -1 + z - \bar{z} + |z|^2 \\ &\iff |z| = 1. \end{aligned}$$

2. we set $z = re^{i\theta}$. Then $z^3 = r^3 e^{3i\theta}$ and $\bar{z} = re^{-i\theta}$. Hence

$$z^3 = \bar{z} \iff \begin{cases} r^3 = r \\ 3\theta = -\theta + 2n\pi \end{cases} \iff \begin{cases} (r = 0) \vee (r = +1) \vee (r = -1 \text{ exclusive}) \\ (\theta = 2k\pi) \vee (\theta = \pi + 2k\pi) \end{cases}$$

Therefore there are three solutions :

- If $r = 0$ then $z = 0$.
- If $r = +1$ then $z = e^{i\pi} = 1$ or $z = e^{2in\pi} = -1$

Exercise 17. Let $\theta \in \mathbb{R}$. Calculate:

$$A = \cos\theta + \cos(2\theta) + \cdots + \cos(n\theta)$$

$$B = \sin\theta + \sin(2\theta) + \cdots + \sin(n\theta)$$

Solution: If $\theta = 2n\pi$, then $A = n$, $B = 0$. If not we have

$$\begin{aligned} A + iB &= \sum_{k=}^{n} (e^{i\theta})^k = e^{i\theta}\frac{e^{in\theta} - 1}{e^{i\theta} - 1} = e^{i\theta}\frac{e^{in\theta/2}(e^{in\theta/2} - e^{-in\theta/2})}{e^{i\theta/2}(e^{i\theta/2} - e^{-i\theta/2})} \\ &= e^{i(n+1)\theta/2}\frac{2i\sin(n\theta/2)}{2i\sin(\theta/2)} = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\left(\cos((n+1)\theta/2) + i\sin((n+1)\theta/2)\right). \end{aligned}$$

therefore

$$A = Re(A+iB) = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\cos((n+1)\theta/2) \text{ and } B = Im(A+iB) = \frac{\sin(n\theta/2)}{\sin(\theta/2)}\sin((n+1)\theta/2)$$

Exercise 18. Simplify the following expression:

$$z = \frac{3 + 2i}{1 - i}$$

Exercise 19. Solve the equation for z:

$$z^2 + 4z + 5 = 0$$

Exercise 20. Calculate the modulus and argument of the complex number $w = 2 + 2i$.

Exercise 21. Express $z = 3e^{i\pi/4}$ in the form $x + yi$, where x and y are real numbers.

Solution:

$$z = 3(\cos(\pi/4) + i\sin(\pi/4)) = 3/\sqrt{2} + 3i/\sqrt{2}.$$

Exercise 22. Given two complex numbers $u = -1 + 2i$ and $v = 3 - i$, calculate $u \cdot v$ and $\frac{u}{v}$.

Exercise 23. Prove that for any complex number z, $|z + 1| \geq |z|$.

Solution: **False**

3 Sequences

3.1 Definitions

Definition 3.1. A real sequence (or sequence) is a mapping

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{R} \\ n & \longmapsto & u_n \end{array}$$

It is denoted by $(u_n)_{n\in\mathbb{N}}$ and u_n is called general term of the sequence.

Definition 3.2.

- A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded below if

$$\exists a \in \mathbb{R}, \forall n \in \mathbb{N} : u_n \geq a$$

- A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded above if

$$\exists b \in \mathbb{R}, \forall n \in \mathbb{N} : u_n \leq b$$

- A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded if it is bounded below and bounded above. In other words if

$$\exists C > 0, \forall n \in \mathbb{N} : |u_n| \leq C$$

Definition 3.3. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence.

1. We say $(u_n)_{n\in\mathbb{N}}$ is increasing (resp. decreasing) if

$$\forall n \in \mathbb{N} : u_n \leq u_{n+1} \text{ (resp. } u_{n+1} \leq u_n)$$

2. We say $(u_n)_{n\in\mathbb{N}}$ is constant if $\forall n \in \mathbb{N} : u_n = u_{n+1}$

3. We say $(u_n)_{n\in\mathbb{N}}$ is monotone if it increasing or decreasing.

☞ **Example 3.1.**

- The sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n = (-1)^n$ is not monotone.
- The sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n = \frac{n+1}{2n+1}$ is not monotone. Indeed, for all $n \in \mathbb{N}$:

$$\begin{aligned} u_{n+1} - u_n = \frac{n+2}{2n+3} - \frac{n+1}{2n+1} &= \frac{(n+2)(2n+1) - (2n+3)(n+1)}{(2n+3)(2n+1)} \\ &= \frac{-1}{(2n+3)(2n+1)} < 0 \end{aligned}$$

- Consider the geometric sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n = a^n$ is monotone if and only if $a \geq 0$. Indeed, if $a > 0$, we have for all $n \in \mathbb{N}$:

$$\frac{u_{n+1}}{u_n} = \frac{a^{n+1}}{a^n} = a.$$

Thus, if $a < 1$, the sequence is decreasing, if $a > 1$ it is increasing and if $a = 1$ or $a = 0$, it is constant. Now if $a < 0$, then $u_{n+1} - u_n = a^n(a-1)$ which is positive if n is even and negative if n is odd.

3.2 Convergence

Definition 3.4. Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence and $\ell \in \mathbb{R}$.

1. We say ℓ is a limit of the sequence $(u_n)_{n\in\mathbb{N}}$ and we write $\lim_{n\to+\infty} = \ell$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N},\ \forall n \geq N : |u_n - \ell| \leq \varepsilon$$

$$(\text{ or } \forall \varepsilon > 0, \exists N \in \mathbb{N} : n \geq N \Longrightarrow |u_n - \ell| \leq \varepsilon)$$

Definition 3.5. 1. We say the sequence $(u_n)_{n\in\mathbb{N}}$ has $+\infty$ as a limit and we write $\lim_{n\to+\infty} = +\infty$, if

$$\forall A > 0,\ \exists n \in \mathbb{N},\ \forall n \geq N : u_n \geq A.$$

2. We say the sequence $(u_n)_{n\in\mathbb{N}}$ has $-\infty$ as a limit and we write $\lim_{n\to+\infty} = -\infty$, if

$$\forall B < 0,\ \exists n \in \mathbb{N},\ \forall n \geq N : u_n \leq B.$$

Definition 3.6. Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence and $\ell \in \mathbb{R}$. We say the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent (or converges to ℓ) if it has a limit $\ell \in \mathbb{R}$. Otherwise, we say it is divergent.

☞ **Example 3.2.** 1. If $u_n = c$, $\forall n \in \mathbb{N}$, then $(u_n)_{n\in\mathbb{N}}$ is convergent to the limit c. Indeed, we have

$$\forall \varepsilon > 0, \exists N = 0,\ \forall n \geq N : |u_n - c| = |c - c| = 0 < \varepsilon.$$

2. If $u_n = \frac{1}{n}$ $\forall n \in \mathbb{N}^*$, then $(u_n)_{n\in\mathbb{N}^*}$ converges to 0. Indeed. Let $\varepsilon > 0$ and $n \in \mathbb{N}^*$. We have

$$|u_n - 0| \leq \varepsilon \Longleftrightarrow \frac{1}{n} \leq \varepsilon \Longleftrightarrow n \geq \frac{1}{\varepsilon}.$$

Hence it suffices to choose $N \geq \frac{1}{\varepsilon}$, that is for example $N = [1/\varepsilon] + 1$. Then $n \geq N \Longrightarrow |u_n - 0| \leq \varepsilon$.

3. If $u_n = \frac{n+1}{2n+1}$ $\forall n \in \mathbb{N}^*$, then $(u_n)_{n\in\mathbb{N}^*}$ converges to $1/2$. Indeed. Let $\varepsilon > 0$ and $n \in \mathbb{N}^*$. We have

$$|u_n - \tfrac{1}{2}| \leq \varepsilon \Longleftrightarrow \tfrac{1}{4n+2} \leq \varepsilon \Longleftrightarrow n \geq \tfrac{1}{4\varepsilon} - \tfrac{1}{2}.$$

Hence it suffices to choose $N \geq \frac{1}{4\varepsilon} - \frac{1}{2}$, that is for example $N = [1/4\varepsilon] + 1$. Then $n \geq N \Longrightarrow |u_n - 1/2| \leq \varepsilon$.

4. $u_n = (-1)^n$. Then the sequence $(u_n)_{n\in\mathbb{N}}$ is not convergent. Indeed, if not there exists $\ell \in \mathbb{R}$ such that $\lim_{n\to\infty} u_n = \ell$. Hence taking $\varepsilon = 1/2$. according to the the definition of the limit,

$$\exists N \in \mathbb{N},\ \forall n \geq N : |u_n - \ell| \leq \tfrac{1}{2}.$$

but we have for $n = 2N$

$$2 = |u_{n+1} - u_n| \leq |u_{n+1} - \ell| + |u_n - \ell| \leq 1/2 + 1/2 = 1$$

which is a contradiction.

