Solution of tutorial series N° 1 vib and waves

2024/2025

Exo1

- $\sum_i \vec{F_i} = m\vec{\gamma}$
- Kinetic momentum theorem

$$\frac{d}{dt}\left[L_{/\Delta}(\text{solid body})\right] = \sum_i \mathcal{M}_{/\Delta}(\vec{f_i}) = J_{/\Delta}\ddot{\theta}$$

then: $$\boxed{\sum_i \mathcal{M}_{/\Delta}(\vec{f_i}) = J_{/\Delta}\cdot\ddot{\theta}}$$

. $L_{/\Delta}(\underbrace{\text{Solid body}}_{= S})$ = kinetic momentum of a solid body / Δ

. $\mathcal{M}_{/\Delta}(\vec{F_i})$ = torque of $\vec{f_i}$ / Δ

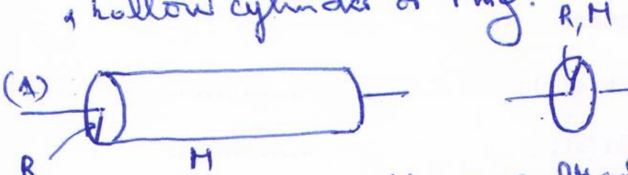
. $J_{/\Delta}$ = inertial moment of (S) / Δ (moment of inertia)

. $\ddot{\theta}$ = angular acceleration / Δ

$J_{/\Delta}$ for some bodies of different shapes

. cylindric shape:

* hollow cylinder or ring:



$J_{/\Delta} = MR^2$

M: mass of the cylinder or the ring

* solid cylinder or disk:

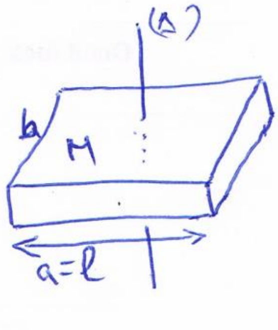
$J_{/\Delta} = \frac{1}{2}MR^2$

. parallelepiped

$J_{/\Delta} = \frac{1}{12}M(a^2+b^2)$

if: a >> b then

$J_{/\Delta} = \frac{1}{12}Ml^2$



. Huygens theorem:

$$J_{/\Delta} = J_{/G} + Ma^2$$

(G): axis (which) passing through the center of gravity of the solid

(Δ): axis // to (G)

a: distance between Δ and (G)

M: mass of the solid

. Kinetic energy of a solid body (S)

. In translation:

$$T(S) = \frac{1}{2}MV_G^2$$

M: mass of (S)

V_G: translation velocity of the centre of gravity of (S)

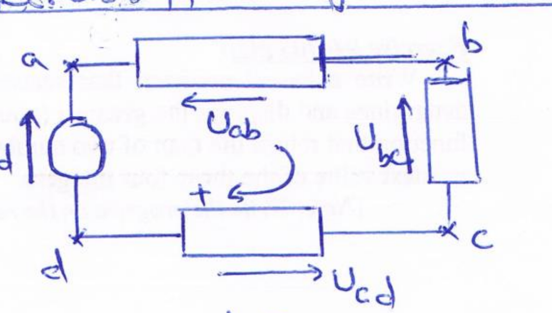
. In rotation / (G)

$$T = \frac{1}{2}J_G\dot{\theta}^2$$

J_G = moment of inertia of (S) / (G)

$\dot{\theta}$ = angular velocity of (S) in rotation / (G)

- Kirchhoff's voltage law (KVL)



 = generator

 receptor

3) KVL states that the sum of all voltages (electrical potential differences) around any closed network (loop) is zero.

matematical is expressed as:

$$\sum_i v_i = 0$$

voltages in the same chosen direction are counted positivelly, the other one are counted negativelley.

The sign convention will be as follows: when traversing the loop (clockwise or not) if the positive terminal is incountred first (in a given component) before the negative one the potential difference (voltage) will be interpreted as positive. In the opposite case it will be negative

note. the arrow in the loop of the above circuit means that the potential rises in this direction

and then:

$$U_{ab} = U_a - U_b > 0$$

$$U_{ad} = U_a - U_d > 0$$

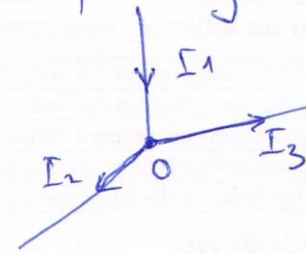
The Kirchhoff's voltage law would be come:

$$U_{ab} + U_{bc} + U_{cd} - U_{ad} = 0$$

- Kirchhoff's current Law (KCL)

a nod in electrical circuit is a point, where two or more circuit elements are connected.

