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Module : Analysis I
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Chapter N°1 : Field of real and complex numbers

0.1 Representation of the set $\mathbb{R}$

At the foundation of Analysis are the real numbers, and there are different methods to introduce this. So in this section we will recall with some definitions and properties.

Definition 1 *It is admitted that there exists a set called the set of* ***natural numbers*** $\mathbb{N}$ *having the following properties :*

- *there exists a smallest element in* $\mathbb{N}$*, designed by* 0.
- *for all integers natural* n *there exists natural element* $n^* = n+1$ *(*n^* *is called the next of* n*.)*
- *for all integers natural* n, $n^* \neq 0$.
- *for all integers natural* n^*, m^* *if* $n^* = m^*$ *then* $n = m$.
- ***recurrence property :*** *let* P *be a property defined on* $\mathbb{N}$, *if* $P(0)$ *is verified (checked) and* $P(n+1)$ *then* P *is checked for all* n *natural*

We write $\mathbb{N} = \{0, 1, 2, 3, ...\}$. We note by $\mathbb{N}^* = \mathbb{N} - \{0\}$ is the set of natural elements non-zero.

Exercise *:* show that for all n natural $2^n > 0$.

FIGURE 1 – The German mathematician Leopold Kronecker (1831–1916).

Leopold Kronecker (1831–1916) has famously said that the natural numbers were created by god and all the rest of mathematics is human creation. Humans have learned to add natural numbers n, m and also to multiply them. The sum $m + n$ and the product $n \times m$ are again members of the set $\mathbb{N}$.

The difference $n - m$ is only a natural number if n is larger than m. Also, one can not divide two arbitrary natural numbers p, q as the quotient $\frac{p}{q}$ is only in $\mathbb{N}$ if q is divisor of p.

Remark 2 *The set of natural numbers* $\mathbb{N}$ *has an obvious defect, because if* n *and* m *two integers such that* $n > m$ *the algebraic equations*

$$n + x = m, \tag{1}$$

or

$$q.x = p \tag{2}$$

do not have solutions in $\mathbb{N}$.

<u>Idea</u> : we will extend $\mathbb{N}$ towards another set called the set of **whole numbers**. To remedy the situation (that to say the first equation has a solution).
The set of whole numbers, we note by $\mathbb{Z}$. The symbol $\mathbb{Z}$ comes from the German word *Zahl* for number.

Remark 3 *The set of whole numbers, also have a defect because to solve equations of the type* (2) *for every* $p, q \in \mathbb{Z}$, *we have to introduce another kind of numbers.*

If $GCD(p, q) = r$ than it exists a and b such that $p = r.a\ q = r.b$ and the solution is $x = \frac{a}{b}$ and $GCD(a, b) = 1$, x called rational number, and the set of rational numbers. This set is denoted by $\mathbb{Q}$ which comes from the German word (of Latin origin) Quotient for quotient. We have

$$\mathbb{Q} = \{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}^*,\ GCD(a, b) = 1\}$$

In this set, one can add, subtract, multiply, and divide without any restrictions following the well known rules. Furthermore, one can solve linear algebraic equations of the form

$$ax + b = 0 \tag{3}$$

for arbitrary $a,\ b \in \mathbb{Q}$ with $a \neq 0$ uniquely by an $x \in \mathbb{Q}$.

Remark 4

1. *For all* $n \in \mathbb{N}^*$ *the equation* $n.x = 1$, *admits an only solution* $x = \frac{1}{n}$.
2. *For all* $n \in \mathbb{N}^*$ *the equation* $n.x = -1$, *admits an only solution* $x = \frac{-1}{n}$.
3. *Any rational number can be represented by a periodic decimal expansion.*

For example : $\frac{2}{7} = 0.285714285714...$, $\frac{2}{7} = 0.\underline{285714}$ *also* $\frac{11}{7} = 1.5714285...$, $\frac{11}{7} = 1.\underline{5714285}$.