5. If $u_n = n^2$, then $\lim_{\to+\infty} u_n = +\infty$. Indeed, given any $A > 0$. Then

$$u_n \geq A \Longleftrightarrow n \geq \sqrt{A}.$$

Therefore, $\forall n \geq [\sqrt{A}] + 1$, we have $n \geq A$ which implies that $u_n = n^2 \geq A$.

☞ **Example 3.3.** Calculate the limit of the sequence $(u_n)_{n\in\mathbb{N}}$ in the following cases

1. $u_n = a^n$, $a \in \mathbb{R}_+$
2. $u_n = n(e^{1/n} - 1)$.
3. $u_n = \sum\limits_{k=0}^{n} \frac{1}{2^k}$.
4. $u_n =$

1. For $u_n = a^n$, where $a \in \mathbb{R}_+$:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} a^n = \begin{cases} +\infty, & \text{if } a > 1 \\ 0, & \text{if } 0 < a < 1 \\ 1, & \text{if } a = 1 \end{cases}$$

2. For $u_n = n(e^{1/n} - 1)$:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} n(e^{1/n} - 1) = 0$$

3. For $u_n = \sum_{k=0}^{n} \frac{1}{2^k}$:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{2^k} = 2$$

Proposition 3.1 (Uniqueness). A convergent sequence has a unique limit.

Proof. Suppose that $(u_n)_{n\in\mathbb{N}}$ has two limits $\ell_1,\ \ell_2$ such that $\ell_1 \neq \ell_2$. Take $\varepsilon = |\ell_1 - \ell_2| > 0$. From the definition of the limit, there are $N_1,\ N_2 \in \mathbb{N}$ such that

$$\forall n \geq N_1 : |u_n - \ell_1| < \varepsilon/2, \quad \forall n \geq N_2 : |u_n - \ell_2| < \varepsilon/2.$$

Hence for $n \geq \max\{N_1, N_2\}$, we have

$$\varepsilon = |\ell_1 - \ell_2| \leq |u_n - \ell_1| + |u_n - \ell_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

contradiction. □

Proposition 3.2. A convergent sequence is bounded.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a convergent sequence to the limit ℓ. Hence taking $\varepsilon = 1$. According to the the definition of the limit,

$$\exists N \in \mathbb{N},\ \forall n \geq N : |u_n - \ell| \leq 1.$$

Then

$$\forall n \geq N : |u_n| \leq |u_n - \ell| + |\ell| \leq 1 + \ell := M_0 \tag{3.1}$$

and we have

$$\forall n \leq N : |u_n| \leq M_1 := \max\{|u_0|, |u_1|, ..., |u_N|\} \tag{3.2}$$

From (3.1) and (3.2), we deduce that $\forall n \in \mathbb{N} : |u_n| \leq M := \max\{M_0, M_1\}$. Hence the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded. □

Proposition 3.3. Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two sequences converging respectively to u and v. Then

1. $\lim\limits_{n\to+\infty} (\lambda u_n) = \lambda u,\ \forall \lambda \in \mathbb{R}$.

2. $\lim_{n\to+\infty}(u_n + v_n) = u + v$.

3. $\lim_{n\to+\infty}(u_n v_n) = uv$.

4. $\lim_{n\to+\infty} \frac{1}{u_n} = \frac{1}{u}$, if $u \neq 0$ and $u_n \neq 0,\ \forall n \in \mathbb{N}$.

Proof. 1. If $\lambda = 0$, then $\lambda u_n = 0 \to 0$, as $n \to +\infty$. If not, let $\varepsilon > 0$. According to the the definition of the limit, $\exists N \in \mathbb{N}$:

$$\boxed{\forall n \geq N : |u_n - u| \leq \tfrac{\varepsilon}{|\lambda|}.}$$

Hence

$$\boxed{\boxed{\forall n \geq N : |\lambda u_n - \lambda u| = |\lambda||u_n - u| \leq \varepsilon.}}$$

2. Let $\varepsilon > 0$. According to the the definition of the limit, there are $N_1, N_2 \in \mathbb{N}$:

$$\boxed{\forall n \geq N_1 : |u_n - u| \leq \tfrac{\varepsilon}{2}, \quad \forall n \geq N_2 : |v_n - v| \leq \tfrac{\varepsilon}{2}}$$

Hence, taking $N := \max\{N_1, N_2\}$

$$\boxed{\boxed{\forall n \geq N : |(u_n + v_n) - (u + v)| \leq |u_n - u| + |v_n - v| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.}}$$

3. Let $\varepsilon > 0$.

$$\boxed{|u_n v_n - uv| = |u_n(v_n - v) + v(u_n - u)| \leq |u_n||v_n - v| + |v||u_n - u|}$$

Since $(u_n)_{n\in\mathbb{N}}$ converges, it is bounded (see Proposition 3.2). Therefore, there is $M > 0$ such that

$$\boxed{\forall n \in \mathbb{N} : |u_n| \leq M.}$$

$$\boxed{|u_n v_n - uv| \leq M|v_n - v| + v|u_n - u| \leq M'(|v_n - v| + |u_n - u|), \;\; M' = \max\{M, v\}.}$$

Otherwise, according to the the definition of the limit, there are $N_1, N_2 \in \mathbb{N}$:

$$\boxed{\forall n \geq N_1 : |u_n - u| \leq \tfrac{\varepsilon}{2M'}, \quad \forall n \geq N_2 : |v_n - v| \leq \tfrac{\varepsilon}{2M'}.}$$

Hence, $\forall n \geq N := \max\{N_1, N_2\}$, we have

$$\boxed{\boxed{|u_n v_n - uv| \leq M'(\frac{\varepsilon}{2M'} + \frac{\varepsilon}{2M'}) = \varepsilon}.}$$

4. The proof is left as an exercise.

□

Proposition 3.4. Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.

1. If $(u_n)_{n\in\mathbb{N}}$ is increasing and bounded above, it is convergent.
2. If $(u_n)_{n\in\mathbb{N}}$ is decreasing and bounded below, it is convergent.

Proof.
1. Let $(u_n)_{n\in\mathbb{N}}$ be an increasing sequence that is bounded above. Consider the set
$$A = \{u_n : n \in \mathbb{N}\}.$$
Since A is a set of real numbers and is bounded above, it has a least upper bound (supremum) denoted by α. We claim that $\lim_{n\to\infty} u_n = \alpha$.
Given $\varepsilon > 0$, since $\alpha = \sup A$, there exists an element u_N in A such that
$$\boxed{\alpha - \varepsilon < u_N.}$$
Since (u_n) is increasing,
$$\forall n \geq N : \alpha - \varepsilon < a_N \leq u_n.$$
This implies
$$\forall n \geq N : |u_n - \alpha| = \alpha - u_n \leq \varepsilon.$$
which satisfies the definition of the limit. This proves that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to α.
2. The proof is left as an exercise.

□

Proposition 3.5. Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}}$ be three sequences such that
$$\boxed{\forall n \in \mathbb{N} : u_n \leq v_n \leq w_n.}$$
Then
$$\boxed{\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} w_n = \ell \Longrightarrow \lim_{n\to+\infty} v_n = \ell.}$$

Proof. Assume that $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} w_n = \ell$. We want to show that $\lim_{n\to+\infty} v_n = \ell$.

Given any $\varepsilon > 0$, since $\lim_{n\to+\infty} u_n = \ell$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|u_n - \ell| < \varepsilon$.

Similarly, since $\lim_{n\to+\infty} w_n = \ell$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $|w_n - \ell| < \varepsilon$.

Let $N = \max\{N_1, N_2\}$. For all $n \geq N$, we have $u_n \leq v_n \leq w_n$, which implies
$$\ell - \varepsilon < u_n \leq v_n \leq w_n < \ell + \varepsilon.$$

Therefore, for all $n \geq N$, we have $|v_n - \ell| < \varepsilon$. This shows that $\lim_{n\to+\infty} v_n = \ell$, as desired.

Hence, we have proved the proposition. □

Proposition 3.6. Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ be two sequences such that

$$\forall n \in \mathbb{N} : u_n \leq v_n$$

Then

1. $\lim_{n\to+\infty} u_n = +\infty \Longrightarrow \lim_{n\to+\infty} v_n = +\infty$.
2. $\lim_{n\to+\infty} v_n = -\infty \Longrightarrow \lim_{n\to+\infty} u_n = -\infty$.

Proof. 1. Assume that $\lim_{n\to+\infty} u_n = +\infty$. We want to show that $\lim_{n\to+\infty} v_n = +\infty$.

Given any $M > 0$, since $\lim_{n\to+\infty} u_n = +\infty$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $u_n > M$.

