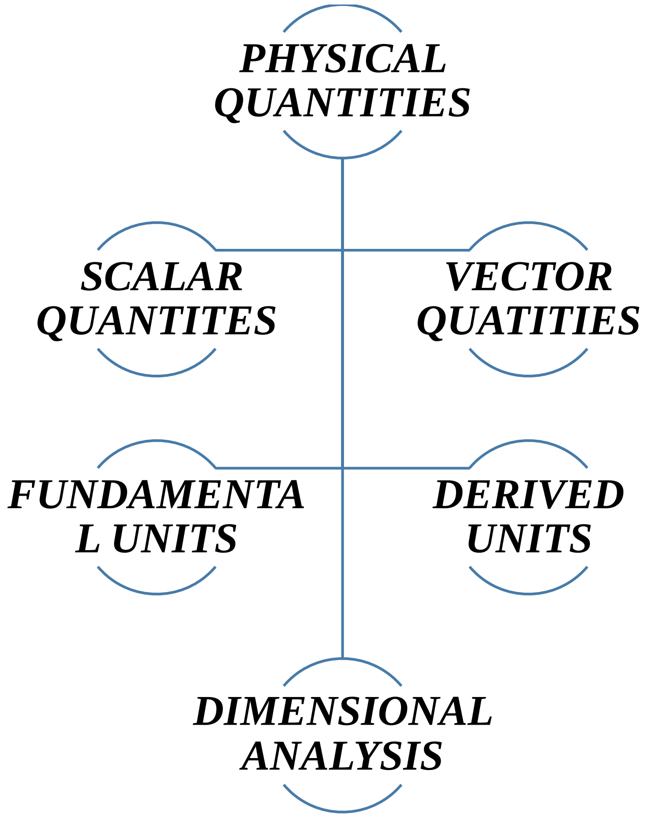
COURSE MIND MAP



Understanding physics easily, especially in classical mechanics, requires giving significant importance to building accurate concepts and providing a physical perspective on mathematical relationships. Therefore, our goal through this compilation of lessons is to reanalyse and review many of the physical principles and connect some mathematical concepts with their physical meanings.

1) PHYSICAL QUANTITY

In physics, a ***physical quantity****is any* ***measurable property****that describes a specific aspect of an object or a system.*

Examples of physical quantities:*mass, time, distance, area, volume, energy, density, pressure...velocity, acceleration, force and temperature …. Etc*

The physical quantitiescan be categorized into two types:

1-1)Scalar quantities

Scalar quantities are properties described just by their ***magnitude*** *(numerical value) and the appropriate* ***unit of measurement****.Examples of scalar quantities include:*

Mass (e.g., 15 kg), Temperature (e.g., 30°C), Energy (e.g., 100 J), Time (e.g., 10 seconds).

1.2)Vector quantities

Vector quantities are properties described by specifying their ***direction****, in addition to their* ***magnitude*** *(numerical value) and their appropriate* ***unit of measurement.*** *Examples of vector quantities include:*

Velocity, Force, Displacement, Acceleration.

2) Units of measurement

Units of measurement are symbols added to values to give them specific physical meanings. Units are divided into two main groups:

2-1) Fundamental Units: *These are the basic building blocks of measurement and are inherently independent, meaning they can't be expressed in terms of other units. They include crucial physical quantities like mass, time, energy, electric charge, length, and distance.*

2-2) Derived Units:*Derived units are units that can be expressed as a combination of fundamental units. Examples include:*

The newton (N), derived from kilogram, meter, and second, measures force.

The joule (J), a combination of newton and meter, quantifies energy.

The watt (W), obtained from joule and second, is used for power measurement.

The coulomb (C), derived from ampere and second, quantifies electric charge.

The hertz (Hz), based on the reciprocal of the second (1/s), represents frequency.

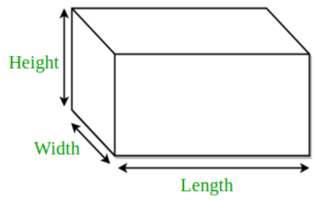
3) The International System of Units (SI)

Measurement units for physical quantities can vary from one country to another. For example, in the United States, units like inches (1 inch = 2.54 cm) and pounds (1 pound = 0.45 kg) are used to measure distance and mass. These units may not be widely recognized in many other countries, leading to challenges in scientific communication. Therefore, to establish a uniform unit system, a consensus has been reached, resulting in the International System of Units (SI), which consists of seven fundamental units as outlined in the table below.