We can easily see that rational numbers have a defect, because the equation $x^2 - 2 = 0$, does not have a solution in $\mathbb{Q}$

Definition 5 *The set of real numbers is the set of x-coordinate points on the line linear* (O, i)

— *The positive real numbers are the x-coordinate points on the right of O.*
— *The negative real numbers are the x-coordinate points on the left of O.*

Remark 6 *A no-rational number is said to be irrational and the set of these numbers is the irrational set denoted by* $\mathbb{R}/\mathbb{Q}$.

For example : $\sqrt{2},\ e, \pi$, are irrational numbers.

0.1.1 The algebraic structure of $\mathbb{R}$

There exist two operations on $\mathbb{R}$, called **addition** and **multiplication**, which assign to every pair $a,\ b$ of elements from $\mathbb{R}$ two new elements $a+b \in \mathbb{R}$ and $ab \in \mathbb{R}$ (we set $ab = a.b$). They are called the **sum** and the product of $a,\ b$. The operations addition and multiplication satisfy the following rules.

1. $(a+b)+c = a+(b+c)$ (Associativity)
2. $a+b = b+a$ (commutativity)
3. There is exactly one element in $\mathbb{R}$, called the zero and denoted by 0, such that
$$a + 0 = a \text{ for all } a \in \mathbb{R}$$
4. For all $a \in \mathbb{R}$ there exists exactly one $b \in \mathbb{R}$ such that $a+b=0$. The element b denoted by $-a$ and we will call it the negative to a.
5. $(ab)c = a(bc)$ (Associativity)
6. $ab = ba$ (commutativity)
7. There is exactly one element in $\mathbb{R} \setminus \{0\}$ called the **one** and denoted by 1, such that $a.1 = a$ for all $a \in \mathbb{R}$.
8. For every $a \in \mathbb{R} \setminus \{0\}$ there is exactly one element $b \in \mathbb{R}$, such that $ab = 1$. We denote b bya^{-1} or $\dfrac{1}{a}$ and we say a^{-1} is the inverse element to a.
9. $a(b+c) = ab + ac$. (Distributivity)

Notation. We set
$$a - b = a + (-b) \text{ and } \frac{a}{b} = ab^{-1} = b^{-1}a$$
and call $a-b$ the difference of a and b, and $\dfrac{a}{b}$ the quotient of a and b. The operation $a, b \longmapsto a-b$ respectively $\dfrac{a}{b}$ subtraction and division.

Proposition 7 *For all $a,\ b$ in $\mathbb{R}$*
- $0.a = a.0 = 0$.
- $a.b = 0$ *if and only if* $a = 0$ *or* $b = 0$.
- $(-1).a = -a$

0.1.2 Order relation in $\mathbb{R}$

Between the elements of $\mathbb{R}$ there is a relation $\leq$, that is, for elements $x, y \in \mathbb{R}$. one can determine whether $x \leq y$ or not. Here the following conditions must hold :

1. For all $x \in \mathbb{R}$: $x \leq x$ ($\leq$ reflexive). In other words, every element relates to itself.
2. For all $x,\ y \in \mathbb{R} : x \leq y$ and $y \leq x$ implies $x = y$ ($\leq$ antisymmetric)
3. For all $x,\ y,$ and z if $x \leq y$ and $y \leq z$ implies that $x \leq z$ ($\leq$ transitive)
4. For all $x,\ y \in \mathbb{R} : x \leq y$ or $y \leq x$

The relation $\leq$ on $\mathbb{R}$ is called **inequality**.

A set on which there is a relation between pairs of elements satisfying axioms 1, 2, and 3, as you know, is said to be **partially ordered.** If in addition axiom 4 holds, that is, any two elements are **comparable**, the set is linearly ordered. Thus the set of real numbers is linearly ordered by the relation of inequality between elements.