Since $u_n \leq v_n$ for all $n \in \mathbb{N}$, it follows that $v_n \geq u_n > M$ for all $n \geq N_1$. This implies that $\lim_{n\to+\infty} v_n = +\infty$.

2. Assume that $\lim_{n\to+\infty} v_n = -\infty$. We want to show that $\lim_{n\to+\infty} u_n = -\infty$.

Given any $M < 0$, since $\lim_{n\to+\infty} v_n = -\infty$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $v_n < M$.

Since $u_n \leq v_n$ for all $n \in \mathbb{N}$, it follows that $u_n \leq v_n < M$ for all $n \geq N_2$. This implies that $\lim_{n\to+\infty} u_n = -\infty$.

Hence, both parts of the proposition have been proved. □

3.3 Subsequence

Definition 3.7. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence and $k_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers. Then the sequence $(u_{k_n})_{n\in\mathbb{N}}$ is called a subsequence of $(u_n)_{n\in\mathbb{N}}$.

☞ **Example 3.4.**

- The sequences $(u_{2n})_{n\in\mathbb{N}}$, $(u_{2n+1})_{n\in\mathbb{N}}$ are sub-sequences of $(u_n)_{n\in\mathbb{N}}$ (with $k_n = 2n$, $k_n = 2n+1$ respectively).
- The sequence $(u_{6n})_{n\in\mathbb{N}}$ is a subsequence of $(u_n)_{n\in\mathbb{N}}$, with $k_n = 6n$ and it is a subsequence of $(u_{2n})_{n\in\mathbb{N}}$ with $k_n = 3n$.

Proposition 3.7. If the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent, then every subsequence $(u_{k_n})_{n\in\mathbb{N}}$ is also convergent and we have $\lim_{n\to+\infty} u_{k_n} = \lim_{n\to+\infty} u_n$.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a convergent sequence with limit ℓ, and let $(u_{k_n})_{n\in\mathbb{N}}$ be a subsequence (indexed by natural numbers k_n, where $k_0 < k_1 < k_2 < k_3 < \dots$). Since $(u_n)_{n\in\mathbb{N}}$ converges to ℓ, for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\forall n \geq N : |u_n - \ell| \leq \varepsilon$$

Now, since $(u_{k_n})_{n\in\mathbb{N}}$ is a subsequence, then $k_n \to +\infty$ as $n \to +\infty$. so we can find N' such that

$$\forall n \geq N' : k_n \geq N.$$

By the convergence of $(u_n)_{n\in\mathbb{N}}$, we have

$$n \geq N' \Longrightarrow k_n \geq N \Longrightarrow |u_{k_n} - \ell| \leq \varepsilon$$

This satisfies the definition of convergence of the sub sequence. □

Theorem 3.8. Every bounded sequence $(u_n)_{n\in\mathbb{N}}$ has convergent subsequence.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence. This means that there exists a constant $M > 0$ such that $|u_n| \leq M$ for all $n \in \mathbb{N}$.

Consider the closed interval $[u_1 - M, u_1 + M]$. Since the sequence is bounded, all of its terms must lie within this interval. Now, divide this interval into two closed subintervals of equal length: $[u_1 - M, u_1]$ and $[u_1, u_1 + M]$.

At least one of these subintervals must contain infinitely many terms of the sequence $(u_n)_{n\in\mathbb{N}}$. Let's denote the chosen subinterval as I_1.

Next, divide I_1 into two equal subintervals and proceed similarly: choose the one that contains infinitely many terms of the sequence. Denote this subinterval as I_2.

Continue this process recursively. At the k-th step, divide the current interval into two equal subintervals and choose the one containing infinitely many terms of the sequence. Denote this subinterval as I_k.

We now have a nested sequence of closed intervals:

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

By the nested interval property of real numbers, there exists a unique point c that belongs to all of these intervals:

$$c \in \bigcap_{k=1}^{\infty} I_k$$

Since each interval I_k contains infinitely many terms of the sequence, it follows that c is a limit point of the sequence. Therefore, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converging to c.

Thus, every bounded sequence has a convergent subsequence. □

3.4 Cauchy sequence

Definition 3.8. The sequence $(u_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence, if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall p, q \geq N : |u_p - u_q| \leq \frac{\varepsilon}{2}$$

Proposition 3.9. *A convergent sequence is Cauchy.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a convergent sequence with limit ℓ. This means that for any $\varepsilon > 0$, there exists an N such that

$$\forall n \geq N : |u_n - \ell| < \frac{\varepsilon}{2}$$

Now, let's choose two arbitrary indices p and q such that $p, q \geq N$. Then, by the triangle inequality,

$$|u_p - u_q| \leq |u_p - \ell| + |\ell - u_q| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that for any $\varepsilon > 0$, there exists an N such that for all $p, q \geq N$, $|u_p - u_q| \leq \varepsilon$, which is the definition of a Cauchy sequence. Hence, a convergent sequence is a Cauchy sequence. □

Proposition 3.10. *Every Cauchy sequence is convergent*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Let $\varepsilon > 0$. Then, there exists an $N_1 \in \mathbb{N}$ such that

$$\forall p, q \geq N_1 : |u_p - u_q| < \frac{\varepsilon}{2}. \tag{3.3}$$

Since $(u_n)_{n\in\mathbb{N}}$ is Cauchy, it is also bounded. By the Bolzano-Weierstrass theorem 3.8, there exists a convergent subsequence $(u_{k_n})_{n\in\mathbb{N}}$ of (u_n). Let ℓ be the limit of this subsequence. Then there exists an $N_2 \in \mathbb{N}$ such that

$$\forall n \geq N_2 : |u_{k_n} - \ell| < \frac{\varepsilon}{2} \tag{3.4}$$

Now, we will show that the entire sequence (u_n) converges to ℓ. By the definition of the subsequence, we have $k_n \to +\infty$ as $n \to +\infty$, Hence, there exists $N_3 \in \mathbb{N}$ such that

$$\forall n \geq N_3 : k_n \geq N_1. \tag{3.5}$$

Choose $N = \max\{N_1, N_2, N_3\}$. Then, for all $n \geq N$, we have from (3.3),(3.4) and (3.5) :

$$|u_n - \ell| \leq |u_n - u_{k_n}| + |u_{k_n} - \ell| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which proves that $\lim_{n\to\infty} u_n = \ell$ and the proposition is proved. □

3.5 Exercises

Exercise 24. Using the definition of limit, show that

$$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}, \quad \lim_{n\to\infty} \frac{\sin n}{n} = 0, \quad \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0.$$

Exercise 25.

1. Show that every constant sequence is convergent.
2. Let $(u_n)_n$ be a sequence with terms in $\mathbb{Z}$. Show that if $(u_n)_n$ is convergent, then it is constant.

Exercise 26. Calculate the following limits

$$\lim_{n\to+\infty} (\sqrt{n^2+n} - n), \quad \lim_{n\to+\infty} \frac{1}{n}\sin(n^2), \quad \lim_{n\to+\infty} \frac{n+(-1)^n}{2n+(-1)^n}, \quad \lim_{n\to+\infty} \sqrt[n]{a} \quad (a > 0)$$

$$\lim_{n\to+\infty} \frac{1+2+\cdots+n}{n^2}, \quad \lim_{n\to+\infty} \frac{1+2^2+\cdots+n^2}{n^3}, \quad \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)}$$

(**ind.** $1+2+\cdots+n = \frac{1}{2}n(n+1), \quad 1+2^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$)

Exercise 27. Using the squeeze theorem, show that the following sequences are convergent.

$$\sum_{k=1}^{n} \frac{n}{n^2+k}, \quad \frac{1}{n^2}\sum_{k=1}^{n} [kx], \ (x \in \mathbb{R}).$$

Exercise 28. Let $(u_n)_n$ be the sequence defined by $u_n = \sum_{k=1}^{n} \frac{1}{k^2}$, for all $n \geq 1$.

1. Show that $(u_n)_n$ is increasing.
2. Show that it is bounded (**Ind.** $\frac{1}{k^2} \leq \frac{1}{k(k-1)} := \frac{1}{k-1} - \frac{1}{k}, \ \forall k \geq 2$).
3. Deduce.