Table01: The seven independent SI base units

Base quantity	Unit name	Unit Symbol
Length	Meter	m
Time	Second	s
Mass	Kilogram	Kg
Temperature	Kelvin	K
Electric current	Ampere	A
Amount of substance	Mole	mol
Luminous intensity	Candela	cd

Example01: *Find the derived unit for the following quantities*



- *Volume = Lenght × Width × Hight*

Volume unit ≡ Length unit × Width unit × Hight unit≡ m×m×m ≡m^3

- *Velocity = distance / time ⇒ Velocity unit ≡ distance unit / time unit = m/s*
- *Acceleration = velocity/ time ⇒ Acceleration unit ≡ velocity unit / time unit ≡* m/s^2
- *Force = mass× acceleration ⇒ Force unit ≡ mass unit × acceleration unit ≡* $Kg.m/s^2$ ≡ **N**

4) DIMENSIONAL ANALYSIS

4-1) Dimension definition

In physics, the term "dimension" signifies the inherent property of a physical quantity. It classifies the measurement units associated with a specific physical quantity. For instance, the *measurement of distance between two points can be done using units like feet, meters, or kilometres, with each unit representing a unique aspect of the dimension known as length.Similarly, when measuring the mass of an object, it can be quantified using units like kilogram, gram, pound, or milligram. These units all share a common nature, as they represent mass, and therefore they have a mass dimension.To denote the dimension, we typically write the symbol of the quantity within square brackets* ***[quantity].***

Table02: Base Quantities and Their Dimensions

Base quantity	***Symbol for dimension***
Length	*L*
Time	*T*
Mass	*M*
Temperature	*Θ*
Electric current	*I*
Amount of substance	*N*
Luminous intensity	*J*

The dimension of the derived physical quantity (Q) can be expressed as:

$$[Q] = L^a M^b L^c T^d I^e \Theta^f N^g J^h$$

In this expression, the variables a, b, c, d, e, f, and g represent the powers associated with the fundamental physical dimensions.

4-2) The dimensional equation

The dimensional equation is an equation that expresses the relationship between different physical quantities through their dimensions side. We can obtainthe dimensional equation by rewriting the original equation in a way that represents each physical quantity in terms of its dimension.In other words, the symbols used for physical quantities are replaced with symbols representing their dimensions.

Through this process, we can verify that the relationship between the physical quantities in the equation is correctly aligned in terms of dimensions and units, ensuring the accuracy and compliance of the equation with the laws of physics.

Example02: *let as assume we have a physics equation that relates force (F) to mass (m) and acceleration (a). This dimensional equation can be represented as follows:*$\vec{F} = m \cdot \vec{a}$

In this equation:

$\vec{F}$ represents force and can be expressed in units of Newton (N).

m represents mass and can be expressed in units of kilograms (kg).

$\vec{a}$ represents acceleration and can be expressed in units of meters per second squared (m/s^2).

If we analyze the dimensions in this dimensional equation, we find that they align correctly:

$[F] = [m]\,[a] = M.L.T^{-2} \equiv N$

This indicates that the unit of Newton for force correctly corresponds to the units of mass and acceleration. Thus, the equation is dimensionally and physically correct.

4-3) Properties of dimensional equation

"Let F, A,B, and Care different physical quantities, and 'm' and 'n' are real numbers.

- *The equation **F + C = B + A** is dimensionally correct only if all the quantities F, C, B, and A, have the same dimensions(**[F] =[C] = [B] =[A])**.*
- *The dimension of any constant real number is dimensionless (If **F =n⇒[F] = 1**).*
- ***F = n × A** we can analyse its dimension as **[F] = [n] × [A]⇒[F] = 1 × [A]**. This demonstrates that F possesses the same dimension as A."*
- ***F = B × A × C**, we can determine its dimension as '**[F] = [B] × [A] × [C]**.' This equation displays how the dimension of F is a product of the dimensions of A, B, and C."*
- ***$F = A^m \Rightarrow [F] = [A]^m$**. This illustrates that F's dimension is related to A raised to the power of 'm'*
- ***F = dA/dx⇒ [F] = [A]/[x]**.*
- ***F = ∫(A dy)⇒ [F] = [A] × [y]**.*
- $F = \sqrt{A} \Rightarrow |F| = |\sqrt{A}| = \sqrt{|A|}$

Example03:

Find the dimension of the following quantities: velocity, acceleration, force, Charge quantity, , Kinetic Energy , and Gravitational potential energy.

- ***Velocity:*** $V=x/t \Rightarrow [V]= [x]/[t]= L/T$
- ***Acceleration:*** $a=v/t \Rightarrow [a]= [v]/[t]=(L/T)/T = L.T^{-2}$
- ***Force:*** $\vec{F} = m\,\vec{a} \Rightarrow [F]= [m].[a]= M.L.T^{-2}$
- ***Charge quantity:*** $Q = i \times t \Rightarrow [Q] = [i] \times [t] \Rightarrow [Q]= I.T$
- ***KineticEnergy:*** $E = \frac{1}{2}\,mV^2 \Rightarrow [E] = \left[\frac{1}{2}\right][m][V^2] = M\,L^2T^2$
- ***Gravitational potential energy:*** $E = m.g.h \Rightarrow [E] = [m][g][h] \Rightarrow [E] = M.L.T^{-2}.L = M\,L^2T^2$

Exercise 01:

The following equation expresses the change in displacement with respect to time and the acceleration of a moving object.

$$x = \frac{1}{2} at^2$$

1 - Write the homogeneity condition of the equation.

2 - Prove its dimensional consistency using the dimensional analysis method.

Solution

1. *The homogeneity condition of the equation :*

$$[dimenion\ of\ the\ left\ side\ ] = [dimension\ of\ the\ right\ side\ ]$$

2- Prove its dimensional consistency using the dimensional analysis method.

