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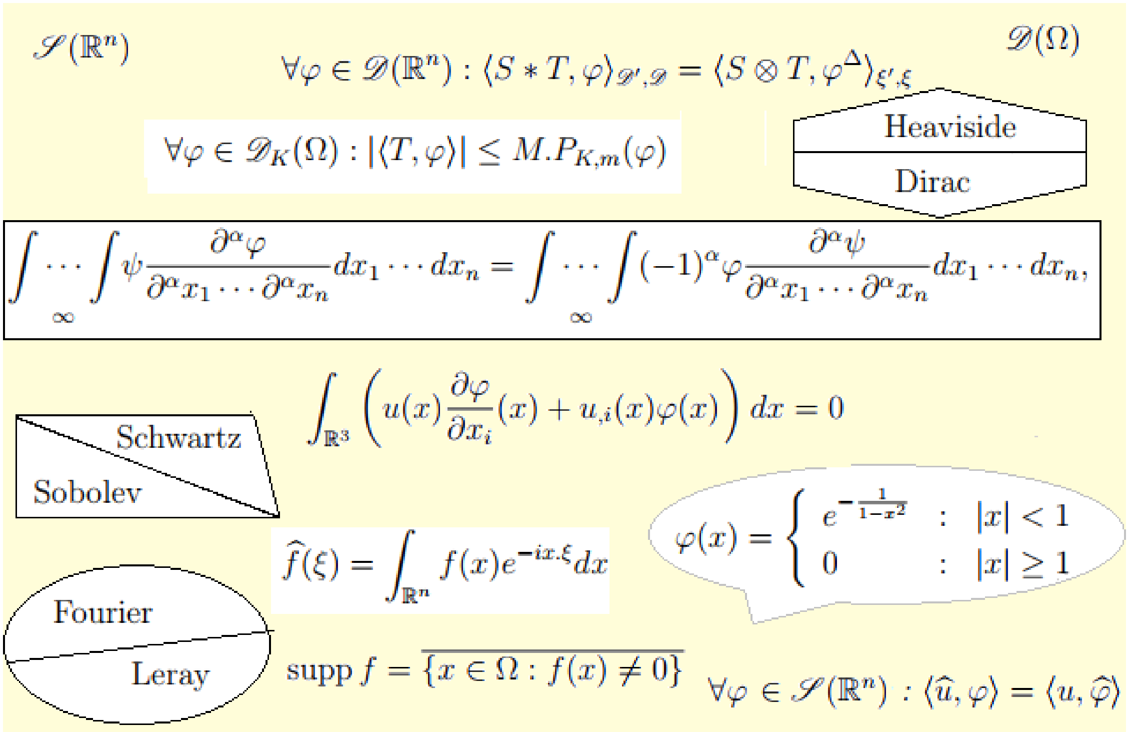

Written by

SAADI ABDERACHID

Intended for first-year Master's students

* Partial Differential Equations and applications.
* Functional analysis.
* Mathematical and Numerical Analysis.

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NOTATIONS

.

$x = (x_1, x_2, \cdots, x_n)$: Element of $\mathbb{R}^n$ $(n \in \mathbb{N}^*)$.

$|x| = \sqrt{x_1^2 + \cdots + x_n^2}$: Norm of $x \in \mathbb{R}^n$.

Ω : Non-empty open set of $\mathbb{R}^n$.

$\Gamma = \partial\Omega$: Boundary of Ω.

$\overline{\Omega}$: Closure of Ω.

$|u|_E$: Norm of a vector u in the normed vector space E.

$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$: Multi-index.

$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$: Length of $\alpha \in \mathbb{N}^n$.

$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, $\alpha! = \alpha_1! \cdots \alpha_n!$: Multi-index notation and factorial.

$\alpha, \beta \in \mathbb{N}^n$: $\alpha \leq \beta$ if and only if for all $i \in \{1, \cdots, n\}$: $\alpha_i \leq \beta_i$.

$\alpha, \beta \in \mathbb{N}^n$ such that $\alpha \leq \beta$: $C_\alpha^\beta = \dfrac{\alpha!}{\beta!(\alpha! - \beta!)}$.

$x, y \in \mathbb{R}^n$: $(x+y)^\alpha = \displaystyle\sum_{\beta \leq \alpha} C_\alpha^\beta x^{\alpha-\beta} y^\beta$.

$\alpha \in \mathbb{N}^n, f : \Omega \longrightarrow \mathbb{R}$ $|\alpha|$-differentiable: $D^\alpha f = \partial^\alpha f = \partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n f$.

$\alpha \in \mathbb{N}^n, f, g : \Omega \longrightarrow \mathbb{R}$ $|\alpha|$-differentiable: $D^\alpha(f.g) = \displaystyle\sum_{\beta \leq \alpha} C_\alpha^\beta D^{\alpha-\beta} f D^\beta g$.

$a.e$: Almost everywhere.

E': Dual of a vector space E.

$\langle , \rangle$: Dual pairing.

$\rightharpoonup$: Weak convergence.

$\hookrightarrow$: Continuous injection.

$\operatorname{supp} f$: Support of a function f.

$\check{f}$: The symmetry of the function f.

τ_a: Translation operator with vector a.

$*$: Convolution product.

$\otimes$: Tensor product.

$\mathcal{F}(f) = \widehat{f}$: Fourier transform of the function f.

$\overline{\mathcal{F}}(f)$: Conjugate Fourier transform of the function f.

INTRODUCTION

The theory of distributions, as well as Sobolev spaces, are powerful mathematical tools for studying functions and solving partial differential equations in cases where classical methods of differentiation and integration do not apply or where there is difficulty in applying them. They play an essential role in many areas of mathematical physics, from wave theory to quantum mechanics, through numerical analysis.

Distribution theory generalizes the concept of a function by allowing the consideration of mathematical objects that are more general than continuous or differentiable functions. Distributions can include impulses, step functions, discontinuous functions, and other mathematical objects. Distributions are defined using continuous linear operators that associate a test function with a distribution.

Sobolev spaces are function spaces that allow quantifying the regularity of functions, especially those that are not necessarily continuous or differentiable in the classical sense. They are defined by introducing norms that take into account the derivatives of the function. More precisely, Sobolev spaces, denoted as $W^{k,p}$, include functions whose first k derivatives in the distributional sense are in the space L^p.

In 1893 and 1894, O. Heaviside proposed symbolic calculus rules for operators used to solve problems in mathematical physics. These symbolic calculations worked well for engineers who used them in a broad sense but were not always mathematically rigorous.

In this context, P. Dirac published an article in 1926 titled «**L'interprétation physique de la dynamique quantique**», where he introduced his famous symbol denoted as δ. Dirac stated that δ is a function defined as follows:

$$\begin{cases} \delta(x) = 0 \text{ if } x \neq 0, \\ \displaystyle\int_{-\infty}^{+\infty} \delta(x)dx = 1. \end{cases} \tag{1}$$

Furthermore, for any smooth function φ and any real number a, we can write:

$$\int_{-\infty}^{+\infty} \varphi(x)\delta(a-x)dx = \varphi(a). \tag{2}$$

P. Dirac acknowledged that what he called a "function" is not in the strict sense of a function. Indeed, when δ is treated as a function, it is equal to 0 almost everywhere, leading to $\int_{-\infty}^{+\infty} \delta(x)dx = 0$, which contradicts $\int_{-\infty}^{+\infty} \delta(x)dx = 1$.

To resolve this dilemma, P. Dirac suggested that the quantity δ could be interpreted as a limit of a sequence of functions. This seems more reasonable. For example, the sequence of functions $\{f_j\}_{j=1}^{+\infty}$ which defined as:

$$f_j(x) = \begin{cases} 0 & : \ |x| > j, \\ \dfrac{j}{2} & : \ |x| \leq j. \end{cases} \tag{3}$$

verify the condition $\int_{-\infty}^{+\infty} f_j(x)dx = 1$. Following L. Schwartz (1945), we choose a continuous function φ that is zero outside a finite interval $]-a, a[$ $(a > 0)$ and contains $\left[-\frac{1}{j}, \frac{1}{j}\right]$ for sufficiently large j. We have:

$$\int_{-\infty}^{+\infty} f_j(x)\varphi(x)dx = \frac{j}{2}\int_{-\frac{1}{j}}^{\frac{1}{j}} \varphi(x)dx.$$

Let ψ be an antiderivative of φ on $]-a, a[$. Then:

$$\int_{-\infty}^{+\infty} f_j(x)\varphi(x)dx = \frac{j}{2}\left[\psi\left(\frac{1}{j}\right) - \psi\left(-\frac{1}{j}\right)\right] = \frac{\psi\left(\frac{1}{j}\right) - \psi\left(-\frac{1}{j}\right)}{\frac{2}{j}}.$$

Let $h = \frac{1}{j}$, and we obtain:

$$\lim_{j\to+\infty}\int_{-\infty}^{+\infty} f_j(x)\varphi(x)dx = \lim_{h\to 0}\frac{\psi(h) - \psi(-h)}{2h} = \psi'(0) = \varphi(0) = \int_{-\infty}^{+\infty} \varphi d\delta,$$

where δ is the Dirac measure, defined in (2.1).

This means that the density measure of $\{f_j\}_{j=1}^{+\infty}$ converges to the Dirac measure, and through translation, we can establish equation (2).

P. Dirac also defined the successive derivatives of δ, denoted as $\delta', \delta'', \ldots$ L. Schwartz justified how to find these successive derivatives but in a more general framework than measures, which is what we call distributions. He published this in 1946 in an article titled «**Généralisation de la notion de fonction, de dérivation, de transformation de Fourier et applications mathématiques et physiques**». In this article, he provided the following two definitions:

Définition 1: Φ sera l'ensemble des fonctions $\varphi(x_1, \cdots, x_n)$ de n variables réelles, indéfiniment dérivables et nulles en dehors d'ensemble bornés. A chaque fonction φ correspond un «noyau», ensemble compact, dont le complémentaire est le plus grand ensemble ouvert sur lequel $\varphi \equiv 0$.

