Chapter 2

Sets and Functions

2.1 Definitions and Examples

2.1.1 Sets and Elements

* Intuitively, a set is a collection of objects. The objects in a set are called elements of that set, and an element a belongs to E (written as $a \in E$) or does not belong to E (written as $a \notin E$).

* An empty set, denoted by $\emptyset$, is a set that does not contain any elements.

* A set $E = \{a\}$, consisting of a single element, is called a singleton.

* Let E be a set. If a set A is contained in E, we say that A is a subset or a sub-set of E. The elements of E that do not belong to set A form a new set called the complement of A in E, denoted as A^c or $C_E(A)$. Formally, $C_E(A) = \{x \in E \mid x \notin A\}$.

2.1.2 Set Operations

Given two sets A and B, we can construct other sets.

* We say that A is included in B (A is a subset of B or a part of B) and we denote it as $A \subset B$ if every element of A is also an element of B.

$$A \subset B \Leftrightarrow (\forall x \in A \Rightarrow x \in B)$$

* We say that A and B are equal if and only if $A \subset B$ and $B \subset A$.

* Given two sets A and B, the union of A and B, denoted as $A \cup B$ (read as "A union B"), is the set of elements that belong to either A or B.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

* Given two sets A and B, the intersection of A and B, denoted as $A \cap B$ (read as "A intersect B"), is the set of elements that belong to both A and B.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

* We say that A and B are disjoint sets if $A \cap B = \emptyset$.

Example In $\mathbb{N}$ (the set of natural numbers), if we denote by $\mathcal{D}(n)$ the set of divisors of the natural number n, we have

$$\mathcal{D}(24) \cup \mathcal{D}(16) = \{1, 2, 3, 4, 6, 8, 12, 16, 24\} \quad \text{and} \quad \mathcal{D}(24) \cap \mathcal{D}(16) = \{1, 2, 3, 4, 8\}$$

2.1.3 Properties and Rules of Calculations

Here are some properties and rules of calculations on sets.

Proposition 2.1 Let A, B, C be subsets of a set E. Then:

1. $A \cup A = A$, $A \cap A = A$.

2. $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$.

3. $A \cup B = B \cup A$, $A \cap B = B \cap A$ (Commutativity).

4. $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$ (Associativity).

5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributivity).

Proof. We prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$$\begin{aligned} \text{Let } x \in A \cup (B \cap C) &\Leftrightarrow x \in A \text{ or } x \in (B \cap C) \\ &\Leftrightarrow x \in A \text{ or } (x \in B \text{ and } x \in C) \\ &\Leftrightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\ &\Leftrightarrow (x \in A \cup B) \text{ and } (x \in A \cup C) \\ &\Leftrightarrow x \in (A \cup B) \cap (A \cup C). \end{aligned}$$

□

Definition 2.1 (Power Set) Let E be a set. We admit the existence of a set denoted by $\mathcal{P}(E)$ such that the following equivalence holds:

$$X \in \mathcal{P}(E) \Leftrightarrow X \subset E$$

$\mathcal{P}(E)$ is called the power set of E.

Remark 2.1 If $\text{card}(E) = n$, then $\text{card}(\mathcal{P}(E)) = 2^n$.

Example If $E = \{1, 2, 3\}$, then $\text{card}(\mathcal{P}(E)) = 2^3 = 8$ and

$$\mathcal{P}(E) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Definition 2.2 (Set Difference) Let A, B be two subsets of E.

1. The difference of A and B, denoted $A \setminus B$, consists of elements that are in A but not in B, i.e., $A \setminus B = A \cap C_E(B)$.

2. The symmetric difference of A and B, denoted $A \triangle B$, is the set $(A \setminus B) \cup (B \setminus A)$ or $(A \cup B) \setminus (A \cap B)$.

Example 1. In $\mathbb{N}$, we have $\mathcal{D}(24) \setminus \mathcal{D}(16) = \{3, 6, 12, 24\}$ and $\mathcal{D}(16) \setminus \mathcal{D}(24) = \{16\}$. Also, $\mathcal{D}(24) \triangle \mathcal{D}(24) = \{6, 12, 16, 24\}$.

2. The set $\mathbb{R} \setminus \mathbb{Q}$ contains irrational numbers like π.

Remark 2.2 When $A \subset E$, we have $E \setminus A = C_E(A)$.

Proposition 2.2 Let A, B be two subsets of E. Then:

1. $A \setminus A = \emptyset$.
2. $A \setminus \emptyset = A$.
3. $A \cup C_E(A) = E$.
4. $A \cap C_E(A) = \emptyset$.
5. $C_E(C_E(A)) = A$.
6. $C_E(A \cap B) = C_E(A) \cup C_E(B)$.
7. $C_E(A \cup B) = C_E(A) \cap C_E(B)$.