The left side of the equation $[x\ ] = L$

The right side of the equation $\left[\frac{1}{2}\right][a][t^2] = [a][t]^2 = L\,T^{-2}T^2 = L$

The dimension of the left side equals the dimension of the right side, indicating that the equation is dimensionally correct.

Exercise 02:

What are the dimensions of both k and v_0 *constants in the following equation?*

$v = kt + v_0$ *where v represents velocity and t represents time*

Solution

The equation is dimensionally correct if both sides have the same dimension.

$$[v] = [k][t] = [v_0]$$

$$LT^{-1} = [k]T = [v_0]$$

$$LT^{-1} = [v_0]\ and\ LT^{-2} = [k]$$

Exercise 03:

Determine the physical dimension of the spring constant and then verify the homogeneity of the equation that relates the spring constant to the elastic potential energy

Solution

1) *The physical dimension of the spring constant (elasticity constant) can be found using the formula related the force exerted by a spring to its displacement*

$$F = -Kx$$

Where

F *is the force applied to the spring.*

K *is the spring constant.*

x *is the displacement from the equilibrium position.*

To find the dimension of K*, we can rearrange the equation as follows:*

$$K = -\frac{F}{x}$$

So, we can find the dimension of the spring constant K *as follows:*

$$[K] = \frac{[F]}{[x]} = \frac{M.L.T^{-2}}{L} = M.T^{-2}$$

So, the physical dimension of the spring constant (K) is $M.T^{-2}$

2) To verify the homogeneity of the equation connecting the spring constant to elastic potential energy, you can follow these steps:

- Express the equation for elastic potential energy (U) *in terms of the spring constant* (K) *and the displacement* (x)*.*

$$U = \frac{1}{2}Kx^2$$

- *Determine the dimensions of each variable involved:*

The dimension of elastic potential energy $[U]$ *is energy, which is represented as* $[U] = M\,L^2T^2$

We already found the dimension of the spring constant $[K]$ *in a previous response:* $[K] = M.T^{-2}$

The dimension of displacement (x) is length, $[x] = L$

Now, substitute K and x dimensions into the equation for elastic potential energy:

$$[K][x]^2 = M.T^{-2}.L^2 = [U]$$

The equation is homogeneous because the dimensions on both sides match. Therefore, the equation connecting the spring constant to elastic potential energy is consistent in terms of dimensions.

B- VECTORS

1) Definition of vector:

A vector is an oriented segment that has both magnitude (length) and direction. Graphically, vectors are often represented by arrows. The length of the arrow represents the magnitude of the vector, and the direction of the arrow indicates the direction of the vector.In handwritten equations or mathematical expressions, vectors are typically represented by placing a letter with an arrow above itsuch as $\vec{V}$*.In physics, vectors offer a comprehensive representation of certain quantities that cannot be adequately described solely by stating their values and units. Vectors enable us to model and represent these quantities, allowing us to understand how they change in various spatial directions, essentially giving them a descriptive form associated with spatial orientations.*

2) The key characteristics of a vector are:

*1- **Magnitude (Length)**: Magnitude represents the length of the vector and expresses the scalar quantity of the vector. Its value is always positive.The symbol used to represent the magnitude or absolute value of a vector is indeed typically written as double vertical bars (‖ ‖)surrounding the vector* $\vec{V}$*, like this:* $\|\vec{V}\|$

*2- **Direction**: Direction indicates the orientation of the scalar quantity of the vector, geometrically defined by the angle between the vector and a specific axis or the arrow's direction.*

*3- **Support**: Support refers to the line connecting the starting point (tailor origin) of the vector to its endpoint (head or tip).*

3) Specifying a vector

The vector is typically defined using one of the following methods:

- ✓ *By specifying its length (magnitude) and the angle it makes with a specific axis (see figure 01).*
- ✓ *By determining its starting point (tail) and endpoint (head) using their coordinates in space (see figure 02).*
- ✓ *By describing its components in space, such as its three-dimensional coordinates (We will address this point later).*

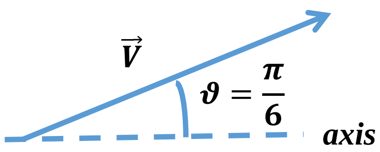


figure 01

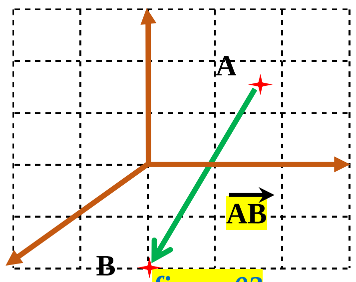
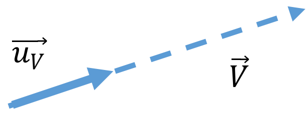


figure 02

4) The Unit Vector

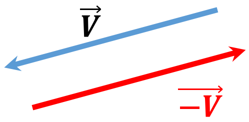


The unit vector notation for vector $\vec{V}$, which is denoted as $\vec{u_V}$, exhibits the following characteristics:

***Magnitude**: It has a magnitude of exactly one, denoted as $\|\vec{u_V}\| = 1$. This signifies that its length is normalized to unity, meaning it doesn't carry any specific units.*

