CHAPTER 1

Notions of Logic

1.1 Logic

Definition 1.1. An assertion is a statement that is either true or false, but not both at the same time.

Example 1.1.

- ☞ 2+4=6.
- ☞ For all $x \in \mathbb{R}$, $-x^2 \leq 0$.
- ☞ For all $z \in \mathbb{C}$, $|z| = \sqrt{2}$.

Definition 1.2 (A Proposition)**.** A proposition is a mathematical expression denoted by symbols such as $\{P, Q, F, ...\}$ to which we can assign the truth value of true or false.

Example 1.2.

- ✈ Every prime number is even, this proposition is false.
- ✈ 8 is greater than 5, this proposition is true.
- ✈ $\sqrt{3}$ is an irrational number, this proposition is true.

Notation 1.1.

- We denote that the proposition P is true as $\langle 1 \rangle$.
- We denote that the proposition P is false as $\langle 0 \rangle$.

Definition 1.3 (Negation)**.** Let P be a proposition, the negation of P is a proposition denoting the opposite, which we denote as $\overline{P}$. Here is its truth table.

P	$\overline{P}$
1	0
0	1

Now, if P is an assertion and Q is another assertion, we will define new assertions constructed from P and Q.

1.2 Logical Operators

Let P and Q be two propositions.

1.2.1 Conjunction $\langle and \rangle$

Definition 1.4. Conjunction is the logical operator: and denoted $\langle \wedge \rangle$, the proposition (P and Q) or $(P \wedge Q)$ is the conjunction of the two propositions P and Q.

- ☞ The assertion "$P \wedge Q$" is true if both P and Q are true.
- ☞ The assertion "$P \wedge Q$" is false otherwise.

This can be summarized in a truth table:

P	Q	$P \wedge Q$
1	0	0
0	1	0
1	1	1
0	0	0

Table 1.1: Truth table for $P \wedge Q$

Example 1.3. We present two cases:

☞ 6 is an even number and 3 is an odd number, this proposition is true.

☞ $\sqrt{3} < 1$ and $\sqrt{2} \geq 2$, this proposition is false.

1.2.2 Disjunction $\langle or \rangle$

Definition 1.5. Disjunction is the logical operator: or denoted $\langle \vee \rangle$, the proposition $(P$ or $Q)$ i.e., $(P \vee Q)$ is the disjunction of the two propositions P and Q.

☞ The assertion "$P \vee Q$" is false if both P and Q are false.

☞ The assertion "$P \vee Q$" is true otherwise.

This can be summarized in a truth table:

P	Q	$P \vee Q$
1	0	1
0	1	1
1	1	1
0	0	0

Table 1.2: Truth table for $P \vee Q$

Example 1.4. We present two cases:

☞ 2 is an even number or 3 is a prime number, this proposition is true.

☞ $3 \leq 1$ or $2 \geq 4$, this proposition is false.

1.2.3 Implication $\langle \Rightarrow \rangle$

Definition 1.6. The implication of two propositions P and Q is denoted as $P \Rightarrow Q$, we say P implies Q or if P then Q.
Therefore,

☞ $P \Rightarrow Q$ is false if P is true and Q is false,

☞ $P \Rightarrow Q$ is true in all other cases.

This can be summarized in a truth table:

P	Q	$P \Rightarrow Q$
1	0	0
0	1	1
1	1	1
0	0	1

Table 1.3: Truth table for $P \Rightarrow Q$

Example 1.5. We present two cases:

☞ $0 \leq x \leq 25 \Rightarrow \sqrt{x} \leq 5$ this proposition is true.

☞ $\sin(\theta) = 1 \Rightarrow \theta = \frac{\pi}{2}$, this proposition is false. Because, for $\theta = \frac{5\pi}{2}$, $\sin(\theta) = 1$.

1.2.4 Equivalence $\langle\Leftrightarrow\rangle$

Definition 1.7. The equivalence of two propositions P and Q is denoted as $P \Leftrightarrow Q$, we can also write $(P \Rightarrow Q)$ and $(Q \Rightarrow P)$.
We say that:

☞ $P \Rightarrow Q$ is true if P and Q have the same truth value.

☞ $P \Rightarrow Q$ is false for all other cases.

This can be summarized in a truth table:

P	Q	$P \Leftrightarrow Q$
1	0	0
0	1	0
1	1	1
0	0	1

Table 1.4: Truth table for $P \Leftrightarrow Q$

Example 1.6. We present two cases:

☞ $x - 1 = 1 \Leftrightarrow x = 2$, this proposition is true.

☞ $e^x = 1 \Leftrightarrow x = 0$, this proposition is true.