Kirchhoff's current law says that the sum of all currents flowing int a nod equals the sum of all current flowing out of the node

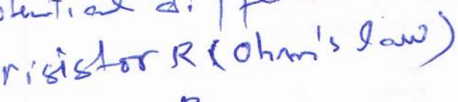


the point O is a node:

we have:

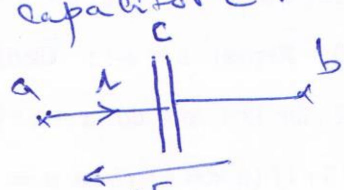
$$I_1 = I_2 + I_3$$

- potential difference across a risistor R (Ohm's law)



$$V_R = U_a - V_b = Ri$$

- potential difference across a capacitor C:



$$V_C = U_a - U_b = \frac{1}{C}\int i\,dt = \frac{q}{C}$$

q = the amount of charge in C

- potential difference across a coil L:

$$V_L = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$$

- Electrostatic energy in C:

$$W_C = \frac{1}{2} C V_C^2 = \frac{1}{2}\frac{1}{C} q^2$$

- Magnetic energy in L:

$$W_L = \frac{1}{2} L i^2$$

- Energy dissipated by Joule effect

$$W_R = R i^2 \quad \text{per unit of time}$$

Exo2

Taylor development of function $f(x)$ near x_0 is:

$$f(x) = f(x_0) + \frac{(x-x_0)^1}{1} f'(x_0) + \dots\dots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \dots$$

then:

- $\sin x = \sin x_0 + \frac{(x-x_0)^1}{1!}\cos x_0 - \frac{(x-x_0)^2}{2!}\sin x_0 + \dots$

$$\dots \frac{(x-x_0)^n}{n!}\sin^{(n)}(x_0) + \dots$$

- $\cos x = \cos x_0 - \frac{(x-x_0)^1}{1!}\sin x_0 - \frac{(x-x_0)^2}{2!}\cos x_0 + \dots$

$$\dots + \frac{(x-x_0)^n}{n!}\cos^{(n)} x_0 + \dots$$

- $e^x = e^{x_0} + \frac{(x-x_0)^1}{1!} e^{x_0} + \frac{(x-x_0)^2}{2!} e^{x_0} + \dots + \frac{(x-x_0)^n}{n!} e^{x_0} + \dots$

for $x_0 = 0$ it would become:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

These series are called MacLaurin series.

Euler's identity near 0:

We have:

$$e^{jx} = \sum_{n=0}^{\infty} \frac{(jx)^n}{n!} = \sum_{p=0}^{\infty} (j)^{2p} \frac{x^{2p}}{(2p)!} + j^{2p+1} \frac{x^{2p+1}}{(2p+1)!}$$

knowing that:

$(j)^{2p} = (-1)^p$ and $(j)^{2p+1} = j(-1)^p$ then

we obtain:

$$e^{jx} = \underbrace{\sum_{p=0}^{\infty} (-1)^p \frac{x^{2p}}{(2p)!}}_{\cos x} + j \underbrace{\sum_{p=0}^{\infty} (-1)^p \frac{x^{2p+1}}{(2p+1)!}}_{\sin x}$$

which gives:

$$e^{jx} = \cos x + j \sin x$$

It's the Euler's identity near zero

Exo3

complex representation of a sinusoidal function is:

$$x(t) = A\cos(\omega t + \varphi) \longrightarrow \bar{x}(t) = \bar{A} e^{j\omega t}$$

where $\bar{A} = A e^{j\varphi}$

then:

$\bar{x}_1(t) = \bar{A}_1 e^{j\omega t}$ and $\bar{x}_2(t) = \bar{A}_2 e^{j\omega t}$

So the sum will be:

$$\bar{x}(t) = \bar{x}_1 + \bar{x}_2 = (\bar{A}_1 + \bar{A}_2) e^{j\omega t}$$

We put: $\bar{A}_1 + \bar{A}_2 = \bar{A}$ then

$$\bar{x}(t) = \bar{x}_1 + \bar{x}_2 = \bar{A} e^{j\omega t}$$

this number complex has the forme of the complex number associated to a sinusoidal function. We conclude that: $x = x_1 + x_2$ is sinusoidal too

let's calculate A and φ:

$$\bar{A} = A e^{j\varphi} \Rightarrow \begin{cases} A = |\bar{A}| \\ \varphi = \arg(\bar{A}) \end{cases}$$