Exercise 29. Let $(u_n)_n$ be the sequence defined by

$$\begin{cases} u_0 = \frac{1}{2} \\ u_{n+1} = u_n^2 + \frac{3}{16}, \quad n \geq 1 \end{cases}$$

1. Show that for all $n \in \mathbb{N} : \frac{1}{4} < u_n \leq \frac{1}{2}$
2. Study the monotonicity of $(u_n)_n$ and deduce its nature.
3. Calculate $\lim_{n\to\infty} u_n$ and deduce $\inf\{u_n / \ n \in \mathbb{N}\}$ and $\sup\{u_n / \ n \in \mathbb{N}\}$

Exercise 30. Let $a > 0$ and $(u_n)_n$ be the sequence defined by

$$\begin{cases} u_0 = 1 \\ u_{n+1} = \frac{1}{2}(u_n + \frac{a}{u_n}), \quad n \geq 1 \end{cases}$$

1. Show that for all $n \in \mathbb{N} : u_n > 0$
2. Suppose that $(u_n)_n$ is convergent. Calculate its limit ℓ
3. Show that for all $n \in \mathbb{N}^* : u_n - \ell \geq 0$
4. Deduce that $(u_n)_{n\in\mathbb{N}^*}$ is decreasing and conclude its nature.
5. Using a calculator, provide an approximation of $\sqrt{2}$ accurate to 10^{-4}.

Exercise 31. Let $(u_n)_n$ be the sequence defined by

$$\begin{cases} u_0 = a, u_1 = b \\ u_{n+1} = \frac{1}{2}(u_n + u_{n-1}) \quad n \geq 2 \end{cases}$$

1. Calculate $u_n - u_{n-1}$ as a function of n for $n \geq 1$.
2. Show that the sequence $(u_n)_n$ is convergent and determine its limit.

Exercise 32. Consider the two sequences $(u_n)_n$ and $(v_n)_n$ defined by

$$\forall n \in \mathbb{N} : u_n = \sum_{k=0}^{n} \frac{1}{k!}, \quad v_n = u_n + \frac{1}{n!}$$

1. Calculate $u_0,\ u_1,\ u_2,\ v_0,\ v_1,\ v_2$.
2. Show that the two sequences are adjacent.
3. Let e be their limit. Show that e is not rational.

Exercise 33. Show that the following sequences are not convergent:

$$(-1)^n, \quad \frac{n}{2} - \left[\frac{n}{2}\right], \quad \sin(\sqrt{n}\frac{\pi}{2})$$

Exercise 34. Let $(u_n)_n$ be the sequence defined by $u_n = \sum_{k=1}^{n} \frac{(-1)^k}{k}$, for $n \geq 1$.

1. Show that the subsequence $(u_{2n})_n$ is decreasing.
2. Show that the subsequence $(u_{2n+1})_n$ is increasing.
3. Calculate $\lim_{n\to\infty}(u_{2n} - u_{2n+1})$.
4. Deduce that the sequence $(u_n)_n$ is convergent.

Exercise 35. Using the Cauchy criterion, show that the sequence with general term $(-1)^n$ is not convergent.

Exercise 36.

1. Show that the sequence with general term $u_n = \sum_{k=1}^{n} \frac{1}{k}$ is not Cauchy. (**Ind.** Choose $p = N$ and $q = 2N$, for any $N \in \mathbb{N}^*$).
2. Deduce.
3. Show that $(u_n)_n$ is increasing and deduce its limit.
4. Repeat the same questions for the sequence $v_n = \sum_{k=2}^{n} \frac{1}{\ln k}$.

4 Real functions

4.1 Preliminaries

Definition 4.1. A function is a relation f between two sets E and F such that, every element $x \in E$ has at most a relation with an element $y \in F$ denoted by $f(x)$ and we write

$$\begin{array}{rcl} f : E & \longrightarrow & F \\ x & \longmapsto & y := f(x) \end{array}$$

The domain of definition of f is the set defined by

$$D_f := \{x \in E : f(x) \text{ exists}\}.$$

4.2 Limits

Definition 4.2. Let I be an open interval, $x_0 \in I$, $\ell \in \mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function.

1. We say the function f has a left limit ℓ at x_0 and we write $\lim\limits_{x \to x_0^-} f(x) = \ell$, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I : -\delta < x - x_0 < 0 \Longrightarrow |f(x) - \ell| \leq \varepsilon.$$

2. We say the function f has a right limit ℓ at x_0 and we write $\lim\limits_{x \to x_0^+} f(x) = \ell$, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I : 0 < x - x_0 < \delta \Longrightarrow |f(x) - \ell| \leq \varepsilon.$$

3. We say the function f has a limit ℓ at x_0 and we write $\lim\limits_{x \to x_0} f(x) = \ell$, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I : 0 < |x - x_0| < \delta \Longrightarrow |f(x) - \ell| \leq \varepsilon.$$

Or equivalently (prove it), if $\lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x) = \ell.$

☞ **Remark 4.1.** • We also denote the limit by "arrow" notation $f(x) \to \ell$ as $x \to x_0$ and say $f(x)$ goes to ℓ as x goes to x_0.

- It follows directly from the above definition that

$$\lim_{x \to x_0} f(x) = \ell \Longleftrightarrow \lim_{x \to x_0} |f(x) - \ell| = 0$$

☞ **Example 4.1.** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function.

1. If $f(x) = 2x$. Show that $\lim_{x\to 1} f(x) = 2$.

2. If $f(x) = x \sin \frac{1}{x}$. Show that $\lim_{x\to 0} f(x) = 0$

3. If $f(x) = \mathbf{sgn}\, x := \begin{cases} +1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$, **(the sign function)**. Show that

$$\lim_{x\to 0^-} f(x) = -1, \quad \lim_{x\to 0^+} f(x) = +1.$$

although the corresponding limit does not exist.

4. If $f(x) = \sin \frac{1}{x}$. Show that $\lim_{x\to 0} f(x)$ does not exist.

Definition 4.3 (Limits as $x \to \pm\infty$). Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function well defined for all $x < -M$ and $x > M$ for certain $M > 0$ and $\ell \in \mathbb{R}$. We say the limit of f equal to ℓ at $+\infty$ (resp. $-\infty$) and we write $\lim_{x\to+\infty} f(x) = \ell$ (resp. $\lim_{x\to-\infty} f(x) = \ell$ if

$$\forall \varepsilon > 0, \exists A > 0, \forall x \in, I : x > A \Longrightarrow |f(x) - \ell| \leq \varepsilon$$

$$(\text{resp. } \forall \varepsilon > 0, \exists B < 0, \forall x \in, I : x < B \Longrightarrow |f(x) - \ell| \leq \varepsilon$$

Proposition 4.1 (Algebraic properties). Let $f, g, h : I \longrightarrow \mathbb{R}$ be functions and $x_0 \in I$. Suppose that

$$\lim_{x\to x_0} f(x) = L, \quad \lim_{x\to x_0} g(x) = M.$$

Then

- $\lim_{x\to x_0} \lambda f(x) = \lambda L$ for every $\lambda \in \mathbb{R}$.
- $\lim_{x\to x_0} (f(x) + g(x)) = L + M$.
- $\lim_{x\to x_0} f(x)g(x) = LM$.
- $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$ and $g(x) \neq 0, \forall x \in I$.

Proof. We will prove each part separately using the epsilon-delta definition of limits.

Part 1: Let $\lambda \in \mathbb{R}^*$ and $\varepsilon > 0$. Since $\lim_{x \to x_0} f(x) = L$, there exists $\delta > 0$ such that for all $x \in I$ with $0 < |x - x_0| < \delta$, we have $|f(x) - L| < \frac{\varepsilon}{|\lambda|}$. Now, for such x, we have

$$|\lambda f(x) - \lambda L| = |\lambda| \cdot |f(x) - L| < |\lambda| \cdot \frac{\varepsilon}{|\lambda|} = \varepsilon.$$

This shows that $\lim_{x \to x_0} \lambda f(x) = \lambda L$.

Part 2: Let $\varepsilon > 0$. Since $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $x \in I$ we have

$$\boxed{0 < |x - x_0| < \delta_1 \implies |f(x) - L| < \tfrac{\varepsilon}{2}}$$

$$\boxed{0 < |x - x_0| < \delta_2 \implies |g(x) - M| < \tfrac{\varepsilon}{2}}$$

.

Let $\delta = \min(\delta_1, \delta_2)$. For all $x \in I$ with $0 < |x - x_0| < \delta$, we have

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\lim_{x \to x_0}(f(x)+g(x)) = L+M$. **Part 3:** Let $\varepsilon > 0$. Since $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $x \in I$ we have

$$\boxed{0 < |x - x_0| < \delta_1 \implies |f(x) - L| < \epsilon}$$

$$\boxed{0 < |x - x_0| < \delta_2 \implies |g(x) - M| < \epsilon}$$

where $\epsilon > 0$ will be chosen later. Let $\delta = \min(\delta_1, \delta_2)$. For all $x \in I$ with $0 < |x - x_0| < \delta$, we have

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &\leq |f(x)||g(x) - M| + |M||f(x) - L| \\ &\leq (|L| + \epsilon)\epsilon + |M|\epsilon = \epsilon^2 + (|L| + |M|)\epsilon. \end{aligned}$$

We can make the expression smaller than ε by appropriately choosing ϵ. **Part 4:** Let $\varepsilon > 0$. Since $M \neq 0$, there exists $\delta_1 > 0$ such that for all $x \in I$ with $0 < |x - x_0| < \delta_1$, we have $|g(x) - M| < \frac{|M|}{2}$.