Properties 1.1. Let P, Q, and R be three propositions. We have the following (true) equivalences:

1. $P \Leftrightarrow \overline{\overline{P}}$
2. $(P \wedge Q) \Leftrightarrow (Q \wedge P)$
3. $(P \vee Q) \Leftrightarrow (Q \vee P)$
4. $\overline{(P \wedge Q)} \Leftrightarrow \overline{P} \vee \overline{Q}$ (De Morgan's Laws)
5. $\overline{(P \vee Q)} \Leftrightarrow \overline{P} \wedge \overline{Q}$ (De Morgan's Laws)
6. $(P \wedge (Q \vee R)) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
7. $(P \vee (Q \wedge R)) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$
8. $P \Rightarrow Q \Leftrightarrow \overline{Q} \Rightarrow \overline{P}$

Proof. To prove these properties, we use truth tables.
For example:

1. **Property 5.**

P	Q	$P \vee Q$	$\overline{(P \vee Q)}$	$\overline{P}$	$\overline{Q}$	$\overline{P} \wedge \overline{Q}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	0	0	1	1	1	1
0	1	1	0	1	0	0

Table 1.5: Truth table for Property 5.

2. **Property 8.**

P	Q	$P \Rightarrow Q$	$\overline{P}$	$\overline{Q}$	$\overline{Q} \Rightarrow \overline{P}$
1	1	1	0	0	1
1	0	0	0	1	0
0	0	1	1	1	1
0	1	1	1	0	1

Table 1.6: Truth table for Property 8.

□

1.3 Quantifiers

1.3.1 Universal Quantifier $\forall$

An assertion P may depend on a parameter x, for example, $e^x > 1$, where the assertion $P(x)$ is true or false depending on the value of x.
Thus, the assertion,

$$\forall x \in E, P(x).$$

We read the above assertion as: For all x belonging to E, P(x).

Example 1.7. We can provide several examples:

- $\forall x \in \mathbb{R},\ x^2 \geq 0$. This is a true assertion.
- $\forall x \in \mathbb{R}_+,\ |x| = x$. This is a true assertion.
- $\forall x \in \mathbb{R},\ e^x \geq 1$. This is a false assertion.

1.3.2 Existential Quantifier $\exists$

An assertion P may depend on a parameter x, where the assertion $P(x)$ is true when we can find at least one x_0 from E for which P(x) is true.
We can write:

$$\exists x \in E, P(x).$$

We read the above assertion as: There exists x belonging to E, such that P(x).

Example 1.8. We can provide several examples:

✓ $\exists x \in \mathbb{R},\ P(x) = x^2 - 2x = 0$. This is a true assertion because there exist two values $x_0 = 0$ and $x_1 = 2$ such that $P(x_i) = 0,\ i = 0, 1$.

✓ $\exists n \in \mathbb{N},\ n\,(n+1)$ is even. This is a true assertion.

✓ $\exists x \in \mathbb{R}^*,\ |x| \leq 0$. This is a false assertion.

1.3.3 Negation of Quantifiers

We can summarize this in the following table:

Assertion P	Negation of Assertion P i.e:$\overline{P}$
$\forall x \in E,\ P(x)$	$\exists x \in E,\ \overline{P}(x)$
$\exists x \in E,\ P(x)$	$\forall x \in E, \overline{P}(x)$

1.4 Methods of Mathematical Reasoning

In this section, we mention several methods of reasoning, including:

1.4.1 Direct Reasoning

To show the assertion $(P \Rightarrow Q)$, we assume that P is true and want to prove that Q is then true.

Example 1.9. Show that:

$$\forall n, m \in \mathbb{N},\ n : even,\ and\, m : odd\ \Rightarrow\ n \times m : even.$$

Proof. We have: n is an even natural number, so: $\exists k_1 \in \mathbb{N},\ n = 2k_1$
m is an odd natural number, so: $\exists k_2 \in \mathbb{N},\ n = 2k_2 + 1$
Therefore,
$n \times m = 2k_1 \times (2k_2 + 1) = 2\,(2k_1 + k_2) = 2k_3,\ , k_3 \in \mathbb{N}, / k_3 = 2k1 + k_2$
So, $n \times m$ is an even number. □

1.4.2 Case-by-Case Reasoning

If we want to verify an assertion $P(x)$ for all x in a set E, we can prove the assertion for x in a part A of E, and then for x in a part B of E. Knowing that A and B are a partition of E. This is the method of disjunction of cases or case-by-case reasoning.

$$\forall x \in E,\ P(x)\text{is true} \equiv \begin{cases} P(x), & \text{is true for all } x \in A \\ & \wedge \\ P(x), & \text{is true for all } x \in B \end{cases} \tag{1.1}$$