Remark 8 *Let $x,\ y \in \mathbb{R}$*
- ⋄ *If $x \geq y$ we can write $y \leq x$*
- ⋄ *If $x \geq y$ and $x \neq y$we can write $y < x$*
- ⋄ *If $x \geq y$ and $x \neq y$we can write $y > x$*

0.2 Existence of the Least Upper (or Greatest Lower) Bound of a Set of Numbers

Definition 9 *Let $X \subset \mathbb{R}$ be nonempty*

— *is said to be **bounded frome above, upper bound or Majorant** if there exists a number $M \in \mathbb{R}$ such that for all $x \in X$, $x \leq M$.*

1. *M is the **supermum** of X exists, it is an upper bound of X and,*
2. *if M' another upper bound of X, than $M \leq M'$ (that's to say M is the **smallest upper bound** of X). If it is exists the supermum of X is denoted by **sup**(X)*

— *is said to be **bounded frome below, lower bound or minorant** if there exists a number $m \in \mathbb{R}$ such that for all $x \in X$, $x \geq m$.*

1. *m is the **infimum** of X exists, it is an lower bound of X and,*
2. *if m' another lower bound of X, than $m' \leq m$ (that's to say m is the **greatest lower bound** of X). If it is exists the infimum of X is denoted by **inf**(X)*

Remark 10 — *If* $\inf(X) \in X$, *we call it **minimum** of* X, *and we denoted it by* $\min(X)$

— *If* $\sup(X) \in X$, *we call it **maximum** of* X, *and we denoted it by* $\max(X)$

For example, the set $X = \{x \in \mathbb{R} | 0 \leq x < 1\}$ has a minimal element. But, as one can easily verify, it has no maximal element.

0.2.1 Characterization of supermum and infimum

Theorem 11 1. *If a set $X \subset \mathbb{R}$ has supremum* $\sup(X) = M$ *if and only if*

— *M is an upper bound of X*

— *for all $\epsilon > 0$there exists $a \in X$ such that* $\sup(A) - \epsilon < a \leq \sup(A)$.

2. *If a set $X \subset \mathbb{R}$ has infimum* $\inf(X) = m$ *if and only if*

— *m is an lower bound of X*

— *for all $\epsilon > 0$there exists $a \in X$ such that* $\inf(X) \leq a < \inf(X) + \epsilon$.

Exercise 1 : Determine the supermum and infimum of the parts A, and$B$ in $\mathbb{R}$

$$A = \{\sin \frac{2n\pi}{7},\ n \in \mathbb{Z}\}, \text{ and } B = \{\cos \frac{2n\pi}{7},\ n \in \mathbb{Z}\}$$

Proposition 12 *The supermum or infimum if there exists is unique*

— *If X no bounded from above, we write for convention* $\sup(X) = +\infty$

— *If X no bounded from below, we write for convention* $\inf(X) = -\infty$

Proposition 13 *Let A; and B two no-empty parts bounded in $\mathbb{R}$*

1. *If $A \subset B$ then* $\sup A \leq \sup B$, *and* $\inf A \geq \inf B$.
2. $\sup(A \cup B) = \max\{\sup A, \sup B\}$.
3. $\inf(A \cup B) = \min\{\inf A, \inf B\}$.

Proof : See the course or tutorials.

Exercise 2 : Find the supremum, infimum, maximum and minimum of the following sets of numbers, whenever they exist, justifying your conclusions :

$$A =]1, 5], \quad B =]-\infty, 2], \quad C = [-1, 0[\cup [3, 4]$$

Theorem 14 *The set of natural numbers is not bounded from above (majorant). The sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}/\mathbb{Q}$ and $\mathbb{R}$ are neither lower nor upper bound.*

0.3 Integer part

Theorem 15 *For all $x \in \mathbb{R}$ there exists a unique $n \in \mathbb{Z}$, such that $n \leq x < n+1$.*

Definition 16 *Let $x \in \mathbb{R}$, the relative integers n verify $n \leq x < n+1$, is said the **integer part** of x, denoted by $[x]$. Also*

$$[x] \leq x < [x] + 1$$

For example : $[1.5] = 1, \quad [0.95] = 0, \quad [-1.75] = -2, \quad ...$

Exercise 3 : Find the integer part of this number $S = \dfrac{1}{\dfrac{1}{1980} + \dfrac{1}{1981} + ... + \dfrac{1}{2001}}$

Exercise 4 : Solve in $\mathbb{R}$ the following equations :

$$[\frac{x-1}{2}] = -2, \quad [x] = x - 1, \quad [x] + |x-1| = x, \quad [\log_2 x] = [\log_3 x].$$