Additionally, since $\lim_{x \to x_0} f(x) = L$, there exists $\delta_2 > 0$ such that for all $x \in I$ with $0 < |x - x_0| < \delta_2$, we have $|f(x) - L| < \frac{\varepsilon |M|}{2}$.

Let $\delta = \min(\delta_1, \delta_2)$. For all $x \in I$ with $0 < |x - x_0| < \delta$, we have

$$\left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| = \frac{|f(x)M - g(x)L|}{|g(x)M|} \leq \frac{|f(x) - L| \cdot |M| + |g(x) - M| \cdot |L|}{|g(x)| \cdot |M|} < \frac{\frac{\varepsilon|M|}{2} \cdot |M| + \frac{|M|}{2} \cdot |L|}{\frac{|M|}{2} \cdot |M|}.$$

Since $|M|$ is not zero, we can choose δ small enough such that the expression becomes smaller than ε. $\square$

4.3 Continuity

In this paragraph, I is an open interval, $x_0 \in I$, $f : I \longrightarrow \mathbb{R}$ is a function well defined for all $x \in I$.

Definition 4.4 (**Continuity**). 1. We say that f is continuous at x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$, that is,

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I : |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| \le \varepsilon.$$

2. We say f is continuous (on I) if it is continuous at every point $x_0 \in I$.

3. We say that f is continuous from the left at x_0 if $\lim_{x\to x_0^-} f(x) = f(x_0)$, that is

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I : -\delta < x - x_0 \le 0 \Longrightarrow |f(x) - f(x_0)| \le \varepsilon.$$

4. We say that f is continuous from the left at x_0 if $\lim_{x\to x_0^+} f(x) = f(x_0)$, that is

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I : 0 \le x - x_0 < +\delta \Longrightarrow |f(x) - f(x_0)| \le \varepsilon.$$

☞ **Remark 4.2.** • It follows from the above definition that f is continuous at x_0 if and only if

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = f(x_0).$$

☞ **Example 4.2.** 1. The function $f(x) = x^2$ is continuous at all points in $\mathbb{R}$. Indeed

2. The function $f(x) = \begin{cases} x\ln x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous at 0^+

3. The sign function $f(x) = \textbf{sgn}\, x := \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$ is not continuous at 0 since $\lim_{x\to x_0} f(x)$ does not exist.

4. The function $f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ is not continuous at 0, since $\lim_{x\to x_0} f(x) = 0 \neq 1 := f(0)$.

5. Study the continuity of the following function

$$f(x) = \begin{cases} x^2 & \text{if } \; x < 1 \\ x^1 + 1 & \text{if } \; x \geq 1 \end{cases}$$

Theorem 4.2. If f and g are continuous functions at x_0, then so are λf, $f + g$ and fg. If in addition $g(x_0) \neq 0$, then f/g is continuous at x_0.

Proof. Exercise □

Theorem 4.3. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function such that $f(a)f(b) \leq 0$. Then, there exists $c \in [a, b]$ such that $f(c) = 0$.

Proof. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a)f(b) \leq 0$. We aim to show that there exists $c \in [a, b]$ such that $f(c) = 0$. Without loss of generality, assume $f(a) \leq f(b)$. If $f(a) = 0$ or $f(b) = 0$, we are done, so let's consider the case where $f(a) < 0$ and $f(b) > 0$. Define the set

$$S = \{x \in [a, b] \mid f(x) \leq 0\}.$$

Notice that $a \in S$ since $f(a) \leq 0$, and $b \notin S$ since $f(b) > 0$. Therefore, S is nonempty and bounded above by b, so $\sup S$ exists.

Let $c = \sup S$. We will show that $f(c) = 0$.

Since c is the supremum of S, for any $\varepsilon > 0$, there exists $x \in S$ such that $c - \varepsilon < x \leq c$. This implies $f(x) \leq 0$.

Because f is continuous, as ε approaches 0, $f(x)$ approaches $f(c)$. Since $f(x) \leq 0$ for all $x \in S$, we have $f(c) \leq 0$.

Suppose, for the sake of contradiction, that $f(c) < 0$. Then by continuity of f, there exists $\delta > 0$ such that for all x with $|x - c| < \delta$, we have $f(x) < 0$. This contradicts the fact that $c = \sup S$.

Hence, we must have $f(c) \geq 0$.

Since we've shown both $f(c) \leq 0$ and $f(c) \geq 0$, it follows that $f(c) = 0$.

Thus, in all cases, there exists $c \in [a, b]$ such that $f(c) = 0$, completing the proof of the Intermediate Value Theorem.

□

Theorem 4.4 ((Weierstrass extreme value)). If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on the closed and bounded interval $[a, b]$. Then f is bounded on $[a, b]$ and attains its maximum and minimum values on $[a, b]$. That is

$$\exists c_1, c_2 \in [a, b] : f(c_1) = \min_{x \in [a,b]} f(x), \; f(c_2) = \max_{x \in [a,b]} f(x)$$

4.4 Uniform continuity

Definition 4.5. Let $f : I \longrightarrow \mathbb{R}$ be a function. We say f is uniformly continuous if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in I : |x - y| \leq \delta \Longrightarrow |f(x) - f(y)| \leq \varepsilon.$$

☞ **Remark 4.3.** In other words, f is uniformly continuous if $f(x) - f(y) \to 0$ as $x - y \to 0$.

☞ **Example 4.3.** 1. $f : [0, 1] \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous. Indeed, given $\varepsilon > 0$. We have

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \leq 2|x - y|.$$

Taking $\delta = \varepsilon/2$, so

$$|x - y| \leq \delta \Longrightarrow 2|x - y| \leq \varepsilon \Longrightarrow |f(x) - f(y)| \leq \varepsilon.$$

2. $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous. Indeed, for $\varepsilon = 2$, taking $x_n = n + 1/n$, $y_n = n$. Then $\forall \delta > 0$, there exists $n \in \mathbb{N}^*$ such that $|x_n - y_n| = 1/n \leq \delta$ and

$$|f(x_n) - f(y_n)| = |(n + 1/n)^2 - n^2| = 2 + 1/n^2 \geq 2 = \varepsilon.$$

3. $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ defined by $f(x) = 1/x$ is not uniformly continuous. Indeed, for $\varepsilon = 1$, taking $x_n = 1/n$, $y_n = \frac{1}{n+1}$. Then $\forall \delta > 0$, there exists $n \in \mathbb{N}^*$ such that $|x_n - y_n| \leq 1/n \leq \delta$ and

$$|f(x_n) - f(y_n)| = |(n + 1) - n| = 1 \geq 1 = \varepsilon.$$

Proposition 4.5. Every uniformly continuous function is continuous

Proof. Let $f : I \longrightarrow \mathbb{R}$ be uniformly continuous function. Given any $x_0 \in I$, then □

Theorem 4.6. Let $f :]a, b[$ be a continuous function such that $\lim\limits_{x \to a+} f(x)$, $\lim\limits_{x \to b-} f(x)$ exist and finite. Then f is uniformly continuous.

Proof. Let $\varepsilon > 0$ be given. We need to show that there exists a $\delta > 0$ such that for all $x, y \in]a, b[$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

Since $\lim_{x \to a+} f(x)$ exists, there exists a $\delta_1 > 0$ such that if $a < x < x + \delta_1 < b$, then $|f(x + \delta_1) - f(x)| < \varepsilon/2$. Similarly, since $\lim_{x \to b-} f(x)$ exists, there exists a $\delta_2 > 0$ such that if $a < x - \delta_2 < x < b$, then $|f(x) - f(x - \delta_2)| < \varepsilon/2$.

Now, choose $\delta = \min(\delta_1, \delta_2)$. Let $x, y \in]a, b[$ such that $|x - y| < \delta$. Without loss of generality, assume $x < y$. Then, we have $|x - (x - \delta_2)| = \delta_2$, and $|(x + \delta_1) - y| = \delta_1$. Therefore, by the triangle inequality, we get

$$|f(x) - f(y)| \leq |f(x) - f(x - \delta_2)| + |f(x + \delta_1) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, for any $x, y \in]a, b[$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$, which shows that f is uniformly continuous. □

4.5 Differentiable functions

I is an open interval, $x_0 \in I$, $f : I \longrightarrow \mathbb{R}$ is a function well defined at all points of I

Definition 4.6. 1. We say that f is differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists and finite.}$$

This limit is denoted by $f'(x_0)$ and called derivative of f at x_0. Thus

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

If f is differentiable at all point of I, we say f is differentiable.

2. We say that f is left-differentiable at x_0 if the left limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists and finite.}$$

This limit is denoted by $f'(x_0^-)$ and called left-derivative of f at x_0.