Example 1.10. Show that:

$$\forall n \in \mathbb{N},\ \frac{n\left(n+1\right)}{2} \in \mathbb{N}$$

Proof. To prove that $\frac{n(n+1)}{2} \in \mathbb{N}$, we need to prove that:

$$\begin{cases} \frac{n(n+1)}{2} \in \mathbb{N}, & \text{if}:\ n\ even, \\ & \wedge \\ \frac{n(n+1)}{2} \in \mathbb{N}, & \text{if}:\ n\ odd. \end{cases} \tag{1.2}$$

1. For, n even, then $\exists k_1 \in \mathbb{N},\ n = 2k_1$. So,

   $$\frac{n\left(n+1\right)}{2} = \frac{2k_1\left(2k_1+1\right)}{2} = k_1\left(2k_1+1\right) \in \mathbb{N}.$$

   Because the product of two natural numbers k_1 and $2k_1+1$ is a natural number. Therefore, the number $\frac{n(n+1)}{2}$ is a natural number for n being an even number.

2. For, n odd, then $\exists k_2 \in \mathbb{N},\ n = 2k_2+1$. So,

   $$\frac{n\left(n+1\right)}{2} = \frac{\left(2k_2+1\right)\left(2k_2+2\right)}{2} = \left(2k_2+1\right)\left(k_2+1\right) \in \mathbb{N}.$$

   Because the product of two natural numbers $(2k_2+1)$ and (k_2+1) is a natural number. Therefore, the number $\frac{n(n+1)}{2}$ is a natural number for n being an odd number.

□

1.4.3 Contrapositive Reasoning

The contrapositive reasoning is based on the following equivalence

$$P \Rightarrow Q \Leftrightarrow \overline{Q} \Rightarrow \overline{P}$$

So, if we want to prove the proposition $(P \Rightarrow Q)$, we actually prove that if $\overline{Q}$ is true then $\overline{P}$ is true.

Example 1.11. Let $n \in \mathbb{N}$. Prove that if n^2 is even, then n is even.

Proof. To prove that n^2 is even $\Rightarrow$ n is even, we use the method of contrapositive reasoning.
Therefore,

$$\overline{n\ is\ even} \Rightarrow \overline{n^2\ is\ even}$$

In other words,

$$n\ is\ odd \Rightarrow n^2\ is\ odd$$

Assume that n is odd. So, $\exists k_1 \in \mathbb{N}$, such that $n = 2k_1$. Then,

$$n^2 = (2k_1)^2 = 4k_1^2 = 2(2k_1^2) = 2k_2,\ \text{where:} k_2 = \left(2k_1^2\right) \in \mathbb{N}$$

And thus n^2 is odd.

We have shown that if n is odd, then n^2 is odd. By the contrapositive proof, this is equivalent to: if n^2 is even, then n is even. □

1.4.4 Proof by Contradiction

To prove that the assertion or proposition R such that $R = P \Rightarrow Q$ is a true proposition.
We assume $\overline{R}$, (i.e:$P \wedge \overline{Q}$) is true and we reach a contradiction, and on the other hand, to prove that the proposition P is true, we easily assume that $\overline{P}$ is true and we reach a contradiction.

Example 1.12. Let $a, b \geq 0$. Prove that if $\frac{a}{1+b} = \frac{b}{1+a}$ then $a = b$.

Proof. By contradiction by assuming that $\frac{a}{1+b} = \frac{b}{1+a}$ and $a \neq b$.
We have, since $\frac{a}{1+b} = \frac{b}{1+a}$, then $a(1+a) = b(1+b)$, so $a + a^2 = b + b^2$.
We obtain

$$a^2 - b^2 = b - a = -(a-b) \Rightarrow (a-b)(a+b) = -(a-b)$$

So,

$$a + b = -1.$$

We reach a contradiction because $a, b \geq 0$.
Finally, we conclude: if $\frac{a}{1+b} = \frac{b}{1+a}$ then $a = b$.

□

1.4.5 Proof by counter-example

The goal of this method is to find the value x for which the proposition $P(x)$ is not always true.

Example 1.13. Prove that: $\forall x \in \mathbb{R}_+,\ f(x) = \ln(1+x) \neq 0$.

Proof. In this example. the proposition is false. Because when we take the counterexample: for $x_0 = 0 \in \mathbb{R}_+$.
We find that $f(x) = 0$. □

1.4.6 Proof by Mathematical Induction

To prove that $P(n) : \forall n \in \mathbb{N}, n \geq n_0, P(n)$ is true, we follow these steps:

1. Show that $P(n_0)$ is true (base case).
2. Assume that $P(n)$ is true for some n (inductive hypothesis).
3. Prove that $P(n+1)$ is true (inductive step). If this is the case, then P is true for all $n \geq n_0$.