Proof. We prove that $C_E(A \cap B) = C_E(A) \cup C_E(B)$.

$$\text{Let } x \in C_E(A \cap B) \Leftrightarrow x \notin (A \cap B)$$

$$\Leftrightarrow \overline{x \in (A \cap B)}$$

$$\Leftrightarrow \overline{x \in A \text{ and } x \in B}$$

$$\Leftrightarrow \overline{x \in A} \text{ or } \overline{x \in B}$$

$$\Leftrightarrow x \notin A \text{ or } x \notin B$$

$$\Leftrightarrow x \in C_E(A) \cup C_E(B).$$

□

Definition 2.3 (Partition) Let E be a set. A partition of E is a set $\{E_i\}$ of subsets of E that satisfies the following two conditions:

1. $E = \bigcup_{i \in I} E_i$
2. $E_i \cap E_j = \emptyset$ for all $i \neq j \in I$.

Example Let A be a subset of E. Then the set $\{A, C_E(A)\}$ is a partition of E.

Definition 2.4 (Cartesian Product) Let A, B be two sets. The Cartesian product, denoted $A \times B$, is the set of pairs (x, y) where $x \in A$ and $y \in B$.

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

Example

1. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$.

2. Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$.

Generalization If we consider sets $A_1, A_2, \ldots, A_n$, we can similarly define n-tuples $(x_1, x_2, \ldots, x_n)$ where $x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n$.

$$A_1 \times A_2 \times \ldots \times A_n = \{(x_1, x_2, \ldots, x_n) \mid x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n\}.$$

Proposition 2.3 Let A, B, C, D be four subsets of E. Then:

1. $(A \times C) \cup (B \times C) = (A \cup B) \times C$.

2. $(A \times C) \cup (A \times D) = A \times (C \cup D)$.

3. $(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D)$.

Proof. We prove that $(A \times C) \cup (B \times C) = (A \cup B) \times C$.

$$\begin{aligned}(A \times C) \cup (B \times C) &= \{(x, y) \mid (x, y) \in A \times C \text{ or } (x, y) \in B \times C\} \\ &= \{(x, y) \mid (x \in A \text{ and } y \in C) \text{ or } (x \in B \text{ and } y \in C)\} \\ &= \{(x, y) \mid (x \in A \text{ or } x \in B) \text{ and } y \in C\} \\ &= (A \cup B) \times C.\end{aligned}$$

□

2.1.4 Definitions and Examples

Definition 2.5 Let E, F be two sets. We say that f is a function from E to F if for every element $x \in E$, there exists a unique element $y \in F$ such that $f(x) = y$, and we write

$$f : E \longrightarrow F \quad \text{or} \quad E \xrightarrow{f} F$$

* The set E is called the domain and F is called the codomain. The element x is called the pre-image and y is called the image of x under f.

* We denote by $\mathfrak{F}(E, F)$ the set of all functions from E to F.

Example

1. $\begin{array}{rcl} f:\{1,2,3\} & \longrightarrow & \{2,4,5\} \\ x & \mapsto & x^2 \end{array}$ is not a function.

2. The identity function $\begin{array}{rcl} f:E & \longrightarrow & E \\ x & \mapsto & x \end{array}$ is a function and will be very useful in the following.

3. The projections $\begin{array}{rcl} P_x:E\times F & \longrightarrow & E \\ (x,y) & \mapsto & P_x(x,y)=x \end{array}$ $\begin{array}{rcl} P_y:E\times F & \longrightarrow & F \\ (x,y) & \mapsto & P_y(x,y)=y \end{array}$ are also functions.

Definition 2.6 (Restrictions and Extensions) Let f be a function from E to F.

1. The restriction of f to a subset $A \subset E$ is the function denoted $f_{|A} : A \longrightarrow F$ defined by

$$f_{|A} = f(x), \quad \forall x \in A$$

2. The extension of f to a set E' containing E is any function g from E' to F whose restriction is f.

Example If f is the identity function from $\mathbb{R}^+$ to itself, it has infinitely many extensions to $\mathbb{R}$, among which:

1. The identity function on $\mathbb{R}$.

2. The absolute value function from $\mathbb{R}$ to itself.

3. The function h defined by $h(x) = \frac{1}{2}(x + |x|)$, which is identically zero on $\mathbb{R}^-$.

2.1.5 Direct Image and Inverse Image

Definition 2.7 Let E, F be two sets.

1. For $A \subset E$ and $f : E \longrightarrow F$, the direct image of A under f is a subset of F defined by

$$f(A) = \{f(x) \mid x \in A\}$$

2. For $B \subset F$ and $f : E \longrightarrow F$, the inverse image of B under f is a subset of E defined by

$$f^{-1}(B) = \{x \mid f(x) \in B\}$$

Example Let f be a given function:
$$\begin{array}{rccc} f: & \mathbb{N} & \longrightarrow & \mathbb{N} \\ & n & \mapsto & 2n+1 \end{array}$$

1. Let $A = \{0, 1, 2\}$, then $f(A) = \{f(n) \mid n \in A\} = \{f(0), f(1), f(2)\} = \{1, 3, 5\}$.

2. Let $B = \{5\}$, then $f^{-1}(B) = \{n \in \mathbb{N} \mid f(n) \in B\} = \{n \in \mathbb{N} \mid f(n) = 5\} = \{2\}$.