7) so:

$$A = \sqrt{\left[Re(\bar{A}_1+\bar{A}_2)\right]^2 + \left[Im(\bar{A}_1+\bar{A}_2)\right]^2}$$

$$= \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_1-\varphi_2)}$$

knowing that:

$$\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2 = \cos(\varphi_1-\varphi_2)$$

so: $\boxed{A = \sqrt{A_1^2+A_2^2+2A_1A_2\cos(\varphi_1-\varphi_2)}}$

$$\varphi = \arg(\bar{A}) = \arg(\bar{A}_1+\bar{A}_2) = \arctg\frac{Im(\bar{A}_1+\bar{A}_2)}{Re(\bar{A}_1+\bar{A}_2)}$$

$\Rightarrow \boxed{\varphi = tg^{-1}\left(\frac{A_1\sin\varphi_1 + A_2\sin\varphi_2}{A_1\cos\varphi_1 + A_2\cos\varphi_2}\right)}$

EX04.

the differential equation is:

$$\ddot{y} + 5\dot{y} + 4y = f(t)$$

non-homogeneous diff equation of second ordre with constant parameters. the solution will be:

$$y(t) = y_h(t) + y_p(t)$$

$y_h(t)$ = solution of homogeneous equation

therefore:

$$\ddot{y}_h + 5\dot{y}_h + 4y_h = 0$$

the caracteristic equation is:

$$r^2 + 5r + 4 = 0$$

this equation admet two roots:

$$r_{1,2} = \frac{-5 \mp \sqrt{\Delta}}{2} \; ; \; \Delta = 9$$

$\Rightarrow r_{1,2} = -1$ and -4

then:

$$y_h(t) = c_1 e^{-t} + c_2 e^{-4t}$$

$y_p(t)$ = particular solution

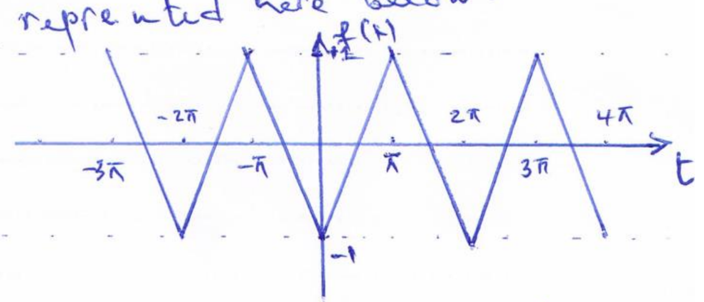
we must take in account the second side which is periodi function

then we must first find the fourier's series of $f(t)$:

$$f(t) = \begin{cases} -\frac{2}{\pi}t - 1 & -\pi \le t \le 0 \\ \frac{2}{\pi}t - 1 & 0 \le t \le \pi \end{cases}$$

$\Rightarrow T = \pi - (-\pi) = 2\pi \Rightarrow \omega = 1\,rad\,s^{-1}$

The shape of this function is represented here below:



let's calculate a_0, a_n and b_n

It's clear that $f(t)$ is an even function ($f(-t) = f(t)$) then:

$\forall n$ $\boxed{b_n = 0}$

$$a_0 = \frac{1}{T}\int_{-\pi}^{+\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(t)\,dt$$

$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{\pi} f(t)\,dt$ because $f(t)$ even function

$$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{\pi}\left(\frac{2}{\pi}t - 1\right)dt = 0$$

$\Rightarrow \boxed{a_0 = 0}$

(9) $a_n = \frac{1}{T}\int_0^1 f(t)\cos(n\omega t)\,dt$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\cos(nt)\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{2}{\pi}t-1\right)\cos(nt)\,dt$$

after having calculat this integral we find:

if $n=2p$: $a_{2p}=0$

if $n=2p+1$: $a_{2p} = -\frac{8}{(2p+1)^2\pi^2}$

the fourier's series will be:

$$f(t) = p(t) = \sum_{p=0}^{\infty} -\frac{8}{(2p+1)^2\pi^2}\cdot\cos(2p+1)t$$

the first two termes of the series are: $-\frac{8}{\pi^2}\cos t$, $-\frac{8}{9\pi^2}\cos 3t$

$$\Rightarrow f(t) \simeq -\frac{8}{\pi^2}\left(\cos t + \frac{1}{9}\cos 3t\right)$$

$$= \frac{8}{\pi^2}\left[\cos(t+\pi) + \frac{1}{9}\cos(3t+\pi)\right]$$

then the equation will be written as:

$$\ddot{y}_p + 5\dot{y}_p + 4y_p = \frac{8}{\pi^2}\left(\cos(t+\pi) + \frac{1}{9}\cos(3t+\pi)\right)$$

because of linearity of this function we ~~put~~ :