***Direction**: $\vec{u_V}$ aligns precisely in the same direction as the original vector $\vec{V}$. In mathematical terms, this can be expressed as $\vec{u_V} = \vec{V}/\|\vec{V}\|$, where $\|\vec{V}\|$represents the magnitude (length) of vector $\vec{V}$.*

5) The Negative of a Vector

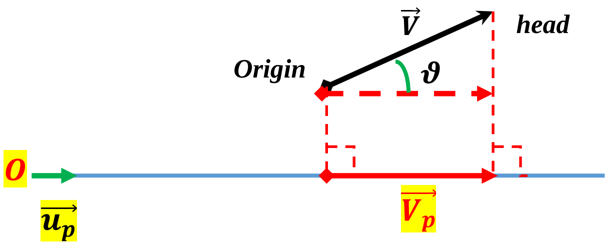


The negative vector is a vector with an equal magnitude to the original vector but directed in the opposite way.

6) The vertical projection of a vector onto an axis

The vertical projection of a vector $\vec{V}$onto an axis(OP) is a vector $\vec{V_p}$aligned with that axis. Its starting point (origin or tail) is the projection of the vector's originpoint onto the axis, and its endpoint is the projection of the vector's head(tip) onto the axis. Its length is the result of multiplying the vector's length by the cosecant of the angle (ϑ)enclosed between them.

$$\|\vec{V_p}\| = \|\vec{V}\| \times \cos(\vartheta)$$

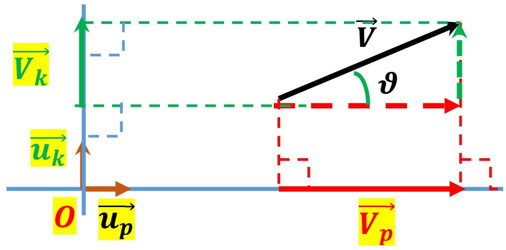


If the axis onto which a vector is projected is oriented with a specific direction, such as being represented by a unit vector, we can express the projection by taking the scalar product (dot product) between the original vector and the unit vector of the axis. We'll delve into the concept of the dot *product between two vectors in more detail later on.,and therefore, we can write the following:*$\overrightarrow{V_p} = \|\overrightarrow{V_p}\|\overrightarrow{u_p} = \|\vec{V}\| \times \cos(\vartheta)\overrightarrow{u_p}$

*The length of the vector obtained through the projection can be called the '**component** of the original vector on the axis.This component represents a part of the original vector and allows you to analyze the original vector into its **subcomponents** along multiple axes.*

7) The analytical representation of a vector

The analytical representation of a vector means representing the vector using its individual components relative to specific axes. These components are typically the outcomes of projecting the vector orthogonally onto these axes. In other words, the analytical representation of a vector is the process of rewriting it using its components and unit vectors extending in specific directions along particular axes.



Example01:

The vector $\vec{V}$ in the corresponding figure has a magnitude of 2 and makes an angle of 30 degrees with the (OP) axis.We can find its components along the two axes (OP) and (OK), which carry the unit vectors $\overrightarrow{u_p}$ and $\overrightarrow{u_k}$, respectively. We can write its analytical expression along these two axes using the following method:

To find the component along the (op) axis. Firstly, we need to perform orthogonal projections of the original vector onto both axes, as detailed in the figure.

On (OP) axis ⇒ $\overrightarrow{V_p} = \|\overrightarrow{V_p}\|\overrightarrow{u_p} = \|\vec{V}\| \times \cos(\vartheta)\overrightarrow{u_p} = \frac{2\sqrt{3}}{2}\overrightarrow{u_p} = \sqrt{3}\overrightarrow{u_p}$

On (OK) axis ⇒ $\overrightarrow{V_k} = \|\overrightarrow{V_k}\|\overrightarrow{u_k} = \|\vec{V}\| \times \sin(\vartheta)\overrightarrow{u_k} = 2 \times \frac{1}{2}\overrightarrow{u_k} = \overrightarrow{u_k}$

$$\vec{V} = \overrightarrow{V_p} + \overrightarrow{V_k} = \sqrt{3}\overrightarrow{u_p} + \overrightarrow{u_k}$$

So $\sqrt{3}$ *and* 1 *values resulting from projecting the vector* $\vec{V}$ *on the two principal axes (OP, and OK) are called the components of the vector* $\vec{V}$.

- *When vector* $\vec{A}$ *analyzed into its components in a Cartesian coordinate system*$(O,\vec{\imath},\vec{\jmath},\vec{k})$*,the analytical expression for this vector is written as follows:*

$$\vec{A} = a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k} \text{ or } \vec{A} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix}$$

Where the relationship between the components of the vector, its magnitude, and the angle it forms with a given axis is expressed as follows:

$$\begin{cases} a_x = |\vec{A}| \cos\theta \\ a_y = |\vec{A}| \sin\theta \\ |\vec{A}| = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2} \\ tan(\theta) = a_y/a_x \end{cases}$$

8) Belonging of vector to a specific plane defined by two vectors

We classify the vector $\vec{V}$ as part of the plane (P) defined by vectors $\vec{V_1}$ and $\vec{V_2}$ when we can represent it using these two vectors. In other words, if we can find values "a" and "b" such that $\vec{V} = a\,\vec{V_1} + b\,\vec{V_2}$, we can conclude that vector $\vec{V}$ can be expressed in relation to vectors $\vec{V_1}$ and $\vec{V_2}$ and is situated within the specified plane.