> **Définition 2:** On appellera «distribution» de l'espace à n dimensions toute fonctionnelle ou forme linéaire $T(\varphi)$ définie pour toute les φ de Φ, et vérifiant de plus la condition de continuité suivante: Si une suite des fonctions φ_i, ont leurs noyaux contenus dans un compact fixe et si elles convergent uniformément vers 0, ainsi que chacune de leurs dérivées, alors les $T(\varphi_i)$ convergent vers 0.

Later on, L. Schwartz published his famous book «**Théorie des distributions**» (Theory of Distributions) in the mid-1960s of the last century.

In a different context, among the classical results of the calculus of variations, we find the Dirichlet principle: the variational integral $\int_\Omega |\nabla u|^2 dx$ has a minimum for certain functions belonging to the class $\mathscr{C}^1(\Omega)$ where Ω is a bounded domain $\mathbb{R}^n$. This principle was used by B. Riemann without mathematically satisfactory justification, but in 1870, K. Weierstrass noted that the existence of minimizing functions for variational integrals is not always guaranteed.

A first rigorous proof of the Dirichlet principle was introduced in 1900 by D. Hilbert for functions $u \in \mathscr{C}(\overline{\Omega}) \cap \mathscr{C}^1(\Omega)$, taking a trace g on $\partial\Omega$. This marked the first steps in the development of Sobolev spaces. It is worth noting that later on, the Dirichlet principle became related to boundary value problems for the Poisson equation:

$$\begin{cases} -\Delta u = f & : \ \text{in } \Omega, \\ u = g & : \ \text{on } \partial\Omega. \end{cases} \tag{4}$$

If u is a solution of the problem (4), then u minimizes the Dirichlet energy:

$$E(v) = \int_\Omega \left(\frac{1}{2} |\nabla v(x)|^2 - f(x)v(x) \right) dx.$$

There have been further developments on the Dirichlet principle, including the work of Bippo Levy, G. Fubini, L. Tonelli, O. M. Nicodym, K. O. Friedrichs, and others.

In 1934, J. Leray, in his article titled «**Sur le mouvement d'un liquide visqueux emplissant l'espace**», introduced a new term, the «quasi-dérivée» (quasi-derivative):

> Soit deux fonctions de carré sommable dans $\mathbb{R}^3, u$ et $u_{,i}$. Nous dirons que $u_{,i}$ est la quasi-dérivée de u par rapport à x_i quand la relation
>
> $$\int_{\mathbb{R}^3} \left(u(x) \frac{\partial \varphi}{\partial x_i}(x) + u_{,i}(x)\varphi(x) \right) dx = 0,$$
>
> sera vérifiée; rappelons que dans cette relation φ représente une quelconque des fonctions admettant des dérivées premières continues qui sont, comme ces fonctions elles-mêmes, des carrés sommables sur $\mathbb{R}^3$.

In 1935, S. L. Sobolev introduced a theory of general solutions to the wave equation, defined as L^1-limits of $\mathscr{C}^2$ solutions of this equation. He introduced the concept of continuous linear functionals on spaces of continuously differentiable functions (later called «distributions of finite order») and announced an existence theorem for a solution to a large class of hyperbolic equations.

In 1938, S. L. Sobolev provided a clear definition of weak derivatives and introduced the spaces known as Sobolev spaces:

> Appelons espaces L_p^ν l'espace fonctionnel linéaire qui est formé de toute les fonctions de n variables réelles $\varphi(x_1, \cdots, x_n)$ dont les dérivées partielles jusqu'à l'ordre ν existent et sont sommables à la puissance $p > 1$ dans chaque partie bornée de l'espace $x_1, \cdots, x_n$. La dérivée $\dfrac{\partial^\alpha \varphi}{\partial^\alpha x_1 \cdots \partial^\alpha x_n}$ est définie comme une fonction qui satisfait l'équation
>
> $$\int \underset{\infty}{\cdots} \int \psi \frac{\partial^\alpha \varphi}{\partial^\alpha x_1 \cdots \partial^\alpha x_n} dx_1 \cdots dx_n = \int \underset{\infty}{\cdots} \int (-1)^\alpha \varphi \frac{\partial^\alpha \psi}{\partial^\alpha x_1 \cdots \partial^\alpha x_n} dx_1 \cdots dx_n,$$
>
> quelque soit la fonction ψ continue ayant des dérivées jusqu'à l'ordre ν s'annule en dehors d'un domaine borné D.

Later, S. L. Sobolev replaced the notation L_p^ν with W_p^m, which is closer to the current notation $W^{m,p}$. The theory of Sobolev spaces has indeed developed rapidly since the 1950s.

As previously mentioned, Sobolev spaces are constructed from Lebesgue spaces, which are Banach spaces. Therefore, the reader is encouraged to deepen their understanding of the topological and analytical properties inherent in Banach spaces. Additionally, it would be beneficial to become familiar with well-established theories such as the Hahn-Banach theorem, the Banach-Steinhaus theorem, and other related concepts in Hilbert spaces.

On the other hand, the theory of distributions is based on spaces of regular functions and their dualities, presenting a specific topological structure that can be quite complex. If the reader wishes to delve further into this notion, we recommend consulting the two works [10] and [13], as well as other references dealing with topological vector spaces. However, it is entirely appropriate to provide some incentives here to pique the reader's interest in these spaces.

It is evident that all elements of the space $\mathscr{C}^k(K)$, where K is a compact subset of $\mathbb{R}^n$, are bounded functions, along with their partial derivatives up to order k. This space has the structure of a normed vector space with the norm defined as $\displaystyle\sum_{k=0}^{m} \sup_{x \in K, |\alpha| = k} |D^\alpha f(x)|$. Unfortunately, this property is generally not obtained for the spaces $\mathscr{C}^m(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$. We need a topological structure that preserves the properties of these spaces. For example, if a sequence $\{f_j\}_{j=1}^{+\infty}$ consists of functions from $\mathscr{C}^m(\Omega)$ and converges to a function f in this topology, it is necessary that $f \in \mathscr{C}^m(\Omega)$. Furthermore, it is necessary that if F is a neighbourhood of f and G is a neighbourhood of g, then $F+G$ is a neighbourhood of $f+g$ and λF is a neighbourhood of λf, where λ is a real or complex number. The required topology is constructed from families of semi-norms $\displaystyle\sup_{x \in K, |\alpha| \leq m} |D^\alpha f(x)|$, where K are compacts contained in Ω. It is a locally convex topology (i.e., for every $f \in \mathscr{C}^m(\Omega)$, there exists a system of convex neighbourhoods of f, which is equivalent in this case to the existence of a system of convex neighbourhoods of 0). For more details, the reader is encouraged to consult Section 1.5, as well as the two works [10] and [13].

In addition to topological concepts, we motivate the reader to explore fundamental notions of algebra, mathematical analysis, integration theory, as well as Lebesgue spaces for a deeper understanding of distribution theory and Sobolev spaces.

This booklet is organized into five chapters. The first chapter provides reviews and supplementary information on essential concepts necessary for understanding the subsequent chapters. The second chapter introduces definitions and properties related to distributions. The third chapter discusses convolution and its properties. The fourth chapter is dedicated to the Fourier transform. Finally, the fifth chapter covers Sobolev spaces. Each chapter concludes with a series of solved exercises.

I sincerely hope that this booklet will be of great value to Master's students, especially those taking courses in functional analysis, numerical analysis, and partial differential equations. My dearest wish is that this work may enrich the national university library, even if only to a small extent.

M'sila, September 21, 2023, corresponding to Rabi' al-Awwal 6, 1445 AH.
Saadi Abderachid.

CHAPTER 1

RECAPS AND SUPPLEMENTS

This chapter appears to serve as a foundation for the upcoming chapters by providing a review and supplement of key concepts necessary to understand the material that will be covered later. Here is a summary of what will be covered in this chapter:

Banach Spaces: These are complete normed vector spaces where the notion of convergence is defined.

Topological Vector Spaces: These spaces combine both a vector structure and a topology, allowing for discussions of continuity and convergence in a more general framework.

Duality and Weak Topology: This concept pertains to topological vector spaces and their dual spaces, as well as the notion of weak and weak* topology.

Spaces of Regular Functions: These spaces are often used to study regularity properties of functions.

Test Function Space: This is a space of functions specifically designed for studying distributions and distribution theory.

Lebesgue Spaces: These spaces are used to study measurable functions and Lebesgue integrals, convolution products, and Fourier transformations.

Radon Measure: This is a topological vector spaces of measure, often used in functional analysis and harmonic analysis.

Regular Domains: These are regular subsets of a space, often used in the context of integration.

Boundary Integral: This concept pertains to integrating functions over the boundaries of domains.

It should be noted that this chapter does not delve into proof details but rather briefly presents key definitions and results. Those looking to deepen their knowledge are encouraged to consult the references mentioned in the booklet for more detailed information and complete proofs

1.1 Banach spaces

We will encounter many examples of Banach spaces (Lebesgue spaces, Sobolev spaces, etc.), which necessitates briefly discussing some of the characteristics of Banach spaces.

Let E be a vector space over $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1 : *Let $p : E \to \mathbb{R}$ be a mapping.*

i) *We say that p is a semi-norm if:*

**) p is homogeneous, i.e., for all $\lambda \in \mathbb{R}$ and all $x \in E$: $p(\lambda x) = |\lambda| p(x)$.*
In particular, $p(0) = 0$.