$y_p = y_{p_1} + y_{p_2}$ where:

$$\ddot{y}_{p_1} + 5\dot{y}_{p_1} + 4y_{p_1} = \frac{8}{\pi^2}\cos(t+\pi) \quad \dots (1)$$

$$\ddot{y}_{p_2} + 5\dot{y}_{p_2} + 4y_{p_2} = \frac{8}{\pi^2}\cos(3t+\pi) \quad \dots (2)$$

(10) we will use the complex method to solve (1) and (2)

$$\frac{8}{\pi^2}\cos(t+\pi) \longrightarrow \frac{8}{\pi^2}e^{j\pi}e^{jt}$$

$$y_{p_1}(t) = A_1\cos(t+\varphi_1) \longrightarrow \bar{y}_{p_1}(t) = \bar{A}_1 e^{jt}$$

where: $\bar{A}_1 = A_1 e^{j\varphi_1}$

then we find that:

$$-\bar{A}_1 + 5j\bar{A}_1 + 4\bar{A}_1 = \frac{8}{\pi^2}e^{j\pi}$$

$$\Rightarrow \bar{A}_1 = \frac{\frac{8}{\pi^2}e^{j\pi}}{3+5j}$$

$A_1 = |\bar{A}_1| = \frac{\frac{8}{\pi^2}}{\sqrt{34}}$

$\varphi_1 = \arg(\bar{A}_1)$

$$\Rightarrow \varphi_1 = \arg(\bar{A}_1) = \pi - tg^{-1}(5/3) = \boxed{2,11 \text{ rad}}$$

so: $\boxed{y_{p_1}(t) = \frac{8}{\pi^2\sqrt{34}}\cos(t+2,11)}$

by the same way we find:

$$\boxed{y_{p_2}(t) = \frac{8}{45\pi^2\sqrt{10}}\cos(3t+1,25)}$$

at last we have:

$y_p(t) = y_{p_1} + y_{p_2}$ and:

$$\boxed{y(t) = c_1e^{-t} + c_2e^{-4t} + \frac{8}{\pi^2\sqrt{34}}\cos(t+2,11) + \frac{8}{45\pi^2\sqrt{10}}\cos(3t+1,25)}$$

to determine the values of c_1 and we use initial conditions:

$y(0) = 1$ and $\dot{y}(0) = 0$

we obtain two equations with two unknowns c_1 and c_2:

$\boxed{c_2 = 0,36}$ and $\boxed{c_1 = 0,7}$

(11) Exo5.

The two constraints applid to the pendulum are expressed by the two equations as follows:

$$\begin{cases} x = 0 & \text{planar motion} \\ y^2+z^2 = l^2 & \text{m is always distat by } l \text{ from } O \end{cases}$$

$\Rightarrow \ l = 2 \ \Rightarrow$ the number of degrees of freedom is then:

$$S = N - l = 3 - 2 = 1$$

the syst is a single of degree of freedoms.

so the system is described by one generalized coordinate.

The motion of m is rotationnal around the axix xx'. m can be localized by θ the angle which makes the rod with respect to the vertical $\Rightarrow$

$q(t) = \theta(t)$

to determine the potential U we must first calculate the generalized force f_θ:

$$f_\theta = \frac{\delta W(\vec{P})}{\partial\theta} + \underbrace{\frac{\delta W(\vec{T})}{\partial\theta}}_{=0} = \frac{\delta W(\vec{P})}{\partial\theta}$$

$$= \vec{P}\cdot\frac{\delta\vec{r}}{\partial\theta} : \quad \vec{r} = \text{vector position}$$

$$\vec{r} = \vec{om} = l(\cos\theta\,\vec{k} + \sin\theta\,\vec{j})$$

$$\vec{P} = mg\vec{k} \quad \text{and} \quad \frac{\delta\vec{r}}{\partial\theta} = l(-\sin\theta\,\vec{k} + \cos\theta\,\vec{j})$$

$$\Rightarrow \boxed{f_\theta = -mgl\sin\theta} \ \cdots (3)$$

f_θ derive from the potential U (12)

then:

$$f_\theta = -\frac{dU}{d\theta}$$

at equilibrium position we have:

$$f_\theta = -\frac{dU}{d\theta} = 0 \Rightarrow mgl\sin\theta = 0$$

$$\Rightarrow \sin\theta = 0 \Rightarrow \theta = \begin{cases} 0 \\ \pi \end{cases}$$

two equilibrium position.