Proposition 2.4 Let $f : E \longrightarrow F$ be a function, A_1, A_2 be two subsets of E, and B_1, B_2 be two subsets of F. Then

(1) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2), \quad f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$;

(2) If $A_1 \subset A_2$, then $f(A_1) \subset f(A_2)$;

(3) $A_1 \subset f^{-1}(f(A_1))$;

(4) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2), \quad f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$;

(5) If $B_1 \subset B_2$, then $f^{-1}(B_1) \subset f^{-1}(B_2)$;

(6) $f(f^{-1}(B_1)) \subset B_1$.

Proof: We prove property (2).

Let $y \in f(A_1)$, then $\exists x \in A_1 \mid f(x) = y$, and since $A_1 \subset A_2$, there exists $x \in A_2 \mid f(x) = y$. Therefore, $y \in f(A_2)$.

□

Definition 2.8 (Composition) Let E, F, G be three sets, and f, g be two functions such that

$$E \xrightarrow{f} F \xrightarrow{g} G$$

Then we can obtain a function from E to G, denoted by $h = g \circ f$, and called the composition of f and g, defined as

$$\forall x \in E, h(x) = g \circ f(x) = g[f(x)]$$

Remark 2.3 In general, $f \circ g \neq g \circ f$. This is illustrated by real functions

$$f(x) = x^2, \quad g(x) = 2x + 1$$

$$f \circ g(x) = f[g(x)] = f(2x+1) = (2x+1)^2, \quad g \circ f(x) = g[f(x)] = g\left(x^2\right) = 2x^2 + 1.$$

Therefore, $f \circ g \neq g \circ f$.

* However, function composition is associative: $h \circ (g \circ f) = (h \circ g) \circ f$.

2.1.6 Injection, Surjection, Bijection

Definition 2.9 Let E, F be two sets and $f : E \longrightarrow F$ be a function.

1. f is injective if and only if

$$\forall x, x' \in E, f(x) = f\left(x'\right) \Rightarrow x = x'$$

2. f is surjective if and only if

$$\forall y \in F, \exists x \in E \mid y = f(x)$$

* Another formulation: f is surjective if and only if $f(E) = F$.

3. f is bijective if f is both injective and surjective. In other words,

$$\forall y \in F, \exists! x \in E \mid y = f(x)$$

Remark 2.4 If f is bijective, and only in this case, to each $y \in F$ is associated a unique $x \in E$.

We can define a bijective function, denoted as

$$f^{-1} : F \longrightarrow E$$

and called the inverse function of f. We have the equivalence

$$y = f(x) \Leftrightarrow x = f^{-1}(y)$$

Example Let $f : \mathbb{N} \longrightarrow \mathbb{Q}$ be defined by $f(x) = \frac{1}{1+x}$. Let's show that f is injective. Assume $x, x' \in \mathbb{N}$ such that $f(x) = f\left(x'\right)$. Then $\frac{1}{1+x} = \frac{1}{1+x'}$, which implies $1 + x = 1 + x'$ and thus $x = x'$. Therefore, f is injective.

However, f is not surjective. We need to find an element y that does not have a pre-image under f. Here it is easy to see that we always have $f(x) \leq 1$, so for example $y = 2$ has no pre-image. Hence, f is not surjective and therefore not bijective.

Theorem 2.1 Let E, F, G be three sets and f, g be two functions such that $f : E \longrightarrow F$ and $g : F \longrightarrow G$

1. If f and g are injective, then $g \circ f$ is injective.

2. If f and g are surjective, then $g \circ f$ is surjective.

3. If f and g are bijective, then $g \circ f$ is bijective.

4. If f and g are bijective, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof

1. Since f and g are injective, we have

$$(g \circ f)(x) = (g \circ f)(y) \Rightarrow f(x) = f(y) \Rightarrow x = y.$$

2. Since f and g are surjective, we have

$$(g \circ f)(E) = g[f(E)] = g(F) = G.$$

3. Follows directly from (1) and (2).

4. Let $z \in G$. Since $g \circ f$ is bijective, there exists $x \in E$ such that $(g \circ f)(x) = z$.

We have $(g \circ f)^{-1}(z) = (g \circ f)^{-1}((g \circ f)(x)) = x$.

On the other hand,

$\left(f^{-1} \circ g^{-1}\right)(z) = \left(f^{-1} \circ g^{-1}\right)((g \circ f)(x)) = f^{-1}\left(g^{-1}(g(f(x))\right) = f^{-1}(f(x)) = x$.

Therefore, $(g \circ f)^{-1}(z) = \left(f^{-1} \circ g^{-1}\right)(z) \quad \forall z \in G$. Hence, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

2.2 Exercises with Solutions

Exercise 1.