0.4 Intervals and absolute value

We now introduce the following notation and terminology for the number sets listed below :

$$]a, b[= \{x \in \mathbb{R} : a < x < b\} \quad \text{is the open interval}$$

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \quad \text{is the closed interval}$$

$$]a, b] = \{x \in \mathbb{R} : a < x \leq b\} \quad \text{is the half open interval, containing b}$$

$$[a, b[= \{x \in \mathbb{R} : a \leq x < b\} \quad \text{is the half open interval, containing a}$$

Definition 17 *Open, closed, and half-open intervals are called numerical intervals or simply intervals. The numbers determining an interval are called its endpoints.*

The quantity $b - a$ is called the length of the interval ab. If I is an interval, we shall denote its length by $|I|$.

The sets

$$]a, +\infty[= \{x \in \mathbb{R} : x > a\}, \quad [a, +\infty[= \{x \in \mathbb{R} : x \geq a\}$$

$$]\infty, b[= \{x \in \mathbb{R} : x < b\}, \quad]\infty, b] = \{x \in \mathbb{R} : x \leq b\},$$

and $]-\infty, +\infty = \mathbb{R}[$ are conventionally called unbounded intervals or infinite intervals.

Definition 18 *An open interval containing the point $x \in \mathbb{R}$ will be called a neighborhood of this point.*

So as not to have to investigate which of the points is "left" and which is "right", that is, whether $x < y$ or $y < x$ and whether the length is $y - x$ or $x - y$, we can use the useful function

$$|x| = \sup\{x, -x\}$$

or on other words

$$|x| = \begin{cases} x & \text{when } x > 0 \\ 0 & x = 0 \\ -x & \text{when } x < 0. \end{cases}$$

which is called the **modulus** or **absolute value** of the number.

Definition 19 *The distance between $x, y \in \mathbb{R}$ is the quantity $|x - y|$.*

The distance is non-negative and equals zero only when the points x and y are the same. The distance from x to y is the same as the distance from y to x, since $|x-y| = |y-x|$. Finally, if $z \in \mathbb{R}$, then $|x-y| < |x-z| + |z-y|$. That is, the so-called **triangle inequality** holds.
The triangle inequality follows from a property of the absolute value that is also called the triangle inequality (since it can be obtained from the preceding triangle inequality by setting $z = 0$ and replacing y by $-y$). To be specific, the inequality

$$|x+y| \leq |x| + |y|$$

holds for any numbers x and y, and equality holds only when the numbers x and y are both negative or both positive.
Proof : If $x \geq 0$ and $y \geq 0$, then $x+y \geq 0$, then $|x+y| = x+y$ and $|x| = x$ and $|y| = y$, so that equality holds in this case.
If $x \leq 0$ and $y \leq 0$, then $x+y \leq 0$, then $|x+y| = -(x+y) = -x-y = -x+(-y)$ and $|x| = -x$ and $|y| = -y$, and again we have equality.
Now suppose one of the numbers is negative and the other positive, for example, $x < 0 < y$. Then either $x < x+y \leq 0$ or $0 \leq x+y < y$. In the first case $|x-y| < |x|$, and in the second case $|x-y| < |y|$, so that in both cases $|x+y| \leq |x| + |y|$.

Proposition 20 *For all $x,\ y \in \mathbb{R}$ we have :*

1. $|x| \geq 0, \quad |x| = -|x|$.
2. $|x| = 0$ *if and only if* $x = 0$.
3. $|x| = x$ *if and only if* $x \geq 0$.
4. $|xy| = |x||y|$
5. $|\frac{x}{y}| = \frac{|x|}{|y|}, \quad y \neq 0.$
6. *Let* $\alpha > 0$. *if* $|x| \leq \alpha$ *if and only if* $-\alpha \leq x \leq \alpha$.

Exercise :5 Prove that for all real numbers $a; b$

1. Prove that for all real numbers $a; b$

$$|x| - |y| \leq |x-y| \leq |x| + |y|$$

2. Prove that for all real numbers $a_1; a_2; ...; a_n$

$$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$$

0.5 Complex numbers $\mathbb{C}$

Problem : The equation $x^2 = -4$, for example does not solution in $\mathbb{R}$, that's to say the field of real numbers has a obvious flaw. that's reason to extend the set of real number a an other set, where the previous equation does a solution.