the difference between $\theta = 0$ and $\theta = \pi$ is:

$$\frac{d^2U}{d\theta^2} = mgl\cos\theta \begin{cases} \theta = 0 \Rightarrow \frac{d^2U}{d\theta^2} > 0 \\ \theta = \pi \Rightarrow \frac{d^2U}{d\theta^2} < 0 \end{cases}$$

then: for $\theta = 0$ stable equilibrium

$\theta = \pi$ unstable " "

by integrating (3) we find:

$$\int_{U(0)}^{U(\theta)} dU(\theta) = \int_0^\theta mgl\sin\theta\,d\theta \Rightarrow U(\theta) = mgl(1-\cos\theta) + U$$

if we choice $\theta = 0$ as a potential origine then: $U(0) = 0 \Rightarrow$

$$\boxed{U(\theta) = mgl(1-\cos\theta)}$$

for the case of oscillations with weak amplitudes we have:

$$\cos\theta \simeq 1 - \frac{\theta^2}{2} \quad \text{and then}$$

$$\boxed{U(\theta) = \frac{1}{2} mgl\,\theta^2}$$

Function and equation of Lagrange

- Lagrange's function denoted $\mathcal{L}$:

$$\mathcal{L}(\theta,\dot\theta) = T - U$$

$$T = \frac{1}{2} J_{/\Delta}\,\dot\theta^2 = \boxed{\frac{1}{2} ml^2\dot\theta^2}$$

(13) thus:

$$\boxed{\mathcal{L}(\theta,\dot\theta) = \frac{1}{2} m l^2 \dot\theta^2 - \frac{1}{2} m g l \theta^2}$$

the Lagrange's equation for a single degree of freedom system, is of the form as follows:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot\theta}\right] - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

which is valid only in the absence of friction and external force.

after derivating this equation will provid the diff equation of motion so:

$$\frac{d}{dt}\left[\partial\mathcal{L}/\partial\dot\theta\right] = \frac{d}{dt}\left(m l^2 \dot\theta\right) = m l^2 \ddot\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = - m g l \theta$$

$$\Rightarrow \quad m l^2 \ddot\theta + m g l \theta = 0$$

dividing by $m l^2$ results to:

$$\boxed{\ddot\theta + \frac{g}{l}\theta = 0}$$

which is a differential equation of the form:

$$\ddot q + \omega_0^2 q = 0$$

whose solution is sinusoidal with angular frequency:

$$\omega_0^2 = g/l \Rightarrow \boxed{\omega_0 = \sqrt{g/l}}$$

pendulum consisting of a mass m and a spring instead of the metal rod

one constraint is applied to the system: $x = 0$ (14)

then:

$$S = 3 - 1 = 2 \text{ degrees of freedom.}$$

effectively the system has two types of motions independant of each other:

- translation motion // $\vec{Om}$
- rotational motion around xx'

the syst is then described by two coordinates: $q_1(t)$ and $q_2(t)$

in polar coordinates we have:

$$q_1(t) = \rho(t) = l(t)$$

$$q_2(t) = \theta(t)$$

the Lagrange's function will be function of $\rho, \theta, \dot\rho$ and $\dot\theta$ i.e.:

$$\mathcal{L}(\rho, \theta, \dot\rho, \dot\theta)$$

The two equations of Lagrange in this case will be:

$$\begin{cases} \dfrac{d}{dt}\left[\dfrac{\partial \mathcal{L}}{\partial \dot\rho}\right] - \dfrac{\partial \mathcal{L}}{\partial \rho} = 0 \\ \dfrac{d}{dt}\left[\dfrac{\partial \mathcal{L}}{\partial \dot\theta}\right] - \dfrac{\partial \mathcal{L}}{\partial \theta} = 0 \end{cases}$$

End