1. Let $A = \{1, 2, 3, 4, 5\}$. Determine whether the following statements are true:

$$2 \in A, 3 \subset A, \emptyset \in A, \{\emptyset\} \subset A, A \cup \{\emptyset\} = A$$

2. Let $B = \{1, 2\}$ and $C = \{1, 3\}$ be two sets.

(a) Determine $B \cap C, B \cup C, C_A(B), C_A(C), A \backslash B$, and $B \triangle C$.

(b) Determine $B \times C, B \times \emptyset, B \times \{\emptyset\}$, and $\mathcal{P}(\mathcal{P}(B))$.

Exercise 2. Let A, B, C be three subsets of the set E. Show that:

1. $A \cap B = \emptyset \Leftrightarrow A \subset C_E(B)$

2. $A \subset B \Leftrightarrow C_E(B) \subset C_E(A)$.

3. $C_E(A \cap B) = C_E(A) \cup C_E(B), \quad C_E(A \cup B) = C_E(A) \cap C_E(B)$

4. $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$.

5. $C_E(A) \Delta C_E(B) = A \triangle B, \quad C_E(A \triangle B) = C_E(A) \triangle B(*)$

6. $(A \times C) \cup (B \times C) = (A \cup B) \times C$.

7. $A \subset B \Rightarrow \mathcal{P}(A) \subset \mathcal{P}(B)$.

Exercise 3. Let A, B, C be three subsets of the set E. Show that:

1. $A = B \Leftrightarrow A \cap B = A \cup B$.

2. $A \cup B = A \cap C \Leftrightarrow B \subset A \subset C$.

3. $A \cap B = \emptyset \Leftrightarrow C_E(A) \cup C_E(B) = E$.

4. $A \triangle B = \emptyset \Leftrightarrow A = B$.

5. $(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C) = (A \backslash C) \cap B = (B \backslash C) \cap A$.

Exercise 4. Let $f : E \longrightarrow F$ be a function. Let A, B be two subsets of the set E and C, D be two subsets of the set F. Show that:

1. $f(A \cap B) \subset f(A) \cap f(B), \quad f(A \cup B) = f(A) \cup f(B)(*)$

2. f is injective $\Leftrightarrow f(A \cap B) = f(A) \cap f(B)$.

3. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D), \quad f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)(*)$

4. $f\left(f^{-1}(C)\right) \subset C.$

5. f is surjective $\Leftrightarrow f\left(f^{-1}(C)\right) = C.$

6. $f^{-1}\left(C_F(C)\right) = C_E f^{-1}(C)$. 7. $f^{-1}(C \triangle D) = f^{-1}(C) \Delta f^{-1}(D).$

Exercise 5. Consider the function f defined by

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = \frac{2x}{1+x^2} \end{aligned}$$

1. Is f injective? Surjective?

2. Show that $f(\mathbb{R}) = [-1, 1]$.

3. Show that the function g defined by

$$\begin{aligned} g : [-1,1] &\longrightarrow [-1,1] \\ x &\longmapsto g(x) = f(x) \end{aligned}$$

is a bijection and find its inverse function g^{-1}.

Exercise 6. Let E be a non-empty set. Consider a function f from E to $\mathbb{R}$ such that

$$\begin{cases} \text{i) } f(\phi) = 0, \\ \text{ii) } f(E) = 1, \\ \text{iii) } \forall A, B \in \mathcal{P}(E) : f(A \cup B) = f(A) + f(B), \text{ if } A \cap B = \phi. \end{cases}$$

1. For any subset A of E, express $f\left(C_E^A\right)$ in terms of $f(A)$.

2. Prove that $\forall A, B \in \mathcal{P}(E) : f(A \cup B) = f(A) + f(B) - f(A \cap B)$.

3. Furthermore, suppose that

$$\text{iv) } \forall A \in \mathcal{P}(E) : f(A) \geq 0.$$

(a) Show that $\forall A, B \in \mathcal{P}(E) : A \subset B \Rightarrow f(A) \leq f(B)$.

(b) Show that $\forall A \in \mathcal{P}(E) : 0 \leq f(A) \leq 1$.

2.2.1 Solution

Exercise 1.

1.

* $2 \in A$ means that 2 is an element of A. This is true because the elements of A are 1, 2, and 3.
* $3 \subset A$ means that 3 is a subset of A. This is false because 3 is an element of A and not a subset of A.
* $\phi \in A$ means that ϕ is an element of A. This is false because the elements of A are 1, 2, and 3, but ϕ is not among these elements.
* $\{\phi\} \subset A$ means that the singleton $\{\phi\}$ is a subset of A. This is false because $\{\phi\}$ is a subset of $P(A)$ (the power set of A) and not a subset of A.
* $A \cup \{\phi\} = \{1, 2, 3, \phi\}$. This is false because A has three elements.

2.

a) $B \cap C = \{1\}; B \cup C = \{1, 2, 3\}; C_A(B) = \{3, 4, 5\}; C_A(C) = \{2, 4, 5\}; A \backslash B = \{3, 4, 5\}$.