Idea : We will imagine that there exists a non-real number i such that $i^2 = -1$ then

$$x^2 = i^2 4 \Leftrightarrow x = 2i \notin \mathbb{R} \text{ and } x = 2i \notin \mathbb{R} \text{ which admits two different roots.}$$

0.5.1 Construction the set of complex number

We consider the set $\mathbb{C}$ defined by

$$\mathbb{C} = \{(a,b) : a,\ b \in \mathbb{R}\},$$

provided by two laws : for all $(a,b),\ (c,d) \in \mathbb{C}$

$$(a,b) + (c,d) = (a+c, b+d), \quad (a,b).(c,d) = (ac-bd, ad+bc).$$

So $(\mathbb{C}, +, .)$ its commutative filed, because

1. $+$ is commutative, associative, $(0,0)$ addition identity and $(-a,-b)$ the inverse element to (a,b), and
2. . is commutative, associative and the multiplication distributive over the addition. (1.0) is multiplication identity. the inverse element of (a,b) is $(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$

For $(a,b) \in \mathbb{C}$: $(a,b) = (a,0) + (0,1)(b,0)$, we accept that there is an identification between $(a,0) \in \mathbb{C} \equiv a \in \mathbb{R}$. And we put $i = (0,1)$ so $i^2 = (0,1).(0,1) = (-1,0)$, by identification we have : $(-1,0) \equiv -1$. Therefore

$$(a,b) = (a,0) + (0,1)(b,0) = a + ib \in \mathbb{C}$$

Definition 21 *If a, b are real and $z = a + ib$, then the complex number, $\overline{z} = a - ib$ is called the conjugate of z. a andb are the real part and imaginary part of z, respectively.*

we shall occasionally write $a = Re(z)$; $b = Im(z)$

Theorem 22 *If z, and w are complex, then*

1. $\overline{z+w} = \overline{z} + \overline{w}$
2. $\overline{z.w} = \overline{z}.\overline{w}$
3. $z + \overline{z} = 2Re(z)$; $z - \overline{z} = 2iIm(z)$
4. *$z.\overline{z}$is real and positive.*

Result :
- z is real $\Leftrightarrow$ $z = \overline{z}$
- z is imaginary $\Leftrightarrow$ $z = -\overline{z}$

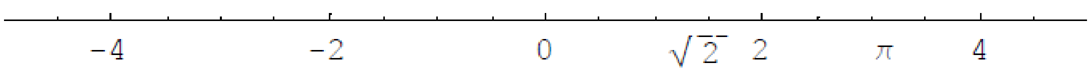
Definition 23 *If z is complex number its absolute value $|z|$ is non-negative square root of $z\overline{z}$; that is $|z| = \sqrt{z\overline{z}} = \sqrt{a^2+b^2}$*

Property : If z, and w are complex, then
- $z = 0 \Leftrightarrow |z| = 0$
- $z\overline{z} = |z|^2$
- $|z.w| = |z|.|w|$
- $|Re(z)| \leq |z|$; $|Im(z)| \leq |z|$

Exercise 6 :

1. Write under algebraic form the following complex numbers $z_1 = \frac{1}{2+3i}$, $z_2 = \frac{1}{3-4i}$
2. Give the conjugate of $\frac{4-5i}{3+i}$.
3. Determine the location points $M(x,y)$ such that : $\frac{iz-1}{z-i}$ is real.