Example02: *We consider the following vectors:*

$$\vec{V_1} = 2\vec{\imath} + \vec{\jmath}\vec{V_2} = \vec{\imath} + 3\,\vec{\jmath} + \vec{k}\vec{V_3} = 4\vec{\imath} + 7\,\vec{\jmath} + 2\vec{k}$$

To prove that the vector $\vec{V_3}$ belongs to the plane (P) formed by vectors $\vec{V_1}$ and $\vec{V_2}$

$$\vec{V_3} = a\,\vec{V_1} + b\,\vec{V_2} = 2a\,\vec{\imath} + a\,\vec{\jmath} + b\,\vec{\imath} + 3b\,\vec{\jmath} + b\,\vec{k} = (2a+b)\,\vec{\imath} + (a+3b)\,\vec{\jmath} + b\,\vec{k}$$

$$(2a+b)\vec{\imath} + (a+3b)\vec{\jmath} + b\,\vec{k} = 4\vec{\imath} + 7\,\vec{\jmath} + 2\vec{k} \Rightarrow \begin{cases} 2a+b=4 \\ a+3b=7 \\ b=2 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=2 \end{cases}$$

$\vec{V_3} = \vec{V_1} + 2\,\vec{V_2}$ *we can write $\vec{V_3}$as function of both $\vec{V_1}$ and $\vec{V_2}$, thus , the vector $\vec{V_4}$ belongs to the plane (P)*

9) The linear dependence of two vectors

We say that two vectors are linearly dependent if one of them can be expressed as the scalar product of the other by a real number. In other term, the vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ are linearly dependent if there exists a real number $\boldsymbol{\alpha}$ that satisfies the following relationship: $\vec{\boldsymbol{A}} = \boldsymbol{\alpha}.\ \vec{\boldsymbol{B}}$

Example03:

Weconsider the following vectors: $\vec{A} = \vec{\imath} + 2\,\vec{\jmath} + \vec{k}\vec{B} = 4\vec{\imath} + 8\,\vec{\jmath} + 2\vec{k}$

- *Verify whether the two vectors are linearly dependent*

$$\vec{A} = \alpha\,\vec{B} \Rightarrow \alpha\left(\vec{\imath} + 2\,\vec{\jmath} + \vec{k}\right) = 4\vec{\imath} + 8\,\vec{\jmath} + 2\vec{k} \Rightarrow \begin{cases} \alpha = 4 \\ 2\,\alpha = 8 \\ \alpha = 2 \end{cases} \Rightarrow = \begin{matrix} \alpha = 4 \\ \alpha = 4 \\ \alpha = 2 \end{matrix}$$

Therefore, It can be said that these two vectors are not linearly dependent, as there is no single real number that satisfies the relation $\vec{A} = \alpha\,\vec{B}$

10) Vector Addition:

The sum of two vectors $\vec{A}$ and $\vec{B}$ *is a vector with its tail at the beginning of the first vector* $\vec{A}$ *and its head at the end of the second vector* $\vec{B}$*, while its length is obtained as follows:*

$$\vec{R} = \vec{A} + \vec{B} \Rightarrow (\vec{R})^2 = (\vec{A} + \vec{B})^2 = (\vec{A})^2 + (\vec{B})^2 + 2\,(\vec{A}.\vec{B})$$

$\|\vec{\mathbf{R}}\|^2 = \|\vec{\mathbf{A}}\|^2 + \|\vec{\mathbf{B}}\|^2 + 2\,\|\vec{\mathbf{A}}\|.\|\vec{\mathbf{B}}\|\,\mathbf{cos}(\boldsymbol{\vartheta})$ *where* $\boldsymbol{\vartheta}$ *denotes the angle enclosed between* $\vec{A}$ and $\vec{B}$ *vectors*

$$\|\vec{\mathbf{R}}\| = \sqrt{\|\vec{\mathbf{A}}\|^2 + \|\vec{\mathbf{B}}\|^2 + 2\,\|\vec{\mathbf{A}}\|.\|\vec{\mathbf{B}}\|\,\mathbf{cos}(\boldsymbol{\vartheta})}$$

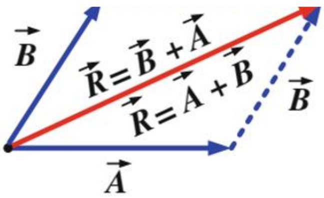
Geometrically*, vectors can be added by parallel displacement of the second vector to the first vector, such that the end of the first vector (the head) matches the starting point of the second vector (the tail). The resulting vector from the addition has its starting point at the origin of the first vector and its endpoint at the head of the second vector, as shown in the following figure where* $\vec{R} = \vec{A} + \vec{B}$

Analytically:

Let us consider $\vec{A}$ *and* $\vec{B}$ *are two vectors have their components in the Cartesian coordinates system (O, XYZ), where "O" is the origin, and "XYZ" represents the three axes with three basis orthogonal unit vectors* $(\vec{\imath}, \vec{\jmath}, \vec{k})$ *that correspond to the three main coordinate axes.We can sum* $\vec{A}$ *and* $\vec{B}$ *analytically by adding all the components multiplied by the same unit vector as follows:*

$$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}\quad \vec{B} = B_x\vec{\imath} + B_y\vec{\jmath} + B_z\vec{k}$$

$$\vec{A} + \vec{B} = (A_x + B_x)\vec{\imath} + (A_y + B_y)\vec{\jmath} + (A_z + B_z)\vec{k}$$



We can use another writing shape for vectors sum

$$\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix} \quad \text{and} \quad \vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix} \Rightarrow \vec{A} + \vec{B} \begin{pmatrix} A_x + B_x \\ A_y + B_y \\ A_z + B_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix}$$

The magnitude of $(\vec{A} + \vec{B})$ *vector is given as follow:*

$$\|\vec{A} + \vec{B}\| = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2 + (A_z + B_z)^2}$$

To find the coordinates of the starting point of the vector sum of vectors $\vec{A}$ and $\vec{B}$, where vector $\vec{A}$ has its tail at point (x_1, y_1) and its head at point (x_2, y_2), and vector $\vec{B}$ has its tail at point (x_3, y_3) and its head at point (x_4, y_4), the resultant of their addition, vector $\vec{C} = \vec{A} + \vec{B}$, will have its starting coordinates at (x_1+x_3, y_1+y_3) and its head at ($x_2 + x_4$, $y_2 + y_4$).

Important Note:

1- *Subtracting two vectors is a special case of vector addition. Geometrically, we add the first vector to the negative of the second vector. Analytically, it is expressed as follows:*

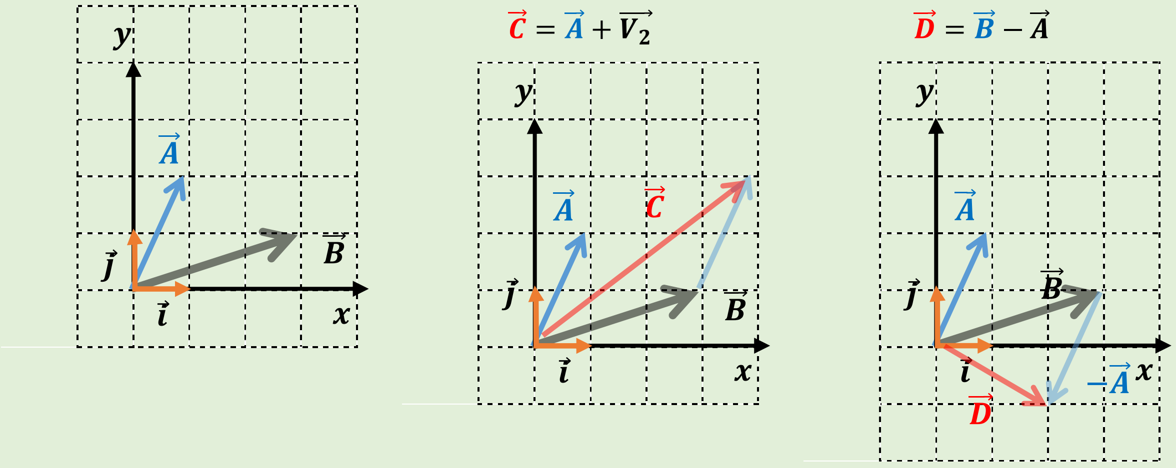
$$\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix} \quad and \quad \vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix} \Rightarrow \vec{A} - \vec{B}\begin{pmatrix} A_x - B_x \\ A_y - B_y \\ A_z - B_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix}$$

- ✓ *Vector addition has the commutativeproperty*$\vec{A} + \vec{B} = \vec{B} + \vec{A}$
- ✓ *Vector addition has the associativeproperty*$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$
- ✓ *Vector addition has thedistributiveproperty*$\lambda \times (\vec{A} + \vec{B}) = \lambda \times \vec{A} + \lambda \times \vec{B}$

Example03:

Weconsider the following vectors:$\vec{A} = 3\vec{\imath} + 1\vec{\jmath}\,\vec{B} = 1\vec{\imath} + 2\vec{\jmath}$

Using the graphical and analytical methods, find the sum and subtraction of these vectors



Addition **Subtraction**

Analytically:

$$\vec{C} = \vec{A} + \vec{B} = 3\vec{\imath} + 1\vec{\jmath} + 1\vec{\imath} + 2\vec{\jmath} = 4\vec{\imath} + 3\vec{\jmath}$$

$$\vec{D} = \vec{B} - \vec{A} = 1\vec{\imath} + 2\vec{\jmath} - (3\vec{\imath} + 1\vec{\jmath}) = -2\vec{\imath} + 1\vec{\jmath}$$