$B \triangle C = \{B \cup C) \backslash (B \cap C) = \{1, 2, 3\} \backslash \{1\} = \{2, 3\}$

b)

* $B \times C = \{(x, y) \mid x \in B \wedge y \in C\} = \{(1, 1), (1, 3), (2, 1), (2, 3)\}$.
* $B \times \phi = \{(x, y) \mid x \in B \wedge y \in \phi\}$, where ϕ does not contain any elements, so $B \times \phi = \phi$.
* $B \times \{\phi\} = \{(x, y) \mid x \in B \wedge y \in \{\phi\}\} = \{(1, \phi), (2, \phi)\}$.
* $P(B) = \{\phi, B, \{1\}, \{2\}\}$,so

$$\begin{aligned} P(P(B)) = \ & \{\phi; P(B); \{\phi\}; \{B\}; \{\{1\}\}; \{\{2\}\}; \{\phi, B\}; \{\phi, \{1\}\}; \{\phi, \{2\}\}; \{B, \{1\}\}; \{B, \{2\}\}; \\ & \{\{1\}, \{2\}\}; \{\phi, B, \{1\}\}; \{\phi, B, \{2\}\}; \{B, \{1\}, \{2\}\}; \{\phi, \{1\}, \{2\}\}. \end{aligned}$$

Exercise 2.

1. $A \cap B = \phi \Leftrightarrow A \subset C_E(B)$.

$\Rightarrow$ We have $A \cap B = \phi$. Let $x \in A$ and assume that $x \notin C_E(B)$.

Then $x \notin C_E(B) \Rightarrow x \in C\left(C_E(B)\right) = B \Rightarrow x \in A \cap B \Rightarrow A \cap B \neq \phi$, which is absurd. Thus, $x \in C_E(B)$.

$\Leftarrow$ We assume that $A \cap B \neq \phi$. Then, $\exists x \in E / x \in A \cap B \Rightarrow x \in A \wedge x \in B$ and since $A \subset C_E(B)$, we have $x \in C_E(B) \wedge x \in B \Rightarrow x \in C_E(B) \cap B = \phi$, which is a contradiction. Therefore, $A \cap B = \phi$.

2. $A \subset B \Leftrightarrow C_E(B) \subset C_E(A)$.

$\Rightarrow$ Let's assume that $A \subset B$ and $x \in C_E(B)$. Then $x \in C_E(B) \Rightarrow x \notin B$ and since $A \subset B$, we have $x \notin A \Rightarrow x \in C_E(A) \Rightarrow C_E(B) \subset (A)$.

$\Leftarrow$ We have $C_E(B) \subset C_E(A)$. Then $x \in A \Rightarrow x \notin C_E(A) \Rightarrow x \notin C_E(B) \Rightarrow x \in B$. Therefore, $A \subset B$.

3. $C_E(A \cap B) = C_E(A) \cup C_E(B)$

$$\begin{aligned} x \in C_E(A \cap B) \Leftrightarrow x \notin (A \cap B) &\Longleftrightarrow x \notin A \vee x \notin B \\ &\Leftrightarrow x \in C_E(A) \vee x \in C_E(B) \\ &\Leftrightarrow x \in C_E(A) \cup C_E(B). \end{aligned}$$

The same applies to the union.

4. $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$.

$$\begin{aligned} A \backslash (B \cup C) &\overset{\text{Def}}{=} A \cap C(B \cup C) \overset{(3)}{=} A \cap \Big(\; C_E(B) \;\; \cap \;\; C_E(C) \; \Big) \\ &= (A \cap C_E(B)) \cap (A \cap C_E(C)) \overset{\text{Def}}{=} (A \backslash B) \cap (A \backslash C). \end{aligned}$$

5. $C_E(A) \Delta C_E(B) = A \triangle B$.

According to the definition: $A \Delta B = (A \backslash B) \cup (B \backslash A) = (A \cap C_E(B)) \cup (B \cap C_E(A))$. By replacing A with $C_E(A)$ and B with $C_E(B)$ in the previous formula

$$\begin{aligned} C_E(A) \Delta C_E(B) &= (C_E(A) \backslash C_E(B)) \cup (C_E(B) \backslash C_E(A)) = \\ & C_E(A) \cap C_E(B) \cup C_E(B) \cap C_E(A) = \\ & (A \cap C_E(B)) \cup (B \cap C_E(A)) = A \Delta B \end{aligned}$$

Since $\cap$ and $\cup$ are commutative laws.

6. $(A \times C) \cup (B \times C) = (A \cup B) \times C$.

$$\begin{aligned}(A \times C) \cup (B \times C) &= \{(x,y)/(x,y) \in A \times C \text{ or } (x,y) \in B \times C\} \\ &= \{(x,y) \mid (x \in A \text{ and } y \in C) \text{ or } (x \in B \text{ and } y \in C\} \\ &= \{(x,y) \mid (x \in A \text{ or } x \in B) \text{ and } y \in C\} \\ &= (A \cup B) \times C.\end{aligned}$$

7. $A \subset B \Rightarrow P(A) \subset P(B)$.