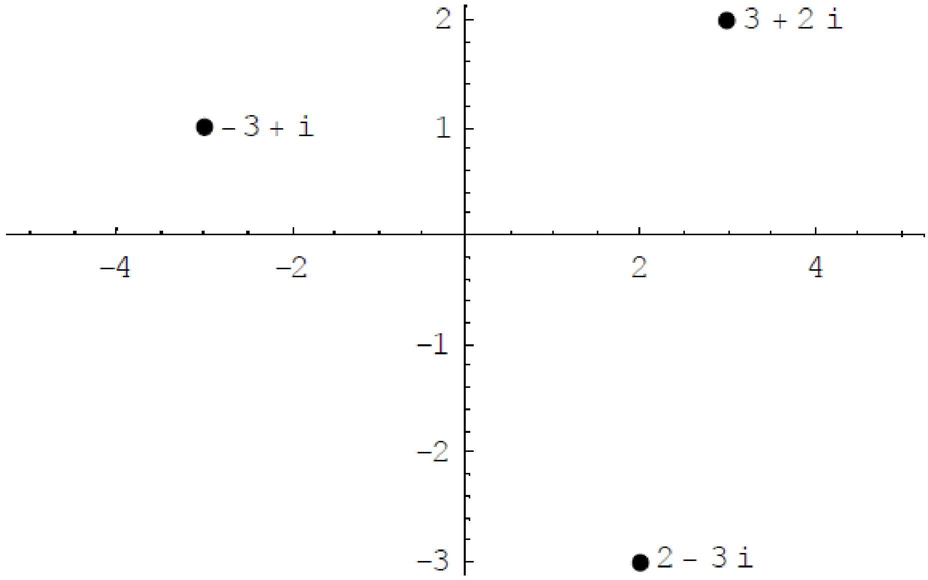
The real numbers are often represented on the real line which increase as we move from left to



right.

The complex numbers, having two components, their real and imaginary parts, can be represented as a plane ; indeed, $\mathbb{C}$ is sometimes referred to as the complex plane, but more commonly, when we represent $\mathbb{C}$ in this manner, we call it an Argand diagram (After the Swiss mathematician Jean-Robert Argand (1768-1822)). The point (a,b) represents the complex number $a + bi$

so that the x-axis contains all the real numbers, and so is termed the real axis, and the y-axis contains all those complex numbers which are purely imaginary (i.e. have no real part), and so is



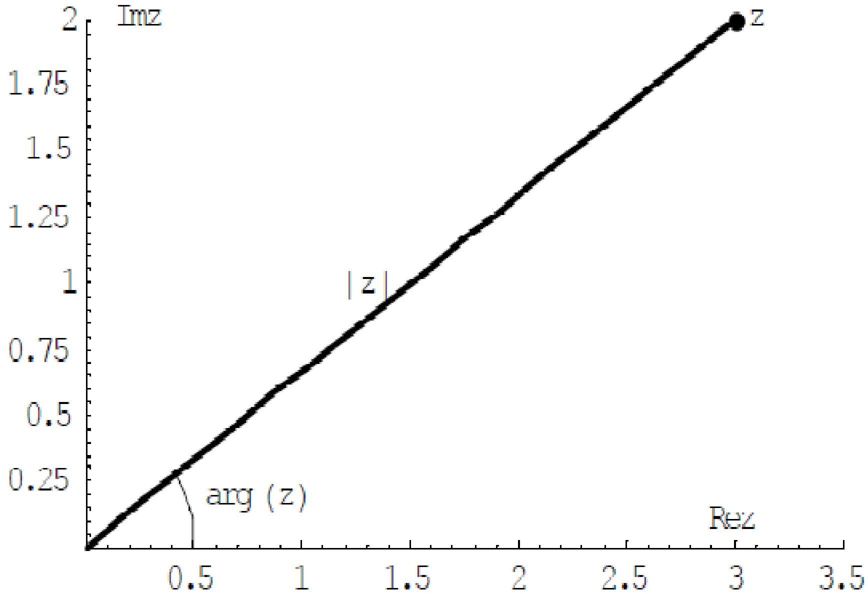
referred to as the imaginary axis.

Definition 24 *we define θ to be the angle that the line connecting the origin to $z = x+iy$, $z \neq 0$ makes with the positive real axis, .The number θ is called* ***the argument*** *of z and is written* $\arg z = \theta$.

We can write $z = r(\cos\theta + \sin\theta)$. (polar or trigonometric form) The relations between z's Cartesian and polar co-ordinates are simple, we see that

$$x = r\cos\theta, \text{ and } \quad y = \sin\theta, \quad r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}.$$

Note that $\arg z$ is defined only up to multiples of 2π. For example, the argument of $1+i$ could be $\frac{\pi}{4}$ or $\frac{9\pi}{4}$ or $\frac{-7\pi}{4}$ etc... For simplicity, in this article we shall give all arguments in the range $0 \leq \theta < 2\pi$, so that $\frac{\pi}{4}$ would be the preferred choice here.