3. We say that f is right-differentiable at x_0 if the right limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists and finite.}$$

This limit is denoted by $f'(x_0^+)$ and called right-derivative of f at x_0.

☞ **Remark 4.4.** • It is sometimes convenient to let $x = x_0 + h$ and the above limit becomes

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

- It is easy to see that f is differentiable at x_0 if and only if it is left and right differentiable at x_0 and $f'(x_0^+) = f'(x_0^-)$.

☞ **Example 4.4.** Study the differentiability of the following functions

1. $f(x) = C$, $C \in \mathbb{R}$. Given $x_0 \in \mathbb{R}$. We have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{C - C}{x - x_0} = 0$$

Thus f is differentiable and $f'(x_0) = 0$.

2. $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$. Given $x_0 \in \mathbb{R}$. We have

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{x^2-x_0^2}{x-x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0$$

Thus f is differentiable and $f'(x_0) = 2x_0$.

3. $f(x) = x^n$, $n \in \mathbb{N}^*$. Given $x_0 \in \mathbb{R}$. We have

$$\begin{aligned} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &:= \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} \\ &= \lim_{x \to x_0} \frac{(x - x_0) \sum_{k=0}^{n-1} x^{n-1-k} x_0^k}{x - x_0} \\ &= \lim_{x \to x_0} \left(\sum_{k=0}^{n-1} x^{n-1-k} x_0^k \right) \\ &= n x_0^{n-1} \end{aligned}$$

Thus, f is differentiable at x_0 and $f'(x_0) = n x_0^{n-1}$. Since this holds for every $x_0 \in \mathbb{R}$, then f is differentiable and $f'(x) = n^{n-1}$.

4. $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x}$.

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{1/(x+h) - 1/x}{h} \\ &= \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}. \end{aligned}$$

Thus f is differentiable and $f'(x) = -\frac{1}{x^2}$.

5. $f :]0, +\infty[$ defined by $f(x) = \sqrt{x}$.

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h}) + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

Thus f is differentiable and $f'(x) = \frac{1}{2\sqrt{x}}$.

6. $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = |x|$.

   - If $x > 0$ then given h such that $-x < h < x$. Then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h} = 1.$$

   Hence f is differentiable at x and $f'(x) = 1$.

- If $x < 0$ then given h such that $-x < h < x$. Then

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad = \quad \lim_{h\to 0}\frac{|x+h|-|x|}{h} = \lim_{h\to 0}\frac{-(x+h)+x}{h} = -1.$$

Hence f is differentiable at x and $f'(x) = -1$.

- If $x = 0$, then, we have

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} \quad = \quad \lim_{h\to 0}\frac{|h|}{h} = \lim_{h\to 0}\frac{h}{h} = +1.$$

and

$$\lim_{h\to 0^-}\frac{k(0+h)-k(0)}{h} \quad = \quad \lim_{h\to 0}\frac{|h|}{h} = \lim_{h\to 0}\frac{-h}{h} = -1.$$

Therefore, the limit of difference quotient does not exist. It follows that f is not differentiable at 0.

Proposition 4.7. If f is differentiable at x_0, then it is continuous at x_0.

Proof. We have

$$|f(x) - f(x_0)| = \left|\frac{f(x)-f(x_0)}{x-x_0}\right| |x - x_0|$$

passing to the limit as $x \to x_0$, taking into account that f is differentiable at x_0, we obtain $\lim\limits_{x\to x_0} |f(x) - f(x_0)| = 0$ which means that f is continuous at x_0. □

Theorem 4.8. Let f, g be a differentiable functions at x_0 then so are λf, $f + g$, fg and f/g if $g(x_0) \neq 0$.

1. $(\lambda f)'(x_0) = \lambda f'(x_0)$
2. $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
3. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
4. If $g(x_0) \neq 0$ then $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g(x_0)^2}$. In particular, we have

$$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}$$

Proof. We will prove each part separately.

1. Let λ be a constant. By the definition of the derivative, we have
$$(\lambda f)'(x_0) = \lim_{h\to 0} \frac{\lambda f(x_0+h) - \lambda f(x_0)}{h}.$$
Using the linearity of the limit, we can factor out λ and obtain
$$(\lambda f)'(x_0) = \lambda \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lambda f'(x_0).$$

2. The derivative of the sum of two functions is the sum of their derivatives:
$$(f+g)'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) + g(x_0+h) - f(x_0) - g(x_0)}{h}.$$
Using the linearity of the limit, we can separate the limit into two parts and apply the definition of the derivatives of f and g:
$$(f+g)'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} + \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} = f'(x_0) + g'(x_0).$$

3. For the product rule, we consider
$$(fg)'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h}.$$
We can rewrite the above expression as
$$(fg)'(x_0) = \lim_{h\to 0} \left(f(x_0+h)\frac{g(x_0+h) - g(x_0)}{h} + g(x_0)\frac{f(x_0+h) - f(x_0)}{h} \right).$$
Applying the definition of derivatives and continuity, we get
$$(fg)'(x_0) = f(x_0)g'(x_0) + g(x_0)f'(x_0).$$

4. Finally, for the quotient rule, we have
$$\begin{aligned}
\left(\frac{f}{g}\right)'(x_0) &= \lim_{h\to 0} \frac{\frac{f(x_0+h)}{g(x_0+h)} - \frac{f(x_0)}{g(x_0)}}{h} = \lim_{h\to 0} \frac{f(x_0+h)g(x_0) - f(x_0)g(x_0+h)}{hg(x_0+h)g(x_0)} \\
&= \lim_{h\to 0} \frac{\frac{f(x_0+h)-f(x_0)}{h} g(x_0) - f(x_0+h)\frac{g(x_0+h)-g(x_0)}{h}}{g(x_0+h)g(x_0)} \\
&= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}
\end{aligned}$$

This completes the proof. □

Theorem 4.9. Let I, J be two open intervals, $x_0 \in I$ and $f : I \longrightarrow J$, $g : J \longrightarrow \mathbb{R}$ be two functions such that $f(x_0) \in J$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$ then $g \circ f$ is differentiable at x_0 and we have

$$\boxed{(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)}.$$

Proof. Since f is differentiable at x_0, by definition, there exists a derivative $f'(x_0)$ given by

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Similarly, since g is differentiable at $y_0 = f(x_0)$, there exists a derivative $g'(f(x_0))$ given by

$$g'(f(x_0)) = \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0}.$$

Now consider the composition of the two functions $g \circ f : I \to \mathbb{R}$. The derivative of this composition at x_0 is given by

$$(g \circ f)'(x_0) = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}.$$

We set $y = f(x)$ which go to $y_0 = f(x_0)$ as $x \to x_0$ since f is continuous. Then, we have

$$\begin{aligned}
(g \circ f)'(x_0) &= \lim_{x \to x_0} \frac{g(y) - g(y_0)}{y - y_0} \frac{y - y_0}{x - x_0} \\
&= \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
&= g'(y_0)f'(x_0) = g'(f(x_0))f'(x_0),
\end{aligned}$$

which completes the proof. □

☞ **Example 4.5.**

1. $f(x) = \sqrt{x^2 + 1}$, calculate $f'(x)$.

$$\boxed{f'(x) = 2x\frac{1}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}}$$

2. $g(t) = f(x)$, $x = e^t$. Calculate $g'(t)$. We have

$$\boxed{g'(t) = (f(e^t))' = e^t f'(e^t) = xf'(x)}$$

4.6 Mean value theorem

$[a, b]$ is a closed bounded interval with $a < b$.

Lemme 4.10. Let $f :]a, b[\longrightarrow \mathbb{R}$ be a differentiable function. Suppose that f has an extreme value at a $c \in]a, b[$. Then $f'(c) = 0$

Proof. Let $f :]a, b[\to \mathbb{R}$ be a differentiable function, and suppose that f has an extreme value at $c \in]a, b[$. We aim to show that $f'(c) = 0$. Since f has an extreme value at c, it means that either $f(c)$ is a maximum or a minimum value. Without loss of generality, let's consider the case where $f(c)$ is a maximum. By the definition of a maximum, for any $x \in]a, b[$, we have $f(x) \leq f(c)$. This implies that the difference quotient

$$\boxed{\frac{f(x) - f(c)}{x - c} \geq 0, \ \forall x < c}, \text{ and } \boxed{\frac{f(x) - f(c)}{x - c} \leq 0, \ \forall x > c}.$$

Then, taking the limit as x approaches c, we have

$$\boxed{\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0}, \quad \text{and} \quad \boxed{\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0}.$$

By the differentiability of f at c, those limits can be expressed as the derivative of f at c:

$$f'(c) \leq 0 \text{ and } f'(c) \geq 0$$

which implies $f'(c) = 0$.