According to the definition: $P(A) = \{X \mid X \subset A\}$, we have:

$X \in P(A) \Rightarrow X \subset A$ and since $A \subset B$, we have $X \subset B \Rightarrow x \in P(B)$. Therefore, the inclusion holds.

Exercise 4.

1. $f(A \cap B) \subset f(A) \cap f(B)$.

Let $y \in f(A \cap B)$, which means there exists $x \in A \cap B$ such that $y = f(x)$. Since $x \in A$, we have $y = f(x) \in f(A)$. Similarly, since $x \in B$, we have $y \in f(B)$. Hence, $y \in f(A) \cap f(B)$.

Therefore, $f(A \cap B) \subset f(A) \cap f(B)$.

2. f is injective $\Leftrightarrow f(A \cap B) = f(A) \cap f(B)$.

$\Leftarrow$ Let's assume that $f(A \cap B) = f(A) \cap f(B)$. We need to prove that f is injective.

Assume that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in E$. Let $A = \{x_1\}$ and $B = \{x_2\}$. We have $f(x_1) = f(x_2) \in f(A) \cap f(B) = f(A \cap B)$, which means $f(A \cap B) \neq \phi$. This implies $A \cap B \neq \phi$, which contradicts the assumption $x_1 = x_2$. Therefore, f is injective.

$\Longrightarrow$ We assume that f is injective. We need to prove that $f(A \cap B) = f(A) \cap f(B)$.

We already proved in part (1) that $f(A \cap B) \subset f(A) \cap f(B)$. Now let's prove the other inclusion. Let $y \in f(A) \cap f(B)$. Then $y \in f(A)$ and $y \in f(B)$.

$$\Rightarrow \exists x \in A | y = f(x) \quad \wedge \quad \exists \bar{x} \in B | y = f\left(x'\right).$$

Since $f(x) = f\left(x'\right)$ and f is injective, we have $x = \bar{x}$.

$$\Rightarrow x \in A \cap B \Rightarrow f(x) \in f(A \cap B) \Rightarrow y \in f(A \cap B).$$

Thus, $f(A) \cap f(B) \subset f(A \cap B)$.

3. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

$$\begin{aligned} f^{-1}(C \cap D) &= \{x; f(x) \in C \cap D\} \\ &= \{x; f(x) \in C \wedge f(x) \in D\} \\ &= \{(x; f(x) \in C) \text{ and } (x; f(x) \in D)\} \\ &= f^{-1}(C) \cap f^{-1}(D). \end{aligned}$$

4. If $f(x) \in f\left(f^{-1}(C)\right)$, then $x \in C$

Therefore, $f\left(f^{-1}(C)\right) \subset C$.

5. f is surjective $\Leftrightarrow f\left(f^{-1}(C)\right) = C$.

$\Longrightarrow$ We need to prove that for every $y \in F$, there exists $x \in E$ such that $y = f(x)$.

For every $y \in F$, we have $y \in \{y\}$ and according to the hypothesis, we can write $\{y\} = f\left(f^{-1}(\{y\})\right)$.

Therefore, there exists an element $x \in E$ with $x \in f^{-1}(\{y\}) \Rightarrow f(x) \in \{y\} \Rightarrow f(x) = y$.

$\Longleftarrow$ We have $f\left(f^{-1}(C)\right) \subset C$ according to (4). Now we need to prove that $C \subset f\left(f^{-1}(C)\right)$.

Let $y \in C$, which means $y \in F$. Since f is surjective, there exists $x \in E$ such that $y = f(x)$.

$$\Rightarrow \exists x \in E | y = f(x) \wedge \exists \bar{x} \in B | y = f\left(x'\right).$$

$$\Rightarrow f(x) = f(x') \quad \text{and since } f \text{ is injective} \quad x = \bar{x}$$

$$\Rightarrow x \in f^{-1}(C) \Rightarrow f(x) \in f\left(f^{-1}(C)\right)$$

Therefore, $y \in f(f^{-1}(C))$. Hence, $f(f^{-1}(C)) \supset C$.

(6) $f^{-1}(C_F(C)) = C_E f^{-1}(C)$.

$$x \in f^{-1}(C_F(C)) \Leftrightarrow f(x) \in C_E(C) \Leftrightarrow f(x) \notin C \Leftrightarrow x \notin f^{-1}(C)$$

$$\Leftrightarrow x \in C_E f^{-1}(C).$$

(7) $f^{-1}(C \triangle D) = f^{-1}(C) \Delta f^{-1}(D)$.

$$\begin{aligned} f^{-1}(C \triangle D) &= f^{-1}((C \backslash D) \cup (D \backslash C)) = f^{-1}(C \backslash D) \cup \hat{f}^{-1}(D \backslash C) \\ &= f^{-1}(C \cap C_F(D)) \cup f^{-1}(D \cap C_F(C)) \\ &= \left(f^{-1}(C) \cap f^{-1}(C_F(D))\right) \cup \left(f^{-1}(D) \cap f^{-1}(C_F(C))\right). \\ &= \left(f^{-1}(C) \cap C_E f^{-1}(D)\right) \cup \left(f^{-1}(D) \cap C_E f^{-1}(C)\right). \\ &= \left(f^{-1}(C) \backslash f^{-1}(D)\right) \cup \left(f^{-1}(D) \backslash f^{-1}(C)\right) \\ &= f^{-1}(C) \Delta f^{-1}(D). \end{aligned}$$

Exercise 5.