Note that the argument of 0 *is undefined.*

Exercise :7 Find the modulus and argument (polar form) of each of the following numbers.

$$(1+\sqrt{3}i), \quad (2+i)(3-i), \quad (1+i)^5$$

Exercise :8 Let α be a real number in the range $0 < \alpha < \frac{\pi}{2}$. Find the modulus and argument of the following numbers.

$$\cos\alpha - i\sin\alpha, \quad \sin\alpha - i\cos\alpha, \quad 1 + i\tan\alpha, \quad 1 + \cos\alpha + i\sin\alpha.$$

Exercise :9 sketch the following sets :

$$|z| < 1, \quad Re(z) = 3, \quad |z-1| = |z+i|, \quad \arg(z-i) = \frac{\pi}{2}$$

Proposition 25 *Let z, and $w \in \mathbb{C}$. Then to multiples of 2π the following equations hold*

- $\arg(zw) = \arg(z) + \arg(w)$ *if* $z,\ w \neq 0$
- $\arg(\frac{z}{w}) = \arg(z) - \arg(w)$ *if* $z,\ w \neq 0$
- $\arg(\bar{z}) = -\arg(z)$ *if* $z \neq 0$.

Theorem 26 *(MOIVRE'S THEOREM) For a real number θ and integer n we have that*

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha$$

Exercise9 : Use De Moivre's Theorem to show that

$$\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

and that

$$\sin(5\theta) = (16\cos^4\theta - 12\cos^2\theta + 1)\sin\theta$$

Definition 27 *(Euler's formula) For any real number θ, we denote $e^{i\theta}$ the complex number $\cos\alpha + i\sin\alpha$ is the modulus 1 and θ argument.*

It follows from Euler's formula that, for any complex number z written in exponential form, $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

Theorem 28 *For any $\theta_1,\ \theta_2 \in \mathbb{R}$, we have*

- $e^{i\theta_1}.e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$
- $\dfrac{e^{i\theta_1}}{e^{i\theta_2}} = e^{i(\theta_1-\theta_2)}$
- $(e^{i\theta})^n = e^{in\theta}$

Example :

- Determine the exponential form for $z = \dfrac{(1+i)^4}{\sqrt{3}-i}$.
- Calculate $(1+i)^{14}$.

0.5.2 The square root of a complex number

Let $w = \alpha + i\beta$, we want to find the square root of w, i.e find $z = x + iy$, such that $z^2 = w$, hence

$$\begin{cases} x^2 - y^2 = \alpha .....(1) \\ 2xy = \beta ....(2) \\ x^2 + y^2 = |w| ......(3) \end{cases}$$

The equations (1) and (3) allowed to find x and y and equation (2) allows us to remove the ambiguity on the signs.

For example $w = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$$\begin{cases} x^2 - y^2 = \frac{\sqrt{3}}{2} \text{.....(1)} \\ 2xy = \frac{1}{2} \text{.......(2)} \\ x^2 + y^2 = 1 \text{......(3)} \end{cases}$$

(1)+(3) : $x = \pm\sqrt{\frac{\sqrt{3}+2}{4}} = \pm\frac{\sqrt{2+\sqrt{3}}}{2}$

(3)-(1) : $y = \pm\frac{\sqrt{2-\sqrt{3}}}{2}$

From (2) : $xy > 0$, so x and y have the same sign. Therefore the roots

$$\frac{\sqrt{2+\sqrt{3}}}{2} + i\frac{\sqrt{2-\sqrt{3}}}{2}, \quad -\frac{\sqrt{2+\sqrt{3}}}{2} - i\frac{\sqrt{2-\sqrt{3}}}{2}$$