***) p satisfies the triangle inequality, i.e., for all $x \in E$ and all $y \in E$ we have: $p(x + y) \leq p(x) + p(y)$.*

ii) *If, in addition: for all $x \in E$, if $p(x) = 0$, then $x = 0$, we say that p is a norm on E.*

Definition 1.2 : *A normed vector space is defined as the pair $(E, \|.\|)$ consisting of a vector space over $\mathbb{R}$ and a norm defined on E.*

Definition 1.3 : *Two norms $\|.\|_1$ et $\|.\|_2$ de E are called equivalent if there exists $\alpha > 0$ and $\beta > 0$ such that*

$$\forall x \in E : \alpha \|.\|_1 \leq \|.\|_2 \leq \beta \|.\|_1$$

Proposition 1.1 : *A normed vector space $(E, \|\cdot\|)$ is a metric space where the distance d is defined by:*

$$d(x, y) = \|x - y\|, \forall x, y \in E.$$

Proposition 1.2 : *Open and closed balls are convex. We say that $(E, \|\cdot\|)$ is locally convex.*

Definition 1.4 : *We say that a normed vector space $(E, \|\cdot\|)$ is uniformly convex if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in B_E$, we have:*

$$\left\| \frac{1}{2}(x + y) \right\| \geq 1 - \delta \Rightarrow \|x - y\| \leq \varepsilon.$$

Proposition 1.3 : *A linear map $f : E \to F$ is continuous if and only if there exists $M > 0$ such that for all $x \in E$ we have: $\|f(x)\|_F \leq M \|x\|_E$.*
As a result, every continuous linear map is Lipschitz.

Definition 1.5 : *We denote by $\mathcal{L}(E; F)$ the space of continuous linear mappings from E to F. The topological dual of E is called the dual space and is denoted by $E' = \mathcal{L}(E; \mathbb{R})$ It is the space of continuous linear functionals on E.*

Proposition 1.4 : *The quantity $\|f\| = \sup\limits_{x \neq 0} \dfrac{\|f(x)\|_F}{\|x\|_E} = \sup\limits_{\|x\|_E = 1} \|f(x)\|_F = \sup\limits_{\|x\|_E \leq 1} \|f(x)\|_F$ is a norm on $\mathcal{L}(E; F)$.*

Definition 1.6 : *We say that a normed vector space is a Banach space if it is a complete metric space.*

Theorem 1.1 : *If F is a Banach space, then $\mathcal{L}(E; F)$ is a Banach space.*
In particular, E' is a Banach space.

Theorem 1.2 (Banach-Steinhaus) : *Let E and F be two Banach spaces, and let $\{f_j\}_{j\in\mathbb{N}}$ be a sequence of linear maps from E to F. If $\{f_j(x)\}_{j\in\mathbb{N}}$ is a bounded sequence, then it is uniformly bounded, i.e.,*

$$\exists M > 0, \forall j \in \mathbb{N} : \sup_{x\neq 0} \frac{\|f_j(x)\|_F}{\|x\|_E} \leq M.$$

Definition 1.7 (compact linear operator) : *In the context of normed vector spaces E and F, an operator A is said to be compact if the image of the unit ball $B_E(0, 1)$ in E is relatively compact in F. This means that the set $A(B_E(0, 1))$ has a compact closure in F.*

Remark 1.1 *: If A is a compact operator from E to F then: for any bounded sequence $\{u_j\}_{j=1}^{+\infty}$ in E, we can extract a subsequence $\{u_{j,k}\}$ such that $A(u_{j,k})$ converge in F.*

1.2 Topological vector spaces

Topological vector spaces are vector spaces equipped with a topological structure that is compatible with the two internal operators, addition $(+)$ and scalar multiplication $(\cdot)$, of these spaces. Among these spaces, we find, for example, spaces of regular functions and their dualities.

Let E be a vector space over $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.8 : *We say that E is a topological vector space if it is equipped with a topological structure having the following properties:*

i) *The addition $(x, y) \to x + y$ is a continuous mapping from $E \times E$ to E.*

ii) *The scalar multiplication $(\lambda, x) \to \lambda \cdot x$ is a continuous mapping from $\mathbb{R} \times E$ $(\mathbb{C} \times E)$ to E.*

Example 1.1 *Every normed vector space is a topological vector space. Open balls form a fundamental system of neighborhoods for this space.*

The topology of a normed vector space is invariant under translation and scaling; therefore, this topology can be generated using neighbourhoods of the origin 0.

Among the methods for constructing topological vector spaces, there are two approaches:

Method 1: We define a fundamental system of neighbourhoods of 0 by specifying a family of semi-norms. An example of such a fundamental system is open balls.

Method 2: We construct a family of subspaces equipped with topologies of the type mentioned above. This method uses the concept of an inductive limit of locally convex spaces. It is the method applied, for example, to the space $\mathscr{D}(\Omega)$.

Definition 1.9 : *We call a topological vector space a locally convex space if 0 has a system of convex neighbourhoods.*

Definition 1.10 : *Let E be a topological vector space. We say that a set $A \subset E$ is bounded if, for every neighborhood V of 0, there exists $n \in \mathbb{N}$ such that*

$$\forall \lambda \in \mathbb{R} : |\lambda| \geq n \Rightarrow A \subset \lambda V.$$

We say that E is locally bounded if it contains at least one non-empty and bounded open set.

1.3 Functionals, topological duals

Definition 1.11 : *We call functional any numeric function f, defined on a vector space E.*

Proposition 1.5 :

1. *Every linear functional on a topological vector space of finite dimension is continuous.*
2. *Every linear functional on a topological vector space that is continuous at a point is continuous over the entire space.*

Theorem 1.3 : *Let f be a linear functional on a topological vector space E. Then, f is continuous on E if and only if there exists a neighbourhood V of 0 such that the functional f is bounded on V.*

Definition 1.12 : *Let E be a topological vector space. We call the dual of E and denote it as E', the space of continuous linear functionals on E.*

Definition 1.13 : *Let E be a locally convex and separated topological vector space. We call the bidual of E and denote it as E'', the dual space of the space E'.*

Let E be a separated, locally convex topological vector space, and let E' be its topological dual.

Definition 1.14 (***strong topology of*** E') : *We equip E' with a separated and locally convex topology, called the strong topology, by considering the fundamental system of neighbourhoods of 0 as follows:*

$$\{x \in A, |f(x)| < M\}, M > 0, A \text{ bounded.}$$

There exists an injection of E into E'', denoted by π. If $\pi(E) = E''$ and π is continuous with respect to the strong topology of E'', we say that E is reflexive. In this case, the spaces E and E'' are isomorphic.

Definition 1.15 (***weak topology of*** E) : *The weak topology on a topological vector space E consists of a fundamental system of open neighborhoods of 0 in the form:*

$$\{x \in E : |f_i(x)| < M\}, M > 0, i = 1, \cdots, n.$$

This topology is the weakest (least fine) topology such that linear functionals are continuous.

We say that a sequence $\{x_j\}_{j=1}^{\infty}$ converges weakly in E to $x \in E$, denoted as $x_j \rightharpoonup x$, if for every $f \in E'$, we have $f(x_j)$ converging to $f(x)$.

Definition 1.16 (***weak topology of*** E' (***weak****)) : *We equip E' with a topology called the weak topology of E' by considering the fundamental system of neighbourhoods of 0 as follows:*

$$\{f \in E', |f(x_i)| < M\}, M > 0,\ i = 1, \cdots, n.$$

We say that a sequence of linear functionals $\{f_j\}_{j=1}^{\infty}$ converges weakly* in E' to $f \in E'$ if for every $x \in E$, we have $f_j(x)$ converging to $f(x)$.

1.4 Fréchet spaces

Definition 1.17 : *A locally convex space is said to be metrizable if it is equipped with an increasing family of semi-norms $\{p_j\}_{j\in\mathbb{N}}$ (i.e. $\forall x \in E, \forall j \in \mathbb{N} : p_j(x) \leq p_{j+1}(x)$), such that: for all $j \in \mathbb{N} : p_j(f) = 0$ if and only if $f = 0$.*

Proposition 1.6 : *Let $\{\alpha_j\}_{j\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that the series $\sum_{j\in\mathbb{N}} \alpha_j$ converges. Let E be a locally convex and metrizable space. We denote by $\{p_j\}_{j\in\mathbb{N}}$ the family of semi-norms on this space. The function $d : E \times E \to \mathbb{R}$ defined by:*

$$\forall (f, g) \in E^2 : d(f, g) = \sum_{j=0}^{+\infty} \alpha_j \min(1, p_j(f - g))$$

is a distance on E.

Definition 1.18 (***Fréchet space***) : *We say that a locally convex, metrizable space equipped with the topology defined by the above distance is a Fréchet space if it is complete.*

Proposition 1.7 : *Let E be a Fréchet space with the family of semi-norms $\{p_j\}_{j\in\mathbb{N}}$, F a Fréchet space with the family of semi-norms $\{q_k\}_{k\in\mathbb{N}}$, and L a linear map from E to F. Then, L is continuous if and only if:*

$$\forall k \in \mathbb{N}, \exists c > 0, j \in \mathbb{N}, \forall x \in E : q_k(L(x)) \leq c p_j(x).$$

Theorem 1.4 (***Hahn-Banach***) : *Let E be a topological vector space, G a subspace of E, and p a map from G to $\mathbb{R}$ satisfying:*

$$\forall \lambda > 0, \forall x \in E : p(\lambda x) = \lambda p(x), \qquad \forall x, y \in E : p(x+y) \leq p(x) + p(y).$$

Let g be a linear map from G to $\mathbb{R}$ satisfying:

$$\forall x \in G : g(x) \leq p(x).$$

Then, there exists a linear extension f of g to E satisfying:

$$\forall x \in G : g(x) = f(x), \qquad \forall x \in E : f(x) \leq p(x).$$

Corollary 1.1 : *Let E be a normed vector space, G a subspace of E, and $g \in G'$ with the norm: $\|g\|_{G'} = \sup\limits_{x \in G, \|x\| \leq 1} |g(x)|$.*
Then, there exists an extension $f \in E'$ de g with $\|f\|_{E'} = \|g\|_{G'}$.

Theorem 1.5 (***Banach-Steinhaus***) : *Let E be a Fréchet space with the family of semi-norms $\{p_j\}_{j \in \mathbb{N}}$, F a Fréchet space with the family of semi-norms $\{q_k\}_{k \in \mathbb{N}}$, and let $\{L_\alpha\} : E \to F$ be a family of continuous linear maps. Suppose that for every $x \in E$, the sequence $\{L_\alpha(x)\}$ is bounded in F. Then:*

$$\forall k \in \mathbb{N}, \exists c > 0, j \in \mathbb{N}, \forall x \in E, \forall \alpha : q_k(L_\alpha(x)) \leq c p_j(x).$$

Corollary 1.2 : *Let E be a Fréchet space, and F a metrizable locally convex space, and let $\{L_j\} : E \to F$ be a sequence of continuous linear maps. Suppose that for every $x \in E$, the sequence $\{L_j(x)\}$ converges in F to an element $L(x)$. Then:*

1. *The map $L : E \to F$, which associates $x \in E$ with the element $L(x)$, is linear and continuous.*
2. *$x_j \to x$ in E implies that $L_j(x_j) \to L(x)$ in F.*

1.5 Regular function spaces

Let Ω a non-empty open set of $\mathbb{R}^n$.