1. f is not injective because $f(2) = f(1/2) = \frac{4}{5}$ but $2 \neq \frac{1}{2}$.

f is not surjective because the value "2" does not have a preimage.

To show this, we can solve the equation $f(x) = 2$ which leads to $x^2 - x + 1 = 0$ and this equation has no real solutions.

2. We know that $f(\mathbb{R}) = [-1, 1]$ if the equation $f(x) = y$ has a unique solution x for every $y \in [-1, 1]$.

$$f(x) = y \Rightarrow yx^2 - 2x + y = 0 \ldots\ldots (*)$$

$$\Delta = 1 - y^2$$

$(*)$ has a solution if and only if $\Delta \geq 0$, so there are solutions if and only if $y \in [-1, 1]$.

Hence, $f(\mathbb{R}) = [-1, 1]$.

3. g is bijective if and only if g is injective and surjective.

$\Longrightarrow$ We assume that g is bijective. We need to prove that for every $y \in [-1, 1]$, the equation $g(x) = y$ has a unique solution.

So for every $y \in [-1, 1]$, there exists a unique $x \in [-1, 1]$ such that $g(x) = y$.

Let's find the solution to $g(x) = x$:

$$\begin{cases} x = \frac{1-\sqrt{1-y^2}}{y}, & \in [-1, 1] \\ x = \frac{1+\sqrt{1-y^2}}{y}, & \notin [-1, 1] \end{cases}$$

We can see that $\frac{1+\sqrt{1-y^2}}{y} \notin [-1, 1]$, so the only solution is $x = \frac{1-\sqrt{1-y^2}}{y}$. Therefore, g is bijective.

$$\begin{aligned} g^{-1} : [-1, 1] &\longrightarrow [-1, 4] \\ y &\longmapsto g^{-1}(y) = \frac{1 - \sqrt{1 - y^2}}{y} \end{aligned}$$

Chapter 3

Binary Relations on a Set

3.1 Basic Definitions

Definition 3.1 (Binary Relation) Let E be a set. A binary relation $\mathcal{R}$ on E is a property that applies to pairs of elements from E. We denote $x\mathcal{R}y$ to indicate that the property is true for the pair $(x, y) \in E \times E$.

Example

1. The inequality $\leq$ is a relation on $\mathbb{N}, \mathbb{Z}$, and $\mathbb{R}$.

2. The inclusion relation in the power set of E: $A\mathcal{R}B \Leftrightarrow A \subset B$.

3. The divisibility relation on the integers: $m\mathcal{R}n \Leftrightarrow m$ divides n.

Definition 3.2 Let $\mathcal{R}$ be a relation on a set E.

1. $\mathcal{R}$ is reflexive if for every $x \in E$, $x\mathcal{R}x$ holds.

2. $\mathcal{R}$ is symmetric if for all $x, y \in E$, $x\mathcal{R}y \Rightarrow y\mathcal{R}x$.

3. $\mathcal{R}$ is antisymmetric if for all $x, y \in E$, $(x\mathcal{R}y \wedge y\mathcal{R}x) \Rightarrow x = y$.

4. $\mathcal{R}$ is transitive if for all $x, y, z \in E$, $(x\mathcal{R}y \wedge y\mathcal{R}z) \Rightarrow x\mathcal{R}z$.

3.2 Equivalence Relations

Definition 3.3 (Equivalence Relation) A binary relation $\mathcal{R}$ on E is an equivalence relation if and only if it is reflexive, symmetric, and transitive.

Example 1 The relation $\mathcal{R}$ of ”being parallel” is an equivalence relation for the set E of affine lines in the plane:

1. Reflexivity: A line is parallel to itself.

2. Symmetry: If line D is parallel to D', then D' is parallel to D.

3. Transitivity: If line D is parallel to D' and D' is parallel to D'', then D is parallel to D''.

Example 2 Consider the following relation on $\mathbb{Z}$:

$$x\mathcal{R}y \Leftrightarrow \exists k \in \mathbb{Z} \mid x - y = 2k$$

1. $\mathcal{R}$ is reflexive because $\exists k = 0 \mid x - x = 2k = 0$, thus $x\mathcal{R}x$.

2. Suppose $x\mathcal{R}y$, then $\exists k \in \mathbb{Z} \mid x - y = 2k \Rightarrow y - x = 2k'$ with $k' = -k \in \mathbb{Z}$. Therefore, $y\mathcal{R}x$. Hence, $\mathcal{R}$ is symmetric.