11) Scalar (Dot) Product, Vector (Cross) Product, And Mixed Product

11-1) Scalar (Dot) Product

- *The scalar (dot) product of two vectors* $\vec{A}$ *and* $\vec{B}$ *is denoted* $\vec{A}\cdot\vec{B}$
- *The scalar (dot) productof two vectors* $\vec{A}$ *and* $\vec{B}$*produces a* ***scalar***.
- *The scalar product of vectors* $\vec{A}$ *and* $\vec{B}$ *is given in terms of their magnitudes and the angle (**ϑ**) enclosed between them as follows:*

$$\vec{\boldsymbol{A}}\,.\,\vec{\boldsymbol{B}} = \left\|\vec{\boldsymbol{A}}\right\|.\left\|\vec{\boldsymbol{B}}\right\|\ \boldsymbol{cos}\,(\boldsymbol{\vartheta})$$

We can distinguish three special cases based on the value of the angle

$$\vec{A}\,.\,\vec{B} = \left\|\vec{A}\right\|.\left\|\vec{B}\right\|\ cos\,(\vartheta) = \begin{cases} \left\|\vec{A}\right\|.\left\|\vec{B}\right\|\ if\ \vartheta = 0\ \ \vec{A}\ \text{and}\ \vec{B}\ are\ parallel\ vectors \\ -\left\|\vec{A}\right\|.\left\|\vec{B}\right\|\ if\ \ \vartheta = \pi\ \ \vec{A}\ \text{and}\ \vec{B}\ are\ anti\ parallel\ vectors \\ 0\ \ if\ \vartheta = \frac{\pi}{2}\ \ \ so\ \vec{A}\perp\vec{B} \end{cases}$$

- *The scalar product of vectors* $\vec{A}$ *and* $\vec{B}$ *is given in terms of the components of* $\vec{A}$ *and* $\vec{B}$*as follows:*

$$\vec{A}\,.\,\vec{B} = (A_x \times\ B_x)\ + (A_y \times\ B_y)\ + (A_z \times\ B_z)$$

- *The scalar product has the commutative property* $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
- *The scalar product has the distributive property* $\vec{C}\cdot\left(\vec{A}+\vec{B}\right) = (\vec{C}\cdot\vec{A}) + (\vec{C}\cdot\vec{B})$
- *The scalar product of the unit vector is given as follow:*

$$\vec{\imath}\,.\vec{\jmath} = \vec{\imath}\,.\vec{\jmath} = \vec{\imath}\,.\vec{k} = \vec{k}\,.\vec{\imath} = \vec{k}\,.\vec{\jmath} = \vec{\jmath}\,.\vec{k} = 0$$

$$\vec{\imath}\,.\vec{\imath} = \vec{\jmath}\,.\vec{\jmath} = \vec{k}\,.\vec{k} = 1$$

- *If the components of vectors* $\vec{A}$ *and* $\vec{B}$ *are time-dependent, the derivative of their scalar product is expressed as follows:*

$$\frac{d(\vec{A}\cdot\vec{B})}{dt} = \frac{d(\vec{A})}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d(\vec{B})}{dt}$$

11-2) Vector (Cross) Product

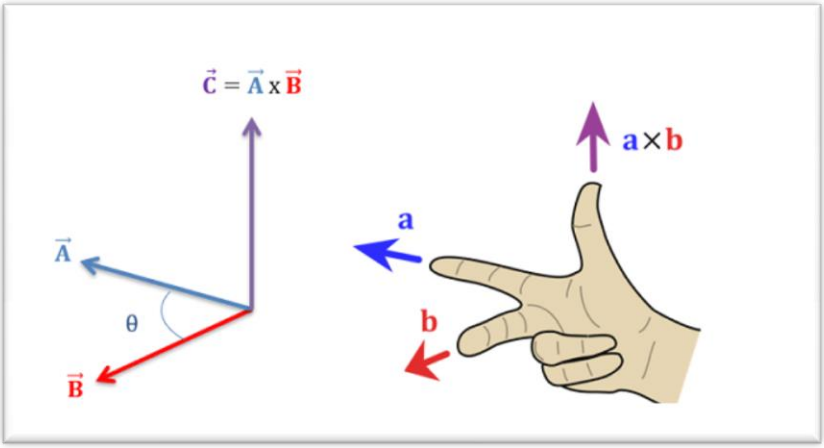
The Vector (Cross) product of two vectors $\vec{A}$ *and* $\vec{B}$ *is denoted* $\vec{A}\wedge\vec{B}$

- *The Vector (Cross) product of two vectors* $\vec{A}$ *and* $\vec{B}$*produces a* ***vector*** *that is perpendicular on the plane formed by* $\vec{A}$ *and* $\vec{B}$*, and its direction is determined using* ***the right-hand rule****. Its magnitude is given in terms of is given in term of* $\vec{A}$ *and* $\vec{B}$*magnitudes and the angle (**ϑ**) enclosed between them as follows:*

$$\|\vec{C}\| = \|\vec{A} \wedge \vec{B}\| = \|\vec{A}\| . \|\vec{B}\| \ \sin(\vartheta)$$