□

Theorem 4.11 (Rolle's theorem). Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and differentiable on $]a, b[$ such that $f(a) = f(b)$. Then

$$\boxed{\exists c \in]a, b[: f'(c) = 0}$$

☞ **Remark 4.5.** It is absolutely necessary to suppose f differentiable at all points of $]a, b[$. Consider the function $f(x) = |x|$ on $[-1, 1]$. Clearly $f(-1) = f(1)$, but there is no point c where $f'(c) = 0$.

Proof. By the Weierstrass extreme value theorem 4.4 f attains its global maximum and minimum values on $[a, b]$. If these are both attained at the endpoints, then f is constant, and $f'((c) = 0$ for all points $c \in]a, b[$. Otherwise, f attains at least one of its global maximum or minimum values at an interior point $c \in]a, b[$. Lemma 4.10 implies that $f'(c) = 0$. □

We extend Rolle's theorem to functions that attain different values at the endpoints.

Theorem 4.12 (Mean value theorem). Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function differentiable on $]a, b[$. Then there exists a point $c \in]a, b[$ such that

$$\boxed{f(b) - f(a) = (b - a) f'(c).}$$

☞ **Remark 4.6.** Graphically, this result says that there is $c \in]a, b[$ such that the slope of the tangent line at the point $(c, f(c))$ is equal to the slope of the chord between the endpoints $(a, f(a))$ and $(b, f(b))$.

Proof. Apply Rolle's theorem 4.11 to the function

$$g(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}\right](x-a).$$

□

Theorem 4.13. Let $f :]a, b[\longrightarrow \mathbb{R}$ be a differentiable function such that $f'(x) = 0$ for all $x \in]a, b[$. Then f is constant.

Proof. f is constant if $f(x) = f(y)$, $\forall x, y \in]a, b[$. Take arbitrary $x, y \in]a, b[$ with $x < y$. As $]a, b[$ is an interval, $[x, y] \subset]a, b[$. Then f restricted to $[x, y]$ satisfies the hypotheses of the mean value theorem 4.12 . Therefore, there is a $c \in]x, y[$ such that

$$f(x) - f(y) = (x - y)f'(c).$$

Since $f'(c) = 0$, we have $f(x) = f(y)$. Hence, f is constant. □

Proposition 4.14. Let $f :]a, b[\longrightarrow \mathbb{R}$ be a differentiable function. Then

- f is increasing if and only if $f'x) \geq 0$ for all $x \in]a, b[$.
- f is decreasing if and only if $f'x) \leq 0$ for all $x \in]a, b[$.

Proof. Let us denote that f is increasing (resp. decreasing) if and only if $\frac{f(x)-f(y)}{x-y} \geq 0$, (resp. ≤ 0), $\forall x \neq y$.

Let us prove the first item. Suppose f is increasing. For all $x, c \in]a, b[$with $x \neq c$,

$$\frac{f(x)-f(c)}{x-c} \geq 0$$

Taking a limit as x goes to c, we see that $f'(c) \geq 0$. For the other direction, suppose $f'(c) \geq 0$ for all $c \in]a, b[$. Take any $x, y \in]a, b[$ with $x < y$, and note that $[x, y] \subset]a, b[$. By the mean value theorem 4.12, there is some $c \in]x, y[$ such that

$$f(x) - f(y) = (x - y)f'(c).$$

Hence

$$\frac{f(x)-f(y)}{x-y} = f'(c) \geq 0$$

and so f is increasing. We leave the second item to the reader as exercise. □

4.7 Exercises

Exercise 37. Find the domain of definition of the following functions

$$f(x) = \sqrt{x^2+3x-4}, \quad g(x) = \ln(x^2+3x-4), \quad h(x) = \frac{\ln(x+1)}{\sqrt{1-x^2}}, \quad k(x) = \frac{1}{[x]-2022}.$$

Exercise 38. Calculate the following limits

$$\lim_{x\to+\infty} (\sqrt[3]{x^3+1}-x), \quad \lim_{x\to1} \frac{\sqrt{x}-1}{\sqrt{x+8}-3}, \quad \lim_{x\to1} \frac{\sqrt[4]{x}-1}{\sqrt{x}-1}, \quad \lim_{x\to1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$$

$$\lim_{x\to+\infty} \frac{e^x}{x^6}, \quad \lim_{x\to+\infty} \frac{\ln(1+x^2)}{x}, \quad \lim_{x\to0} \frac{[|x|]}{x^{10}}, \quad \lim_{x\to+\infty} \frac{1-\cos x}{x^2}, \quad \lim_{x\to\pi} \frac{\sin x}{x-\pi}.$$

Exercise 39. 1. Using the definition of the derivative, calculate the following limits

$$\lim_{x\to0} \frac{\ln(1+x)}{x}, \quad \lim_{x\to0} \frac{e^x-1}{x}$$

2. Deduce the following limits

$$\lim_{x\to+\infty} \left(1+\frac{k}{x}\right)^x, \; k \in \mathbb{R}, \quad \lim_{x\to0} \frac{a^x-b^x}{x}, \; a, b > 0.$$

Exercise 40. 1. Show that

$$\forall x, y \geq 0 : |\sqrt{x}-\sqrt{y}| \leq \sqrt{|x-y|}$$

2. Deduce that the function $x \mapsto \sqrt{x}$ is uniformly continuous on $\mathbb{R}_+$.

3. Show that the function $x \mapsto \frac{1}{x}$ is not uniformly continuous on $(0, \infty)$ (Choose $x = \frac{1}{n}$, $y = \frac{1}{2n}$).

Exercise 41. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^3 + \frac{a}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

1. Calculate $\lim_{x\to0} f(x)$

2. Deduce the value of a for which f is continuous.

Exercise 42. Study the continuity of the function defined on $\mathbb{R}$ by $f(x) = [x]$ (consider the two cases: $x \in \mathbb{Z}$ and $x \notin \mathbb{Z}$).

Exercise 43. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that $f(x) = 0$ for all $x \in \mathbb{Q}$. Show that $f(x) = 0$ for all $x \in \mathbb{R}$.

Exercise 44. 1. Let $f : [0, 1] \longrightarrow [0, 1]$ be a continuous function. Show that f has a fixed point.

2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and decreasing function. Show that f has a unique fixed point.

Exercise 45. 1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and periodic function such that $\lim_{x\to+\infty} f(x)$ exists. Show that f is constant.

2. Deduce that $x \mapsto \sin x$ and $x \mapsto \cos x$ do not have limits at $+\infty$ and $-\infty$.

Exercise 46. Calculate the derivatives of the following functions: $\sqrt{\frac{1+x^2}{x-1}}$, $\ln(1+\cos(x^2-x+1))$

Exercise 47. 1. Using the definition of the derivative, calculate the following limits

$$\lim_{x\to 0} \frac{\ln(1+x)}{x}, \quad \lim_{x\to 0} \frac{e^x-1}{x}$$

2. Deduce the following limits

$$\lim_{x\to+\infty} \left(1+\frac{k}{x}\right)^x, \; k \in \mathbb{R}, \quad \lim_{x\to 0} \frac{a^x-b^x}{x}, \; a, b > 0.$$

Exercise 48. Let f be the function defined on $\mathbb{R}^*$ by $f(x) = x^2 \sin \frac{1}{x^2}$.

1. Show that f can be extended to be continuous on $\mathbb{R}$ and give its extension $\tilde{f}$.
2. Study the differentiability of $\tilde{f}$ and calculate its derivative $\tilde{f}'$
3. Is $\tilde{f}$ of class $\mathcal{C}^1(\mathbb{R})$?

Exercise 49. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that

$$\forall x, y \in \mathbb{R} : |f(x) - f(y)| \leq |x - y|^2.$$

1. Show that f is differentiable and calculate its derivative.
2. Deduce the value of f.

Exercise 50. Show the following inequalities

$$\forall x > -1 : \frac{x}{1+x} \leq \ln(1+x) \leq x,$$

$$\forall x \in]0, 1[: 1 + x \leq e^x \leq \frac{1}{1-x}$$

(Apply the Mean Value Theorem to the functions: $e^x - x - 1$, $(1 - x)e^x - 1$).

Exercise 51. Calculate the nth-order derivatives for $n \in \mathbb{N}$ of the following functions

$$(x^2 + x + 1)e^x, \quad \frac{e^x}{1-x}, \quad \frac{e^{-x}}{1+x}.$$

5 Usual function

5.1 Definition of arcsin and aarccos Functions

Definition 5.1. 1. The arcsine function, denoted as $\arcsin(x)$ or $\sin^{-1}(x)$, is the inverse of the sine function $\boxed{\sin : [-\pi/2, \pi/2] \longrightarrow [-1, 1]}$. In other words, $\arcsin : [-1, 1] \longrightarrow [-\pi/2, \pi/2]$ such that $\forall x \in [-1, 1]$, we have

$$\arcsin(x) = \theta \quad \text{where} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad \text{and} \quad \sin(\theta) = x.$$

2. The arccos function, denoted as $\arccos(x)$ or $\cos^{-1}(x)$, is the inverse of the cosine function $\boxed{\cos : [0, \pi] \longrightarrow [-1, 1]}$. In other words, $\arccos : [-1, 1] \longrightarrow [0, \pi]$ such that $\forall x \in [-1, 1]$, we have

$$\arccos(x) = \theta \quad \text{where} \quad 0 \leq \theta \leq \pi \quad \text{and} \quad \cos\theta = x.$$

Proposition 5.1. (**Properties of the Arcsine and Arccos Function**).