3. Suppose $x\mathcal{R}y$ and $y\mathcal{R}z$. Then, $(\exists k \in \mathbb{Z} \mid x - y = 2k)$ and $(\exists k' \in \mathbb{Z} \mid y - z = 2k')$ by adding these equations, we obtain $x - z = 2k''$ with $k'' = (k + k') \in \mathbb{Z}$. Thus, $x\mathcal{R}z$. Therefore, $\mathcal{R}$ is transitive. Consequently, $\mathcal{R}$ is an equivalence relation.

Definition 3.4 Let $\mathcal{R}$ be an equivalence relation on a set E. The equivalence class of an element $x \in E$ is the set of elements in E that are related to x by $\mathcal{R}$, denoted by $\mathcal{C}(x)$ or $\bar{x}$:

$$\bar{x} = \{y \in E \mid y\mathcal{R}x\}$$

Definition 3.5 Let $\mathcal{R}$ be an equivalence relation on a set E. The quotient set of E by $\mathcal{R}$ is the set of equivalence classes of $\mathcal{R}$, denoted by $E/\mathcal{R}$:

$$E/\mathcal{R} = \{\bar{x} \mid x \in E\}$$

Example In the previous example, we have

$$\begin{aligned}\bar{x} &= \{y \in E \mid y\mathcal{R}x\}\\ &= \{y \in E \mid x - y = 2k\}\\ &= \{x - 2k : k \in \mathbb{Z}\}\\ &= \{\dots, x-4, x-2, x, x+2, x+4, \dots\}.\end{aligned}$$

$\overline{0} = \{y \in E \mid 0\mathcal{R}y\} = \{\dots, -4, -2, 0, 2, 4, \dots\}$**,** $\overline{1} = \{y \in E \mid 1\mathcal{R}y\} = \{\dots, -3, -1, 1, 3, \dots\}$

and $\overline{2} = \overline{0}$.**Therefore,**$\mathbb{Z}/\mathcal{R} = \{\bar{x} \mid x \in E\} = \{\overline{0}, \overline{1}\}$

Proposition 3.1 Let $\mathcal{R}$ be an equivalence relation on E. Then

1. An equivalence class is a subset of the set E, i.e., for all $x \in E$, $\bar{x} \subset E$.

2. An equivalence class is never empty, i.e., for all $x \in E$, $\bar{x} \neq \phi$.

3. The intersection of two distinct equivalence classes is empty, i.e., for all $x, y \in E$, $\bar{x} \cap \bar{y} = \phi$.

4. For all $x, y \in E$, $x\mathcal{R}y \Leftrightarrow \bar{x} = \bar{y}$.

Theorem 3.1 Let $\mathcal{R}$ be an equivalence relation on E. The equivalence classes $(\bar{x})_{x\in E}$ form a partition of E:

$$E = \cup_{x\in E}\bar{x}$$

3.3 Order Relation

Definition 3.6 (Order Relation) A binary relation $\mathcal{R}$ on E is an order relation if and only if it is reflexive, antisymmetric, and transitive. We then say that $(E, \mathcal{R})$ is an ordered set.

Example.

1. The inequality $\leq$ is an order relation on $\mathbb{N}, \mathbb{Z}$, and $\mathbb{R}$.

2. The inclusion relation in the power set of E is an order relation: $A\mathcal{R}B \Leftrightarrow A \subset B$.

Definition 3.7 Let $\mathcal{R}$ be an order relation on E. Two elements x and y of E are said to be comparable if $x\mathcal{R}y$ or $y\mathcal{R}x$.

Definition 3.8 (Total Order and Partial Order) Let $\mathcal{R}$ be an order relation on E. If any two elements x and y are always comparable, we say that $\mathcal{R}$ is a total order relation and the set E is called totally ordered. Otherwise (i.e., if there exist at least two non-comparable elements x and y), we say that $\mathcal{R}$ is a partial order relation and the set E is called partially ordered.

Example.

1. $\leq$ is a total order on $\mathbb{N}, \mathbb{Z}$, and $\mathbb{R}$.
2. The divisibility relation in $\mathbb{N}^*$ is a partial order.

Definition 3.9 Let $\mathcal{R}$ be an order relation on E, and let M, m be two elements of E.

1. M is an upper bound of a subset A of E if $x\mathcal{R}M$ for every $x \in A$.
2. m is a lower bound of a subset A of E if $m\mathcal{R}x$ for every $x \in A$.

Example.

1. The set $\{8, 10, 12\}$ is bounded above by 120 and bounded below by 2 for the divisibility relation "/" on $\mathbb{N}$.
2. $\mathcal{P}(E)$ is bounded below by $\emptyset$ and bounded above by E for the inclusion relation $\subset$.

3.4 Exercises with Solutions

Exercise 1. In $\mathbb{R}$, the binary relation $\mathcal{R}$ is defined as follows:

$$\forall x, y \in \mathbb{R} : x\mathcal{R}y \Longleftrightarrow x^2 - 1 = y^2 - 1$$

1. Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{R}$.
2. Determine the quotient set $\mathbb{R}/\mathcal{R}$.