Definition 1.19 : *Let $f : \Omega \to \mathbb{R}$, and $k \in \mathbb{N}$. We say that f is of class $\mathscr{C}^k(\Omega)$ if and only if $D^\alpha f$ exists and is continuous for every multi-index α such that $|\alpha| \leq k$.*
If $f \in \mathscr{C}^k(\Omega)$ for every $k \in \mathbb{N}$, we say that f is of class $\mathscr{C}^\infty(\Omega)$.

Proposition 1.8 *Let $K \subset \Omega$ be a compact set, and let $m \leq k$ be two natural numbers. The quantity: $P_{K,m}(f) = \sup\limits_{|\alpha| \leq m, x \in K} |D^\alpha f(x)|$, $f \in \mathscr{C}^k(\Omega)$,*
defines a semi-norm on $\mathscr{C}^k(\Omega)$.

We equip the space $\mathscr{C}^k(\Omega)$ with a topological structure compatible with successive derivatives up to k, using a family of semi-norms $\{P_{K,m}\}$, where K ranges over the set of compacts included in Ω, and m ranges over the set $\{0, 1, \ldots, k\}$.
The space $\mathscr{C}^k(\Omega)$, endowed with the above topological structure, is a topological vector space that is locally convex and separated. Moreover, $\mathscr{C}^k(\Omega)$ is a Fréchet space.
Similarly, we equip $\mathscr{C}^\infty(\Omega)$ with a topological structure. Therefore, the topological vector space $\mathscr{C}^\infty(\Omega)$ is a locally convex, separated, and Fréchet space.

We provide a practical definition of convergence in $\mathscr{C}^k(\Omega)$, where $k \in \mathbb{N} \cup +\infty$:

Definition 1.20 :

i) *Let $k \in \mathbb{N}$. We say that a sequence of functions $\{f_j\}$ in $\mathscr{C}^k(\Omega)$ converges to f in $\mathscr{C}^k(\Omega)$ if for every compact set $K \subset \Omega$ and for every natural number $m \leq k$, we have:*

$$\lim_{j \to +\infty} P_{K,m}(f_j - f) = 0.$$

ii) *We say that a sequence of functions f_j in $\mathscr{C}^\infty(\Omega)$ converges to f in $\mathscr{C}^\infty(\Omega)$ if for every compact set $K \subset \Omega$ and for every natural number m, we have:*

$$\lim_{j \to +\infty} P_{K,m}(f_j - f) = 0.$$

1.6 The space $\mathscr{D}(\Omega)$ of test functions

Let Ω a non-empty open set of $\mathbb{R}^n$.

Definition 1.21 : *Let f be a function defined almost everywhere (a.e.) on Ω.*
"The null open set of f is defined as the largest open set $\mathcal{O}_f$ such that $f \equiv 0$ a.e."
"The support of f, denoted as $\operatorname{supp} f$, is defined as $\mathbb{R}^n \setminus \mathcal{O}_f$, the complement of $\mathcal{O}_f$.

Proposition 1.9 *We have:* $\operatorname{supp} f = \overline{\{x \in \Omega : f(x) \neq 0\}}$

Definition 1.22 : *Let $K \subset \Omega$ be a compacte set and let $m \in \mathbb{N}$.*

i) *$\mathscr{D}^m_K(\Omega)$ is the space of functions in $\mathscr{C}^m(\Omega)$ with compact support included in K.*

ii) *$\mathscr{D}^m(\Omega)$ is the space of functions in $\mathscr{C}^m(\Omega)$, with compact support included in Ω.*

iii) *$\mathscr{D}_K(\Omega)$ is the space of functions in $\mathscr{C}^\infty(\Omega)$, with compact support included in K.*

iv) *$\mathscr{D}(\Omega)$ is the space of functions in $\mathscr{C}^\infty(\Omega)$, with compact support included in Ω.*

Remark 1.2 :

1. $\mathscr{D}(\Omega)$ is called the space of test functions.

2. *Sometimes the following notations are used:*

   $\mathscr{K}(\Omega)$ *or* $\mathscr{C}_0^0(\Omega)$ *for the space* $\mathscr{D}^0(\Omega)$.

   $\mathscr{C}_0^m(\Omega)$ *for the space* $\mathscr{D}^m(\Omega)$.

   $\mathscr{C}_0^\infty(\Omega)$ *for the space* $\mathscr{D}(\Omega)$.

Example 1.2 : *Let* φ *be the function defined by:* $\begin{cases} \varphi(x) = e^{-\frac{1}{1-x^2}} & : \ x \in]-1,1[\\ \varphi(x) = 0 & : \ x \notin]-1,1[\end{cases}$

We can prove that $\varphi \in \mathscr{D}(\mathbb{R})$.

Proposition 1.10 : $\mathscr{D}(\Omega) = \bigcup_{K\subset\Omega,\ K\,compact} \mathscr{D}_K(\Omega)$.

Proposition 1.11 :

1. *If* $\varphi \in \mathscr{D}(\Omega)$ *et* $\psi \in \mathscr{C}^\infty(\Omega)$, *then:* $\varphi.\psi \in \mathscr{D}(\Omega)$.

2. *If* $\varphi, \psi \in \mathscr{D}(\Omega)$ *et* $\lambda, \mu \in \mathbb{R}$, *then:* $\lambda\varphi + \mu\psi \in \mathscr{D}(\Omega)$.

We equip the space $\mathscr{D}(\Omega)$ with a topology called the strict inductive limit topology of Fréchet spaces of the type $\mathscr{D}_K(\Omega)$, where K ranges over compacts in Ω. The topology defined on the spaces $\mathscr{D}_K(\Omega)$ is induced by that of $\mathscr{C}^\infty(\Omega)$.

We can then provide a characterization of convergence in the space $\mathscr{D}(\Omega)$ as follows:

Definition 1.23 : *We say that a sequence of test functions* (φ_j) *converges to* φ *in* $\mathscr{D}(\Omega)$ *if there exists a compact set* $K \subseteq \Omega$ *such that:*

1. $\operatorname{supp}\varphi_j \subseteq K$ *for all* j *and* $\operatorname{supp}\varphi \subseteq K$.

2. *For all* $m \in \mathbb{N}$ *we have:* $\lim\limits_{j\to+\infty} P_{K,m}(\varphi_j - \varphi) = 0$.

Definition 1.24 : *We say that a sequence* $\{\rho_j\}_{j=1}^{+\infty}$ *in* $\mathscr{D}(\mathbb{R}^n)$ *is a regularization sequence if for every* $j \in \mathbb{N}$, *there exists* ε_j ($\varepsilon_j \to 0$ *as* $j \to +\infty$) *such that:*

$$\rho_j \geq 0, \qquad \int_{\mathbb{R}^n} \rho_j(x)dx = 1, \qquad \operatorname{supp}\rho_j \subseteq B(0,\varepsilon_j).$$

Such a function φ_j is called a «pic» function on $B(0,\varepsilon_j)$.

Example 1.3 : *Let* $\psi \in \mathscr{D}(\mathbb{R}^n)$ *such that* $\operatorname{supp}\psi \subseteq B(0,1)$, *and we define* $\rho = \dfrac{\psi}{\int_{\mathbb{R}^n}\psi(x)dx}$. *We have* $\operatorname{supp}\rho \subseteq B(0,1)$.

For any positive sequence (ε_j) *tending to* 0, *we define* $\rho_j(x) = \dfrac{1}{\varepsilon_j^n}\rho\left(\dfrac{x}{\varepsilon_j}\right)$.

*It can be verified that this sequence is a regularization sequence. This family is called an «**approximation of the identity**».*

Definition 1.25 : *Let* $T : \mathscr{D}(\Omega) \to \mathbb{R}$ *be a linear functional. We say that* T *is continuous if: For every sequence* $\{\varphi_j\}_{j=1}^{+\infty}$ *converging to* φ *in* $\mathscr{D}(\Omega)$, *the sequence* $\{T(\varphi_j)\}$ *converges to* $T(\varphi)$ *in* $\mathbb{R}$.

1.7 Some main results

Let Ω be a non-empty open set of $\mathbb{R}^n$.

Proposition 1.12 : *There always exists a sequence (called exhaustive) of compacts* $\{K_j\}_{j=1}^{+\infty}$ *in* Ω *such that:*

1. $\forall j \in \mathbb{N} : K_j \Subset K_{j+1}$ $(K_j \subset \overset{0}{K}_{j+1})$,
2. $\Omega = \bigcup_{j=0}^{+\infty} K_j$.

Theorem 1.6 (***Urysohn Lemma***) : *Let* K, F *be two disjoint sets in* $\mathbb{R}^n$*, where* K *is compact and* F *is closed. Then, there exists* $\varphi \in \mathscr{D}(\mathbb{R}^n)$ *such that:*

i) $0 \leq \varphi \leq 1$,

ii) $\varphi = 0$ *in a neighbourhood* V *de* F,

iii) $\varphi = 1$ *in a neighbourhood* W *de* K.

Corollary 1.3 : *Let* K *be a compact set of* Ω*, Then, there exists* $\varphi \in \mathscr{D}(\mathbb{R}^n)$ *such that:*

i) $0 \leq \varphi \leq 1$,

ii) $\varphi = 1$ *in a neighbourhood of* K *(we can choose it to be compact).*

Definition 1.26 : *Let* $\{\Omega_j\}_{j+1}^{+\infty}$ *be an exhaustive sequence of open sets in* Ω*, i.e*

***)** $\forall j \in \mathbb{N} : \Omega_j \Subset \Omega_{j+1}$ $(\overline{\Omega_j} \subset \Omega_{j+1})$,

****)** $\Omega = \bigcup_{j=0}^{+\infty} \Omega_j$.

We say that a sequence $\{\varphi_j\}_{j=1}^{+\infty}$ *in* $\mathscr{D}(\mathbb{R}^n)$ *is a truncation sequence on* Ω *if, for every* j*:*

i) $0 \leq \varphi_j \leq 1$,

ii) $\varphi_j = 1$ *in a neighbourhood of* $\overline{\Omega}$.