1. $\boxed{\arcsin(-x) = -\arcsin(x),\ \arccos(-x) = \pi - \arccos x}$ $\forall x \in [-1, 1]$.

2. Derivative:

$$\boxed{\forall x \in]-1, 1[: \arcsin' x = \tfrac{1}{\sqrt{1-x^2}}, \quad \arccos' x = -\tfrac{1}{\sqrt{1-x^2}}}$$

3. Inverse of sine and cosine:

$$\boxed{\arcsin(\sin\theta) = \theta, \quad \text{for } -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2}}$$

$$\boxed{\arccos(\cos\theta) = \theta, \quad \text{for } 0 \leq \theta \leq \pi}$$

5.2 Definition of hyperbolic functions ch and ash

The hyperbolic sine and hyperbolic cosine functions are defined on $\mathbb{R}$ as follows:

$$\text{sh}(x) = \frac{e^x - e^{-x}}{2}$$
$$\text{ch}(x) = \frac{e^x + e^{-x}}{2}$$

Let us denote that those functions are differentiable and we have

$$\text{sh}'(x) = \frac{e^x - + e^{-x}}{2} = \text{ch}x$$
$$\text{ch}'(x) = \frac{e^x - e^{-x}}{2} = \text{sh}x$$

Definition 5.2. 1. The arcsine function, denoted as $\text{argsh}(x)$ or $\text{sh}^{-1}(x)$, is the inverse of the sine function $\boxed{\text{sh} : \mathbb{R} \longrightarrow \mathbb{R}}$. In other words, $\text{argsh} : [-1,1] \longrightarrow \mathbb{R}$ such that $\forall x \in [-1,1]$, we have

$$\text{argsh}x = y \quad \text{where} \quad \text{sh}y = x.$$

2. The arccos function, denoted as $\text{argch}(x)$ or $\text{ch}^{-1}(x)$, is the inverse of the cosine function $\boxed{\text{ch} : [0,+\infty] \longrightarrow [1,+\infty[}$. In other words, $\text{argch} : [1,+\infty[\longrightarrow [0,+\infty]$ such that $\forall x \in [-1,1]$, we have

$$\text{argch}(x) = y \quad \text{where} \quad \text{ch}y = x.$$

Exercise 52. Calculate $\text{argch}0,\ \text{argsh}0, \text{argch}1,\ args1$

Proposition 5.2 (Derivative). The functions argch and argsh are differentiable and we have

$$\boxed{\forall x \in \mathbb{R} : \text{argsh}'x = \frac{1}{\sqrt{x^2+1}}}$$

$$\boxed{\forall x \in]1,+\infty[: \text{argch}'x = \frac{1}{\sqrt{x^2-1}}}$$

5.3 Exercises

Exercise 53. Show that for all $x \in [-1,1]$, we have

$$\boxed{\sin(\arccos x) = \sqrt{1-x^2} = \cos(\arcsin x)}$$

Exercise 54. Let $f : D \to [-1,1]$ be the function defined by $f(x) = \sin x$ where $D = [\frac{\pi}{2}, \frac{3\pi}{2}]$.

1. Verify that f is bijective and determine its inverse f^{-1} in terms of arcsin.

2. Same question for $f(x) = \cos x$ and $D = [2022\pi,\ 2023\pi]$.

Exercise 55. 1. Calculate $\arcsin(\sin\frac{\pi}{3}),\ \arccos\cos(\frac{\pi}{3}),\ \arccos(\sin\frac{\pi}{3})$.

2. Calculate $\arccos(\cos\frac{4\pi}{3}),\ \arccos(\cos\frac{7\pi}{3}),\ \arcsin(\sin\frac{2\pi}{3}),\ \arcsin(\sin\frac{7\pi}{3})$.

Exercise 56. 1. Show that $\arctan a + \arctan b = \arctan\frac{a+b}{1-ab}$, with $ab < 1$

2. Calculate $\arctan(1/2) + \arctan(1/3)$

Exercise 57. 1. Calculate

$$\boxed{C = \sum_{k=0}^{n} \text{ch}(kx), \quad S = \sum_{k=0}^{n} \text{sh}(kx)}$$

2. Linearize $\text{sh}x.\text{ch}(2x)$, $\text{ch}x.\text{ch}^2x$

3. Verify that $\text{sh}(2x) = 2\text{sh}x\text{ch}x$ and then calculate

$$P = \text{ch}x.\text{ch}(\tfrac{x}{2}).\text{ch}(\tfrac{x}{2^2}).......\text{ch}(\tfrac{x}{2^n}).$$

Exercise 58. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \text{argch}\sqrt{1+x^2}$.

1. Determine the domain of definition of f.
2. Calculate $\text{argch}(\text{ch}t)$, for all $t \in \mathbb{R}$
3. Show that $\forall x \in \mathbb{R} : f(x) = \text{argsh}|x|$.
4. Calculate $f'(x)$, for all $x \in \mathbb{R}^*$.
5. Is f differentiable at 0?.

Exercise 59. (**Assignment**)
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function defined by

$$f(x) = \begin{cases} \arctan \frac{1}{x^2} & \text{if } \quad x \neq 0 \\ \ell & \text{if } \quad x = 0 \end{cases}$$

1. Determine ℓ.
2. Show that f is differentiable on $\mathbb{R}^*$ and calculate f'.
3. Show that f is differentiable at 0 and calculate $f'(0)$ (Apply MVT between 0 and x).
4. Deduce that f is $\mathcal{C}^\infty$.
5. Calculate g' where g is the function defined on $\mathbb{R}$ by $g(x) = \arctan x^2$.
6. Calculate $\arctan x^2 + \arctan \frac{1}{x^2}$, $\forall x \in \mathbb{R}^*$ and deduce $\arctan x + \arctan \frac{1}{x}$, $\forall x \in \mathbb{R}^*$.
7. Show that $g : [0, +\infty[\to [0, \pi/2[$ is bijective and calculate g^{-1}.
8. Calculate $(g^{-1})'$ in two ways.

Reminder

$$\cos(a+b) = \cos a \cos b - \sin a \sin b, \quad \cos(a-b) = \cos a \cos b + \sin a \sin b$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b, \quad \sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$\cos(2x) = \cos^2 x - \sin^2 x, \quad \sin(2x) = 2 \sin x \cos x$$

$$\arccos : [-1,1] \to [0,\pi], \quad \arcsin : [-1,1] \to [-\pi/2, \pi/2]$$

$$\arccos' x = \tfrac{-1}{\sqrt{1-x^2}}, \quad \arcsin' x = \tfrac{1}{\sqrt{1-x^2}}$$

$$\arctan : \mathbb{R} \to]-\pi/2, \pi/2[, \quad \arctan' x = \frac{1}{1+x^2}$$

$$\text{ch}x = \frac{e^x+e^{-x}}{2}, \ \text{sh}x = \frac{e^x-e^{-1}}{2}, \quad \text{ch}x + \text{sh}x = e^x$$

$$\text{ch}'x = \text{sh}x, \quad \text{sh}'x = \text{ch}x, \quad \text{ch}^2x - \text{sh}^2x = 1$$

$$\text{ch}(a+b) = \text{ch}a\text{ch}b + \text{sh}a\text{sh}b, \quad \text{ch}(a-b) = \text{ch}a\text{ch}b - \text{sh}a\text{sh}b$$

$$\text{sh}(a+b) = \text{sh}a\text{ch}b + \text{ch}a\text{sh}b, \quad \text{sh}(a-b) = \text{sh}a\text{ch}b - \text{ch}a\text{sh}b$$

$$\text{ch}(2x) = \text{ch}^2x + \text{sh}^2x, \ \ \text{sh}(2x) = 2\text{sh}x\text{ch}x$$

$$\text{argch} : [1, +\infty[\to [0, +\infty], \quad \text{argsh} : \mathbb{R} \to \mathbb{R}$$

$$\text{argch}'x = \frac{1}{\sqrt{x^2-1}}, \quad \text{argsh}'x = \frac{1}{\sqrt{1+x^2}}$$

$$\text{argth} : [-1, +1] \to \mathbb{R}, \quad \text{argth}'x = \frac{1}{x^2-1}$$