Example 1.14. Prove that: $\forall n \in \mathbb{N},\ 2^n > n$.

Proof. 1. For $n_0 = 0$, we have $1 > 0$, so the proposition $P(n)$ is true for the initial value $n_0 = 0$.

2. Suppose that $P(n)$ is true (i.e., $\forall n \in \mathbb{N},\ 2^n > n$).
3. We will show that $P(n+1)$ is true (i.e., $\forall n \in \mathbb{N},\ 2^{n+1} > n+1$. (?))

$$2^{n+1} = 2^n + 2^n > n + 2^n, \text{ because by } P(n) \text{ we know } 2^n > n.$$
$$> n + 1, \text{ because: } 2^n \geq 1.$$

□

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Série d'exercices N°01.
Algèbre 1

Exercice 01 :

Voici les propositions suivantes, sont-elles **vraies** ou **fausses** ?. Justifiez votre réponse.

1. $2 + 5 = 7$
2. Pour tout $n \in \mathbb{N}, n + 1 = 2$.
3. Il existe $n \in \mathbb{N}, n + 3 = 3$.
4. Pour tout $n \in \mathbb{N}, n$ est pair.
5. Pour tout $z \in \mathbb{C}, |z| = 1$.
6. Pour tout $z \in \mathbb{C}, z = \overline{z},$ si : $z \in \mathbb{R}$.
7. Pour tout $z \in \mathbb{C}$ et, $x, y \in \mathbb{R}$, tq : $z = x + iy,\ |z| = \sqrt{x^2 + y^2}$.

Exercice 02 :

Exprimer les assertions suivantes à l'aide des quantificateurs et répondre aux questions :

a) Le produit de deux nombres pairs est-il pair ?

b) Le produit de deux nombres impairs est-il impair ?

c) Le produit d'un nombre pair et d'un nombre impair est-il pair ou impair ?

d) Un nombre entier est pair si et seulement si son carré est pair ? (*).

Exercice 03 :

Soient les propositions suivantes :

1. $\forall \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R},\ \alpha + \beta > 0$
2. $\exists \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R},\ \alpha + \beta > 0$
3. $\forall \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R},\ \alpha^2 + \beta^2 \geq 0$
4. $\forall \alpha \in \mathbb{R}, |\alpha| \geq 0$
5. $\exists \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R},\ \beta^2 > \alpha$ (*).

Ces propositions sont elles vraies ou fausses ? Donner leur négation.

Exercice 04 :

Dans quels cas les propositions suivantes sont elles vraies ?

a) $\overline{P} \vee P$

b) $(P \Rightarrow Q) \wedge \left(\overline{P} \Rightarrow Q\right)$

c) $\overline{P \wedge \left(\overline{Q \wedge R}\right)} \Leftrightarrow Q$

d) $P \wedge (P \vee Q) \Leftrightarrow P$

e) $\overline{P \wedge Q} \Leftrightarrow (\overline{P} \vee \overline{Q})$

f) $((P \vee Q) \Rightarrow R) \Leftrightarrow (P \Rightarrow R) \wedge (Q \Rightarrow R)$ (*).

Exercice 05 :

1. En utilisant le raisonnement par l'absurde démontrer que
   a) $\sqrt{2}$ n'est pas un nombre rationnel.
   b) Si $n \in \mathbb{N}^*$ alors $n^2 + 1$ n'est pas le carré d'un entier naturel.
2. Monter par récurrence
   a) $\forall n \in \mathbb{N}^*,\ 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$.
   b) $\forall n \in \mathbb{N}^*,\ 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$.
   c) $\forall a \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ (1+a)^n \geq 1 + na$.

Exercice 06 : (À domicile)
On considère les propositions suivantes :

☞ $\mathcal{P}(n)$: $4^n - 1$ est divisible par 3.

☞ $\mathcal{Q}(n)$: $4^n + 1$ est divisible par 3.

1. Montrer que les propositions $\mathcal{P}(n)$ et $\mathcal{Q}(n)$ sont héréditaires.
2. Montrer que $\mathcal{P}(n)$ est vraie pour tout $n \in \mathbb{N}$.
3. Que peut-on dire pour $\mathcal{Q}(n)$?

Exercice 07 : (À domicile)
On rappelle que $\sqrt{2}$ est un nombre irrationnel.

1. Démontrer que si a et b sont deux entiers relatifs tels que $a + b\sqrt{2} = 0$, alors $a = b = 0$.
2. En déduire que si m, n, p et q sont des entiers relatifs, alors

$$m + n\sqrt{2} = p + q\sqrt{2} \Leftrightarrow (m = p \text{ et } n = q).$$