Proposition 1.13 : *Every open set admits a truncation sequence.*

Theorem 1.7 (***partition of unity***) : *Let* K *be a compact set included in a finite union of open sets* $\{\Omega_j\}_{j=1}^{N}$*. Then, there exists a family of functions* $\{\varphi_j\}_{j=1}^{N}$ *such that:*

i) $\varphi_j \in \mathscr{D}(\Omega_j)$,

ii) $0 \leq \varphi_j \leq 1$,

iii) $\sum_{j=1}^{N} \varphi_j = 1$.

1.8 Lebesgue spaces

We equip $\mathbb{R}^n$ with the Borel (or Lebesgue) sigma-algebra and the standard Lebesgue measure $dx = dx_1 dx_2 \cdots dx_n$. Let Ω be a non-empty open set in $\mathbb{R}^n$.

Definition 1.27 *: Let f be a measurable function on Ω. We say that f is Lebesgue integrable if f is measurable and $\int_\Omega |f(x)|dx < +\infty$.*

We denote by $\mathscr{L}^1(\Omega)$ the space of Lebesgue integrable functions on Ω.

We denote by $L^1(\Omega)$ the quotient space $\mathscr{L}^1(\Omega)/\sim$, where $\sim$ is the equivalence relation defined as follows:
For $f, g \in \mathscr{L}^1(\Omega)$, $f \sim g$ if and only if $f = g$ a.e on Ω.
We equip $L^1(\Omega)$ with the following norm: $\|f\|_{L^1(\Omega)} = \int_\Omega |f(x)|dx$.

Definition 1.28 *Similarly, the space $L^p(\Omega)$ (where $p > 1$) is the space of equivalence classes of measurable functions f such that $|f|^p \in L^1(\Omega)$, i.e., $\int_\Omega |f(x)|^p dx < +\infty$*

We equip $L^p(\Omega)$ with the following norm: $\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}$.

Definition 1.29 *The space $L^\infty(\Omega)$ is the space of equivalence classes of measurable functions f such that $ess\sup(f) = \inf\{c \geq 0, |f| < c \text{ a.e on } \Omega\} < +\infty$.*

We equip $L^\infty(\Omega)$ with the following norm: $\|f\|_{L^\infty(\Omega)} = ess\sup(f)$.

Definition 1.30 *The space $L^p_{loc}(\Omega)$ (where $p > 1$) is the space of equivalence classes of measurable functions f such that $f \in L^p(K)$ for every compact subset $K \subset \Omega$.*

Theorem 1.8 :

1. *The space $L^p(\Omega)$ for $1 \leq p \leq \infty$, equipped with the norm $\|.\|_{L^p(\Omega)}$, is a Banach space.*
2. *The space $L^2(\Omega)$ is indeed a Hilbert space, equipped with the inner product:*

$$f, g \in L^2(\Omega) : (f, g) = \int_\Omega f(x).g(x)dx.$$

3. *The space $L^p(\Omega)$ $(1 < p < \infty)$ is uniformly convex.*
4. *The space $L^p(\Omega)$ $(1 < p < \infty)$ is reflexive.*

5. *The space* $L^p(\Omega)$ $(1 \le p < \infty)$ *est separable.*
6. $L^1(\Omega)$ *and* $L^\infty(\Omega)$ *are not reflexive Banach spaces.* $L^\infty(\Omega)$*is not separable.*

Theorem 1.9 (***Hölder's inequality***) : *Let* $p, p' \in]1, +\infty[$ *such that* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then, for all* $f \in L^p(\Omega)$ *and* $g \in L^{p'}(\Omega)$, *we have* $f.g \in L^1(\Omega)$. *Moreover:*
$\|f.g\|_{L^1(\Omega)} \le \|f\|_{L^p(\Omega)}.\|g\|_{L^{p'}(\Omega)}$.
p' *is called the conjugate of* p, *and we have:* $p' = \frac{p}{p-1}$.
In particular, we have the Cauchy-Schwarz inequality:
For all $f, g \in L^2(\Omega)$: $\|f.g\|_{L^1(\Omega)} \le \|f\|_{L^2(\Omega)}.\|g\|_{L^2(\Omega)}$.

Remark 1.3 *:*

i) *Let* $p_1, p_2, \cdots, p_k \in [1, +\infty]$ *and* $p \ge 1$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k}$. *Let* $f_1 \in L^{p_1}(\Omega), f_2 \in L^{p_2}(\Omega), \cdots, f_k \in L^{p_k}(\Omega)$, *and* $f = f_1.f_2.\cdots.f_k$. *Then:* $f \in L^p(\Omega)$, *and we have the generalized Hölder's inequality:*

$$\|f\|_{L^p(\Omega)} \le \|f_1\|_{L^{p_1}(\Omega)}.\|f_2\|_{L^{p_2}(\Omega)}.\cdots.\|f_k\|_{L^{p_k}(\Omega)}.$$

ii) *If* $f \in L^p(\Omega) \cap L^q(\Omega)$ *with* $1 \le p \le q \le +\infty$ *then:* $f \in L^r(\Omega)$ *for all* $p \le r \le q$ *and we have the interpolation inequality:*

$$\forall \theta \in [0, 1] : \|f\|_{L^r(\Omega)} \le \|f\|^{\theta}_{L^p(\Omega)}.\|f\|^{1-\theta}_{L^q(\Omega)}.$$

Theorem 1.10 *: Let* $p, q \in [1, +\infty]$ *be such that* $p \le q$. *Then:*

1. $L^q_{loc}(\Omega) \subset L^p_{loc}(\Omega)$.
2. $L^q(\Omega) \subset L^p(\Omega)$ *if* Ω *is bounded.*

Theorem 1.11 *: Let* f *be a measurable function on* Ω *such that* $\int_A f(x)dx = 0$ *for all compact set (open set)* $A \subset \Omega$. *Then:* $f = 0$ *a.e on* Ω .

Theorem 1.12 (***Representation of Riez***) : *Let* $p \in]1, +\infty[$, *and let* $\varphi \in (L^p(\Omega))'$, *the dual of* $L^p(\Omega)$. *Then, there exists* $g \in L^{p'}(\Omega)$ *(where* p' *is the conjugate of* p*) such that:*

$$\langle \varphi, f \rangle = \int_\Omega f(x).g(x)dx, \forall f \in L^p(\Omega).$$

We can then identify $(L^p(\Omega))'$ with $L^{p'}(\Omega)$.
One can also identify $(L^1(\Omega))'$ with $L^\infty(\Omega)$. For $\varphi \in (L^1(\Omega))'$, there exists $g \in L^\infty(\Omega)$ such that:

$$\langle \varphi, f \rangle = \int_\Omega f(x).g(x)dx, \forall f \in L^1(\Omega).$$

We have: $L^1(\Omega) \subset (L^\infty(\Omega))'$, with strict inclusion.

Theorem 1.13 (***dominate convergence of Lebesgue***) : *Let $\{f_j\}_{j\in\mathbb{N}}$ de a sequence of functions in $L^p(\Omega)$ ($p \in [1, +\infty[$). Assume that:*

1. *$\{f_j\}$ converges a.e. to a function f.*

2. *There exists a function $g \in L^p(\Omega)$ such that $|f_j| \leq g$ a.e for all $j \in \mathbb{N}$.*

Then: $f \in L^p(\Omega)$ and $f_j \overset{L^p(\Omega)}{\longrightarrow} f$.

Remark 1.4 *One can replace the sequence $\{f_j\}_{j\in\mathbb{N}}$ with a family of functions $\{f_t\}_{t\in(a,b)}$ where a, b are in the extended real numbers. The limit will be taken at the point $t_0 \in [a, b]$.*

Theorem 1.14 : *The space $\mathscr{D}(\Omega)$ is dense in the space $L^p(\Omega)$ for all $p \in [1, +\infty[$.*

1.9 Measure of Radon

We equip $\mathbb{R}^n$ with the Borel sigma-algebra $\mathcal{B}(\mathbb{R}^n)$, and let Ω be a non-empty open set in $\mathbb{R}^n$.

Definition 1.31 : *A Borel measure on Ω, finite on compacts, is a measure from $\mathcal{B}(\Omega)$ to $[0, +\infty]$ for which we have $\mu(K) < \infty$ for every compact set $K \subset \Omega$.*

Such a measure is regular, i.e., for any measurable set $A \subset \Omega$, we have:

$$\begin{aligned} \mu(A) &= \inf\{\mu(O), O \supset A \text{ open set}\}, \\ &= \sup\{\mu(K), K \subset A \text{ compact set}\}. \end{aligned}$$

Definition 1.32 :

i) *A Radon measure (signed) is the difference of two Borel measures, both finite on compacts.*

ii) *We denote by $\mathscr{M}(\Omega)$ the space of Radon measures on Ω.*

Proposition 1.14 : *Let $f \in L^1_{loc}(\Omega)$. Then, the function $A \in \mathcal{B}(\Omega) \mapsto \int_A f(x)dx$ defines a Radon measure on Ω. Such a measure is called absolutely continuous, and f is its density.*

Remark 1.5 : *We have: $L^1_{loc}(\Omega) \subset \mathscr{M}(\Omega)$. The space $\mathscr{M}(\Omega)$ is larger than $L^1_{loc}(\Omega)$. For example, the Dirac measure δ_x at the point x, defined by:*

$$\delta_x(A) = \begin{cases} 0 & : \quad x \notin A, \\ 1 & : \quad x \in A, \end{cases} \tag{1.1}$$

is a Radon measure but is not a function.