We can distinguish two special cases based on the value of the angle

$$\|\vec{A} \wedge \vec{B}\| = \|\vec{A}\| . \|\vec{B}\| \ \sin(\vartheta) = \begin{cases} 0 & \text{if } \vec{A} \text{ and } \vec{B} \text{ are parallel vectors } (\vec{A} \parallel \vec{B}) \\ \pm\|\vec{A}\| . \|\vec{B}\| & \text{if } \vec{A} \text{ and } \vec{B} \text{ are orthogonal vectors } (\vec{A} \perp \vec{B}) \end{cases}$$



- ✓ *The vector(Cross) product has not the commutativeproperty* $\vec{A} \wedge \vec{B} = -\vec{B} \wedge \vec{A}$
- ✓ *The vector(Cross) product has the distributiveproperty* $\vec{C} \wedge (\vec{A} + \vec{B}) = (\vec{C} \wedge \vec{A}) + (\vec{C} \wedge \vec{B})$
- ✓ *The vector(Cross) productof vectors* $\vec{A}$ *and* $\vec{B}$ *is given in terms of the determinant as follows:*

$$\vec{A} \wedge \vec{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \vec{\imath} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \vec{\jmath} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \vec{k}$$

$$\vec{A} \wedge \vec{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - B_y A_z)\vec{\imath} - (A_x B_z - B_x A_z)\vec{\jmath} + (A_x B_y - B_x A_y)\vec{k}$$

- ✓ *The vector(Cross) product of the unit vector is given as follow:*

$$\vec{\imath} \wedge \vec{\jmath} = -\vec{\jmath} \wedge \vec{\imath} = \vec{k}$$
$$\vec{\jmath} \wedge \vec{k} = -\vec{k} \wedge \vec{\jmath} = \vec{\imath}$$
$$\vec{k} \wedge \vec{\imath} = -\vec{\imath} \wedge \vec{k} = \vec{\jmath}$$
$$\vec{\imath} \wedge \vec{\imath} = \vec{\jmath} \wedge \vec{\jmath} = \vec{k} \wedge \vec{k} = \vec{0}$$

- ✓ *If the components of vectors* $\vec{A}$ *and* $\vec{B}$ *are time-dependent, the derivative of thevector (cross) product is expressed as follows:*

$$\frac{d(\vec{A}\wedge\vec{B})}{dt}=\frac{d(\vec{A})}{dt}\wedge\vec{B}+\vec{A}\wedge\frac{d(\vec{B})}{dt}$$

11-3) Mixed Product

The result of the mixed product of vectors is a scalar value, calculated by the determinant as follows:

$$\vec{\boldsymbol{C}}\cdot(\vec{\boldsymbol{A}}\wedge\vec{\boldsymbol{B}})=\begin{vmatrix}C_x & C_y & C_z\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}C_x-\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}C_y+\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}C_z$$

$$\vec{\boldsymbol{C}}\cdot(\vec{\boldsymbol{A}}\wedge\vec{\boldsymbol{B}})=(A_yB_z-B_yA_z)C_x-(A_xB_z-B_xA_z)C_y+(A_xB_y-B_xA_y)C_z$$

Exercise 01:

Consider the following vectors:

$$\vec{A}=4\vec{\imath}+2\,\vec{\jmath}\,\vec{B}=2\vec{\imath}+\vec{\jmath}-\vec{k}$$

1- *Calculate the magnitude of* $\vec{A}$ *and* $\vec{B}$, *the dot product* $\vec{A}.\vec{B}$*and the cross product* $\vec{A}\,\vec{B}$

2- *Calculate the angle* θ *formed by the two vectors*

Solution

1- *Calculate the magnitude of* $\vec{A}$ *and* $\vec{B}$, *the dot product* $\vec{A}.\vec{B}$*and the cross product* $\vec{A}\,\vec{B}$

$$\|\vec{A}\|=\sqrt{(4)^2+(2)^2}=\sqrt{20}$$

$$\|\vec{B}\|=\sqrt{(2)^2+(1)^2+(-1)^2}=\sqrt{6}$$

$$\vec{A}\,.\vec{B}=\ 4*2+2*1+0*(-1)=10$$

$$\vec{\boldsymbol{A}}\wedge\vec{\boldsymbol{B}}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ 4 & 2 & 0\\ 2 & 1 & -1\end{vmatrix}=\begin{vmatrix}2 & 0\\ 1 & -1\end{vmatrix}\vec{\imath}-\begin{vmatrix}4 & 0\\ 2 & -1\end{vmatrix}\vec{\jmath}+\begin{vmatrix}4 & 2\\ 2 & 1\end{vmatrix}\vec{k}=-2\vec{\imath}-4\vec{\jmath}+0\,\vec{k}$$

2- *Calculate the angle* θ *formed by the two vectors*

$$\vec{A}\,.\vec{B}=\|\vec{A}\|\times\|\vec{B}\|\,cos(\vartheta)\ =10$$

$$\|\vec{A}\|\times\|\vec{B}\|\,cos(\vartheta)\ =\frac{\vec{A}\,.\vec{B}}{\|\vec{A}\|\times\|\vec{B}\|}=\frac{10}{\sqrt{20\times6}}=0.912$$

$$\vartheta=24.21^0$$