0.5.3 Example :Quadratic equation

We deal this type of equations with a simple numerical example, we consider

$$(1+i)z^2 - (5+i)z + 6 + 4i = 0.$$

Then $\Delta = 16 - 30i$, so the square roots of Δ is $5+3i$ and $-5+3i$. Hence the solutions is

$$z_1 = 2 - 3i, \quad z_2 = 1 + i$$

Exercise 10 : Find the square roots of $-5-12i$, and hence solve the quadratic equation

$$z^2 - (4+i)z + (5+5i) = 0.$$

Exercise 11 : Show that the complex number $1+i$ is a root of the cubic equation

$$z^3 + z2 + (5-7i)z - (10+2i) = 0,$$

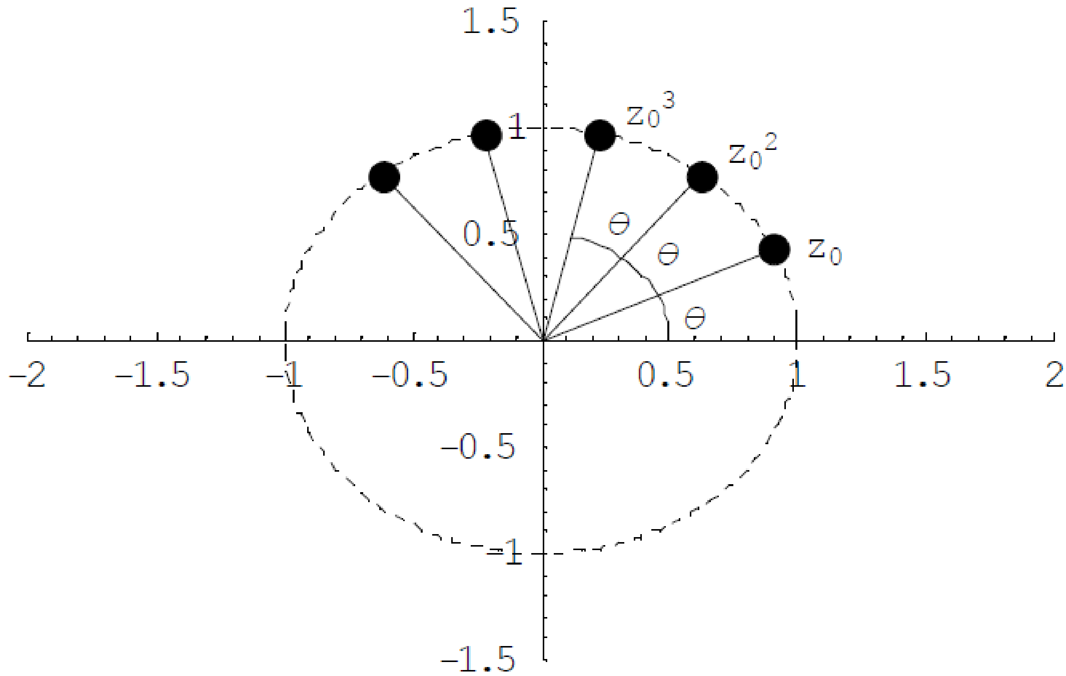
and hence find the other two roots.

0.5.4 Roots of Unity

Consider the complex number

$$z_0 = \cos\theta + i\sin\theta.$$

where $0 \leq \theta < 2\pi$. The modulus of z_0 is 1, and the argument of z_0 is θ.



Problem. Let n be a natural number. Find all those complex z such that $z^n = 1$.
We know from the Fundamental Theorem of Algebra that there are (counting repetitions) n solutions : these are known as the nth roots of unity.

Let's first solve $z^n = 1$ directly for $n = 2, 3, 4$.

— ◦ When $n = 2$ we have $0 = z^2 - 1 = (z - 1)(z + 1)$, and so the square roots of 1are ± 1.

— ◦ When $n = 3$ we can factorise as follows $0 = z^3 - 1 = (z - 1)(z^2 + z + 1)$. So the cube roots of 1 are 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2}$, $-\frac{1}{2} - \frac{\sqrt{3}}{2}$

— ◦ When $n = 4$ we can factorise as follows $0 = z^4 - 1 = (z^2 - 1)(z^2 + 1) = (z - 1)(z + 1)(z - i)(z + i)$, so that the fourth roots of 1 are -1, i and $-i$.