Theorem 1.15 (***Riesz***) : *One can identify the space $\mathscr{M}(\Omega)$ with the space $\mathscr{K}'(\Omega)$, the topological dual of the space $\mathscr{K}(\Omega)$ of continuous functions with compact support in Ω, in such a way that:*

$$\forall \mu \in \mathscr{M}(\Omega), \exists c_\mu > 0, \forall \varphi \in \mathscr{K}(\Omega), \forall K \subset \Omega \text{ (compact)}: \left| \int_K \varphi d\mu \right| \leq c_\mu \sup_{x \in K} |\varphi(x)|.$$

The space $\mathscr{M}(\Omega)$, considered as the dual of $\mathscr{K}(\Omega)$, is a Fréchet space.

1.10 Regular domains , integration on the boundary

Let Ω a bounded open set of $\mathbb{R}^n$ and let $\Gamma = \partial\Omega$ the boundary of Ω. Q, Q_+, and Q_0 are defined as follows:

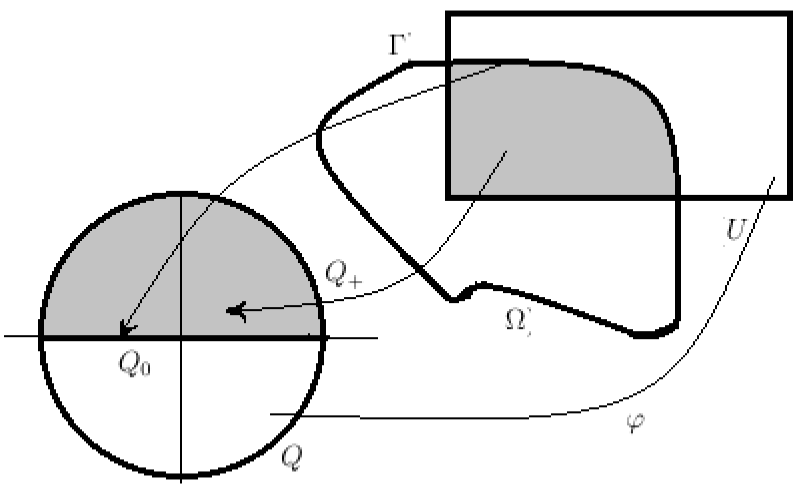
$$Q := \{x \in \mathbb{R}^n : |x'| < 1; |x_n| < 1\}$$

$$Q_+ := \{(x', x_n) \in \mathbb{R}^n : |x'| < 1; 0 < x_n < 1\}$$

$$Q_0 := \{(x', 0) \in \mathbb{R}^{n-1} \times \{0\} : |x'| < 1; \}$$

Definition 1.33 : *We say that Ω is of class $\mathscr{C}^k$ if for every $x \in \Gamma$, there exists a pair (U, φ), where U is an open set in $\mathbb{R}^n$ containing x, and $\varphi \in \mathscr{C}^k(U)$ is a diffeomorphism from U to Q such that for $\psi = \varphi^{-1}$, we have:*

1. $\varphi \in \mathscr{C}^k(\overline{Q})$;
2. $\varphi(U \cap \Gamma) = Q_0$;
3. $\varphi(U \cap \Omega) = Q_+$;



Definition 1.34 : *We denote by $\nu(x)$ the outward unit normal vector at point $x \in \Gamma$. If u is a sufficiently regular function defined on $\overline{\Omega}$, we have the normal derivative of u on Γ:*

$$\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$$

Here, ∇u is the gradient of the function u, and " $\cdot$ " represents the dot product between the gradient and the unit normal vector ν. This expression represents the normal derivative of u with respect to the outward normal direction on the boundary Γ.

Remark 1.6 : *If Ω is of class $\mathscr{C}^k$, one can extract a parameterization $x_i = \phi(y_1, y_2, \ldots, y_{n-1})$, where ϕ is of class $\mathscr{C}^k$. In this case, Γ is the graph of ϕ in an orthonormal coordinate system, and we have:*

$$\nu_y = \frac{(\nabla\phi(y), -1)}{\sqrt{1 + (\nabla\phi(y))^2}}.$$

We can say that the boundary of an open set of class $\mathscr{C}^k$ has a parameterization by a function of class $\mathscr{C}^k$.

Definition 1.35 : *Ω is Lipschitz if Γ has a parameterization by a Lipschitz function.*

Proposition 1.15 : *Suppose that Ω is of class $\mathscr{C}^1$. We can always decompose Γ into a disjoint union, such that Γ_i is the graph of a function ϕ_i in an orthonormal coordinate system as described in the previous remark. We define the line integral for a function f defined on Γ as follows:*

$$\int_\Gamma f d\sigma(x) = \sum_i \int_{\Gamma_i} f(y, \phi(y))\sqrt{1 + (\nabla\phi_i(y))^2} dy$$

Theorem 1.16 [***Ostrogradsky formula***] : *Let Ω be a bounded open set of class $\mathscr{C}^1$, and Γ its boundary. Let F be a vector field, i.e., a function in $\mathscr{C}^1(\overline{\Omega})$ with values in $\mathbb{R}^n$. Then:*

$$\int_\Omega div\, F(x) dx \int_\Gamma F(x).\nu(x) d\sigma(x).$$

This equation represents a relationship between the divergence of the vector field over the domain Ω and its line integral over the boundary Γ.

Corollary 1.4 [***Green formula***] : *Let Ω be a bounded open set of class $\mathscr{C}^1$, and Γ its boundary. Let u be a function in $\mathscr{C}^2(\Omega, \mathbb{R}) \cap \mathscr{C}^2(\overline{\Omega}, \mathbb{R})$, and v be a function in $\mathscr{C}^1(\Omega, \mathbb{R})$. Then:*

$$\int_\Omega v(x)\Delta u(x) dx + \int_\Omega \nabla v(x).\nabla u(x) dx = \int_\Gamma v(x)\frac{\partial u}{\partial \nu}(x) d\sigma(x).$$

This equation represents a relationship involving the Laplacian, gradients, and normal derivatives of the functions u and v over the domain Ω and its boundary Γ.

Exercises

Exercise 1.1 : *Consider the function φ defined on $\mathbb{R}$ as follows:*

$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & : \ |x| < 1 \\ 0 & : \ |x| \geq 1 \end{cases}$$

We define the sequence of functions $\{\varphi_j\}_{j=1}^{+\infty}$ as follows: $\varphi_j(x) = \varphi(jx)$.

1. *Is $\varphi_j \in \mathscr{C}^\infty(\mathbb{R})$?*
2. *Provide the support of φ_j.*
3. *Provide a generalization of φ to $\mathbb{R}^n$.*

Exercise 1.2 : *Let $\varphi \in \mathscr{D}(]0,2[)$ such that $\varphi \geq 0$ and $\varphi = 1$ on $\left[\frac{1}{2}, \frac{3}{2}\right]$. Let sequence of functions $(\varphi_j)_{j\in\mathbb{N}}$ defined by: $\forall j \in \mathbb{Z}, \forall x \in \mathbb{R} : \varphi_j(x) = \varphi(x+j)$.*

1. *Consider the function ψ defined as follows: $\psi(x) = \sum_{j\in\mathbb{Z}} \varphi_j(x)$.*

   Is ψ well-defined? Is $\psi > 0$?

2. *Consider the sequence of functions (u_j) defined as follows:*

$$\forall j \in \mathbb{Z}, \forall x \in \mathbb{R} : u_j(x) = \frac{\varphi_j(x)}{\psi(x)}.$$

   Is u in $\mathscr{D}(\mathbb{R}^n)$?

   Does it satisfy the relation: $\forall x \in \mathbb{R} : \sum_{j\in\mathbb{Z}} u_j(x) = 1$?

Exercise 1.3 : *Let $K \subseteq \mathbb{R}^n$ be a compact set, and let $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Are the following two implications true?*

1. *$\psi = 0$ in a neighbourhood of $K \Rightarrow \operatorname{supp} \psi \subset (\mathbb{R}^n \backslash K)$.*
2. *$\psi = 0$ on $K \Rightarrow \operatorname{supp} \psi \subset (\mathbb{R}^n \backslash K)$.*

Exercise 1.4 : *Let $\varphi \in \mathscr{D}(\mathbb{R}^n), h \in \mathbb{R}^n \backslash \{0\}$.*

For all $t \in \mathbb{R}^*$ we set: $\varphi_t(x) = \dfrac{\varphi(x+th) - \varphi(x)}{t}$.

1. *Show that $\varphi_t \in \mathscr{D}(\mathbb{R}^n)$ for all $t \neq 0$*
2. *Show that as t tends to 0, φ_t converges in $\mathscr{D}(\mathbb{R}^n)$ to a function that we will determine.*

Exercise 1.5 : *Consider the sequence of functions* $\{f_j\}_{j=1}^{+\infty}$ *in* $\mathscr{D}(\mathbb{R})$ *defined by:*

$$f_j(x) = \begin{cases} \frac{1}{2^j} \exp\left(-\frac{1}{1-\frac{x^2}{n^2}}\right) & : \ |x| < j \\ 0 & : \ |x| \geq j \end{cases}$$

Show that, for each $k \geq 0$, *the sequence of functions* $\{f_j^{(k)}\}_{j=1}^{+\infty}$ *converges uniformly on every compact set* K *to a function* $f \in \mathscr{D}(\mathbb{R})$ *which will be specified.*

Do we have convergence in $\mathscr{D}(\mathbb{R})$*?*

Exercise Solutions

Solution 1.1 :

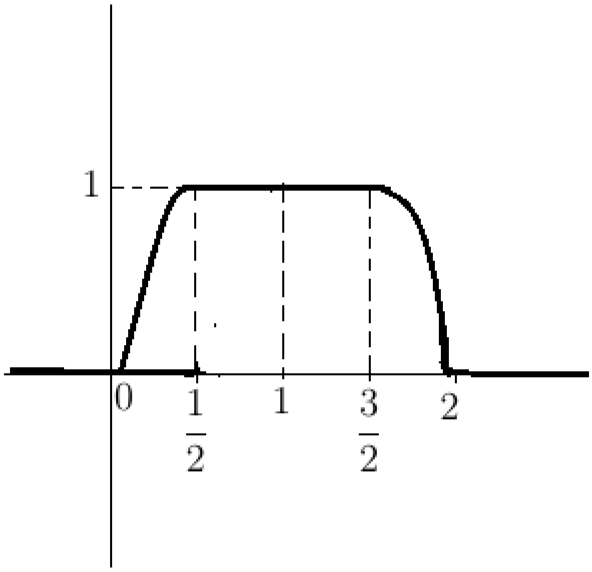
$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & : \ |x| < 1 \\ 0 & : \ |x| \geq 1 \end{cases}, \qquad \varphi_j(x) = \varphi(jx).$$

1. *To show that* $\varphi_j \in \mathscr{C}^\infty(\mathbb{R})$, *it is sufficient to demonstrate that* $\varphi \in \mathscr{C}^\infty(\mathbb{R})$.
   t is clear that φ *is of class* $\mathscr{C}^\infty$ *on* $\mathbb{R}\backslash\{-1, 1\}$. *We show that* φ *is infinitely differentiable at the points* -1 *and* 1. *We have:* $\lim\limits_{x\to -1} \varphi(x) = \lim\limits_{x\to 1} \varphi(x) = 0$.
   Making the change of variable $y = \dfrac{1}{1-x^2}$ *on* $]-1, 1[$, *we find:*
   $\lim\limits_{x\to -1} \varphi(x) = \lim\limits_{x\to 1} \varphi(x) = \lim\limits_{y\to +\infty} e^{-y} = 0.$
   $\lim\limits_{x\to -1} \dfrac{\varphi(x)-\varphi(-1)}{x+1} = \lim\limits_{x\to 1} \dfrac{\varphi(x)-\varphi(1)}{x-1} = 0.$
   Making the change of variable $y = \dfrac{1}{1-x^2}$ *on* $]-1, 1[$, *we obtain:*
   $\lim\limits_{x\overset{>}{\longrightarrow} -1} \varphi(x) = \lim\limits_{x\overset{<}{\longrightarrow} 1} \varphi(x) = \lim\limits_{y\longrightarrow +\infty} e^{-y} = 0.$
   Therefore, φ *is continuous on* $\mathbb{R}$.
   $\lim\limits_{x\overset{<}{\longrightarrow} -1} \dfrac{\varphi(x)-\varphi(-1)}{x+1} = \lim\limits_{x\overset{>}{\longrightarrow} 1} \dfrac{\varphi(x)-\varphi(1)}{x-1} = 0.$
   Making the change of variable $y = \dfrac{1}{1+x}$ *in the right neighbourhood of* -1 *and* $y = \dfrac{1}{1-x}$ *in the left neighbourhood of* -1, *we obtain:*
   $\lim\limits_{x\overset{>}{\longrightarrow} -1} \dfrac{\varphi(x)-\varphi(-1)}{x+1} = \lim\limits_{x\overset{<}{\longrightarrow} 1} \dfrac{\varphi(x)-\varphi(1)}{x-1} = \lim\limits_{y\longrightarrow +\infty} y e^{-\frac{y^2}{2y-1}} = 0.$
   Therefore, φ *est is differentiable on* $\mathbb{R}$ *and we have:* $\varphi'(x) = \begin{cases} -\dfrac{2x}{(1-x^2)^2} e^{-\frac{1}{1-x^2}} & : \ |x| < 1 \\ 0 & : \ |x| \geq 1 \end{cases}$
   Following the same method, we find that $\varphi \in \mathscr{C}^\infty(\mathbb{R})$, *and therefore,* $\varphi_j \in \mathscr{C}^\infty(\mathbb{R})$.

2. *We have:* $\operatorname{supp} \varphi = [-1, 1]$ *and pour tout* $x \in \mathbb{R}$: $x \in \operatorname{supp} \varphi_j$ *iff* $jx \in [-1, 1]$.
   Then, $\operatorname{supp} \varphi_j = \left[-\dfrac{1}{j}, \dfrac{1}{j}\right]$.

3. *Generalization of* φ *to* $\mathbb{R}^n$: $\varphi(x) = \begin{cases} e^{-\frac{1}{1-\|x\|^2}} & : \|x\| < 1 \\ 0 & : \|x\| \geq 1 \end{cases}$.

Solution 1.2 : $\varphi \in \mathscr{D}(]0,2[), \varphi \geq 0, \varphi = 1$ *sur* $\left[\frac{1}{2}, \frac{3}{2}\right]$, $\forall j \in \mathbb{Z}, \forall x \in \mathbb{R} : \varphi_j(x) = \varphi(x+j)$.



1. $\psi(x) = \sum_{j\in\mathbb{Z}} \varphi_j(x), x \in \mathbb{R}$.
   Let $j \in \mathbb{Z}$. *since* $\operatorname{supp}\varphi \subset]0,2[$, *we have for all* $x \in \mathbb{R}$: $\varphi_j(x) = 0$ *if* $j \leq -x$ *ou* $j \geq 2 - x$. *Hence:* $\psi(x) = \sum_{-x<j<2-x} \varphi_j(x)$, *i.e,* ψ *is defined.*
   There exists always $j \in \mathbb{Z}$ *such that* $j + x \in \left[\frac{1}{2}, \frac{3}{2}\right]$. *Then,* $\psi(x) \geq \varphi(1) = 1 > 0$.

2. $\forall j \in \mathbb{Z}, \forall x \in \mathbb{R} : u_j(x) = \frac{\varphi_j(x)}{\psi(x)}$.
   since $\varphi_j \in \mathscr{D}(\mathbb{R}^n)$ *et* $\psi \in \mathscr{C}^\infty(\mathbb{R}^n), \psi > 0$, *Then,* $u \in \mathscr{D}(\mathbb{R}^n)$?
   $$\forall x \in \mathbb{R} : \sum_{j\in\mathbb{Z}} u_j(x) = \sum_{j\in\mathbb{Z}} \frac{\varphi_j(x)}{\psi(x)} = \frac{\sum_{j\in\mathbb{Z}} \varphi_j(x)}{\psi(x)} = \frac{\psi(x)}{\psi(x)} = 1.$$

Solution 1.3 : $K \subseteq \mathbb{R}^n$ *compact,* $\psi \in \mathscr{D}(\mathbb{R}^n)$.

1. $\psi = 0$ *in the neighbourhood of* $K \Rightarrow \operatorname{supp}\psi \subset (\mathbb{R}^n \backslash K)$ *true, indeed:*
   Assume that $\psi = 0$ *in the neighbourhood of* K. *Then: there exists an open set* $O \supset K$ *such that* $\psi = 0$ *on* O. *Then,* O *included in the null open of* ψ.
   Then, $\operatorname{supp}\psi \subset \mathbb{R}^n \backslash O \subset (\mathbb{R}^n \backslash K)$.

2. $\psi = 0$ *on* $K \Rightarrow \operatorname{supp}\psi \subset (\mathbb{R}^n \backslash K)$ *false. Here's a counterexample:*
   Consider $\varphi \in \mathscr{D}(\mathbb{R}^n)$ *such that* $\operatorname{supp}\varphi \subset B(0,2)$ *and* $\varphi = 1$ *on* $B(0,1)$. *Setting:* $\psi(x) = \theta(x).\varphi(x)$, *où* $\theta(x) = x_1^2 + \cdots + x_n^2$.
   $\psi \in \mathscr{D}(\mathbb{R}^n)$ *and* $\psi = 0$ *on the compact set* $K = \{0\}$, *but* $\operatorname{supp}\psi \supset B(0,1)$.

Solution 1.4 : $\varphi \in \mathscr{D}(\mathbb{R}^n), h \in \mathbb{R}^n \backslash \{0\}. \forall t \in \mathbb{R}^* : \varphi_t(x) = \dfrac{\varphi(x+th) - \varphi(x)}{t}$.

1. *The functions:* $x \mapsto \varphi(x+th)$ *and* $x \mapsto \varphi(x)$ *belong to* $\mathscr{D}(\mathbb{R}^n)$ *with* t *being a constant with respect to* x. *Therefore,* $\varphi_t \in \mathscr{D}(\mathbb{R}^n)$ *for all* $t \neq 0$.

2. *We have:* $\lim\limits_{t\to 0} \varphi_t(x) = \varphi'_h(x)$, *where* φ'_h *is the derivative of* φ *in the direction of the vector* h. *Since* $\varphi \in \mathscr{D}(\mathbb{R}^n)$, *then* $\varphi'_h \in \mathscr{D}(\mathbb{R}^n)$. *Furthermore,* $\operatorname{supp} \varphi'_h \subset \operatorname{supp} \varphi$.

   For t *small enough, we can find a compact set* $K \subset \mathbb{R}^n$ *such that* $\operatorname{supp} \varphi_t \subset K$, *and of course,* $\operatorname{supp} \varphi'_h \subset K$.

   Finally, for any $m \in \mathbb{N}$, *we have:*

   $\lim\limits_{t\to 0} P_{K,m}(\varphi_t(x) - \varphi' h(x)) = \lim t \to 0 \sup\limits_{x\in K, |\alpha|\leq m} |D^\alpha(\varphi_t(x) - \varphi'_h(x))| = 0.$

   Therefore, if t *tends to* 0, φ_t *converges in* $\mathscr{D}(\mathbb{R}^n)$ *to* φ'_h.

Solution 1.5 : $\{f_j\}_{j\in\mathbb{N}}$ *sequence of functions de* $\mathscr{D}(\mathbb{R})$ *defined by :*

$$f_j(x) = \begin{cases} \dfrac{1}{2^j} \exp\left(-\dfrac{1}{1-\frac{x^2}{j^2}}\right) & : \ |x| < j \\ 0 & : \ |x| \geq j \end{cases}$$

Consider the function φ, *defined as follows:*

$$\varphi(x) = \begin{cases} \exp\left(-\dfrac{1}{1-x^2}\right) & : \ |x| < 1 \\ 0 & : \ |x| \geq 1 \end{cases}$$

Similar to the exercise 1.1, we can show that $\varphi \in \mathscr{D}(\mathbb{R})$. *Furthermore, we have* $f_j = \dfrac{1}{2^j}\varphi \circ g_n$ *where* $g_j(x) = \dfrac{x}{j}$. *Therefore,* $f_j \in \mathscr{D}(\mathbb{R})$ *and* $\operatorname{supp} f_j = [-j, j]$.

For any $k \in \mathbb{N}$, *we have:* $f_j^{(k)} = \dfrac{1}{j^k.2^j}\varphi^{(k)} \circ g_j$. *Therefore,* $\lim\limits_{j\to+\infty} f_j^{(k)}(x) = 0$.
Let K *be a compact set in* $\mathbb{R}$. *We have:*
$\sup\limits_{x\in K} |f_j^{(k)}(x) - 0| = \dfrac{1}{j^k.2^j} \sup x \in K |\varphi^{(k)} \circ g_j| \longrightarrow 0$ *as* $j \to +\infty$.

Therefore, for each $k \geq 0$, *the sequence of functions* $\{f_j^{(k)}\}$ *converges uniformly on every compact set* K *to the function* $f = 0$.

However, since $\operatorname{supp} f_j = [-j, j]$, *we cannot find a compact set* K *that contains all the supports of the sequence* $\{f_j\}$. *Therefore, there is no convergence in* $\mathscr{D}(\mathbb{R})$.

CHAPTER 2

DISTRIBUTIONS: DEFINITIONS AND PROPERTIES

In this chapter, we provide the definition of a distribution, differentiation, along with some examples and properties. First, we provide a brief motivation.

The Dirac delta function on $\mathbb{R}$ is defined as follows:

$$\delta(A) = \begin{cases} 1 & : \quad 0 \in A, \\ 0 & : \quad 0 \notin A, \end{cases} \tag{2.1}$$

and the function of Heaviside H defined by:

$$H(x) = \begin{cases} 1 & : \quad x \geq 0, \\ 0 & : \quad x < 0, \end{cases} \tag{2.2}$$

can be considered as Radon measures on $\mathbb{R}$, i.e., continuous linear functionals on $\mathscr{K}(\mathbb{R})$.

I) Let $f \in L^1_{loc}(\mathbb{R})$ be a differentiable function such that $f' \in L^1_{loc}(\mathbb{R})$. Then, both f and f' can be considered as Radon measures. Thus, for any $\varphi \in \mathscr{K}(\mathbb{R})$, for ε small enough, we consider the perturbation $f_\varepsilon(x) = f(x+\varepsilon)$ of f, which is also in $L^1_{loc}(\mathbb{R})$. The Dominated Convergence Theorem of Lebesgue (Theorem 1.13 and Remark 1.4) allows us to write:

$$\begin{aligned} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\int_{-\infty}^{+\infty} f(x+\varepsilon)\varphi(x)dx - \int_{-\infty}^{+\infty} f(x)\varphi(x)dx \right] &= \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \varphi(x)dx \\ &= \int_{-\infty}^{+\infty} f'(x)\varphi(x)dx \end{aligned}$$

We have: $\lim\limits_{\varepsilon \to 0} \dfrac{f_\varepsilon - f}{\varepsilon} = f'$, in $\mathscr{K}'(\mathbb{R}) = \mathscr{M}(\mathbb{R})$.

II) $H \in L^1_{loc}(\mathbb{R})$, but it is not differentiable in the usual sense. We will seek an alternative notion of differentiation for H. Let $\varphi \in \mathscr{K}(\mathbb{R})$.

On the other hand, for Φ as an antiderivative of φ and $\varepsilon > 0$ sufficiently small, we have:

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{\langle H_\varepsilon - H,\varphi\rangle}{\varepsilon} &= \lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty}\frac{H(x+\varepsilon)-H(x)}{\varepsilon}\varphi(x)dx \\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_{-\infty}^{+\infty}H(x+\varepsilon)\varphi dx - \int_{-\infty}^{+\infty}H(x)\varphi dx\right] \\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_{-\varepsilon}^{+\infty}\varphi dx - \int_{0}^{+\infty}H(x)\varphi dx\right] \\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\varepsilon}^{0}\varphi(x)dx \\
&= \lim_{\varepsilon\to 0}\frac{\Phi(0)-\Phi(-\varepsilon)}{\varepsilon} \\
&= \varphi(0) \\
&= \delta(\varphi).
\end{aligned}$$

We say that H' exists in the weak sense on $\mathscr{K}(\mathbb{R})$, and we write: $H' = \delta$.

II) Let δ_ε be the perturbation of δ for ε small enough. Then, for $\varphi \in \mathscr{K}(\mathbb{R})$, we have:

$$\lim_{\varepsilon\to 0}\frac{\langle \delta_\varepsilon - \delta,\varphi\rangle}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{\varphi(\varepsilon)-\varphi(0)}{\varepsilon}.$$

This limit exists only for differentiable functions with compact support, i.e., for $\varphi \in \mathscr{D}^1(\mathbb{R})$. Therefore, δ' is not a measure. Specifically, $\delta' \in (\mathscr{D}^1(\mathbb{R}))'$.

Following this pattern, $\delta'', \delta^3, \ldots$ belong to the spaces $(\mathscr{D}^2(\mathbb{R}))', (\mathscr{D}^3(\mathbb{R}))', \ldots$.

The space that encompasses all of these spaces is called the space of **distributions**, it is the topological dual of $\mathscr{D}(\mathbb{R})$.

In the following, Ω is a non-empty open set in $\mathbb{R}^n$.

2.1 Definitions and examples

Definition 2.1 : *We call a distribution on Ω any continuous linear form on the vector space $\mathscr{D}(\Omega)$.*

In other words, a linear form $T : \mathscr{D}(\Omega) \to \mathbb{R}$ is a distribution if and only if:

For every compact set $K \subset \Omega$, there exist $m \in \mathbb{N}$ and $M > 0$ such that:

$$\forall\varphi \in \mathscr{D}_K(\Omega) : |\langle T,\varphi\rangle| \leq M.P_{K,m}(\varphi), \tag{2.3}$$

where $T(\varphi)$ is denoted by $\langle T,\varphi\rangle$ (duality bracket).

We denote by $\mathscr{D}'(\Omega)$ the space of distributions on $\mathscr{D}(\Omega)$.

Definition 2.2 (***convergence in*** $\mathscr{D}'(\Omega)$) : *We say that a sequence of distributions $\{T_j\}_{j=1}^{+\infty}$ converges to T in $\mathscr{D}'(\Omega)$ if:*

$$\forall\varphi \in \mathscr{D}(\Omega), \lim_{j\to+\infty}\langle T_j,\varphi\rangle = \langle T,\varphi\rangle$$

Proposition 2.1 : *Let $T_j{}_{j=1}^{+\infty}$ be a sequence of distributions. Suppose that the numerical sequence $\langle T_j, \varphi\rangle$ converges to a limit $\ell(\varphi)$. We define a linear form T as follows:*

$$\forall \varphi \in \mathscr{D}(\Omega) : \langle T, \varphi\rangle = \ell(\varphi).$$

Then, $T \in \mathscr{D}'(\Omega)$.

Proof: One can simply apply Corollary 1.2 of the Banach-Steinhaus theorem, considering the space $\mathscr{D}(\Omega)$ as a Fréchet space and $\mathbb{R}$ as a Banach space (hence, locally convex and metrizable). ■

Definition 2.3 (***order of distribution***) : *If $T \in \mathscr{D}'(\Omega)$, we know that for every compact set $K \subset \Omega$, there exist $m \in \mathbb{N}$ and $M > 0$ such that*

$$\forall \varphi \in \mathscr{D}_K(\Omega) : |\langle T, \varphi\rangle| \leq M.P_{K,m}(\varphi)$$

If m is independent of K, we say that the distribution T is of finite order.

The order of T is the smallest m that satisfies this property.

Remark 2.1 : *It can be shown that a distribution of order m is a continuous linear form on $\mathscr{D}^m(\Omega)$, and conversely, if we equip the space $\mathscr{D}^m(\Omega)$ with the topological structure generated by the family of semi-norms $P_{k,j}(\Omega)$ (where $0 \leq j \leq m$), it is easy to see that if T is a distribution of order m on Ω, then $T \in (\mathscr{D}^m(\Omega))'$ and the inverse.*

We denote by $\mathscr{E}'^m(\Omega)$ the space of distributions of order m. This space can be equipped with either the strong topology or the weak topology (see §1.3)

Remark 2.2 : *Radon measures on Ω are distributions of order 0 on Ω.*

Definition 2.4 (***positive distribution***) : *We say that a distribution T on Ω is positive if:*

$$\forall \varphi \in \mathscr{D}(\Omega) : \varphi \geq 0 \Rightarrow \langle T, \varphi\rangle \geq 0$$

Example 2.1 : *The functional defined by $\langle T, \varphi\rangle = 0$ for all $\varphi \in \mathscr{D}(\Omega)$ is the zero distribution on Ω.*

Example 2.2 : *Let $c \in \mathbb{R}$, and let T be the functional defined as follows:*

$$\forall \varphi \in \mathscr{D}(\Omega) : \langle T, \varphi\rangle = \int_\Omega c.\varphi(x)dx.$$

T is the constant distribution that equals c on $\mathscr{D}(\Omega)$; it is a Radon measure.

Indeed, consider $K \subset \Omega$ *as a compact set and* $\varphi \in \mathscr{D}_K(\Omega)$*. We have:*

$$\begin{aligned} |\langle T, \varphi\rangle| &= \left|\int_\Omega c.\varphi(x)dx\right| \\ &\leq |c|.\int_\Omega |\varphi(x)|dx \\ &= |c|.\int_K |\varphi(x)|dx \\ &\leq |c|.mes(K).\sup_{x\in K}|\varphi(x)| \\ &= |c|.mes(K).P_{K,0}(\varphi). \end{aligned}$$

Example 2.3 : *The Dirac measure* δ_a *at the point* $a \in \mathbb{R}^n$ *is defined as follows:*

$$\forall \varphi \in \mathscr{D}(\Omega) : \langle \delta_a, \varphi\rangle = \varphi(a)$$

δ_a *is a Radon measure (in particular:* $\delta_0 = \delta$*).*

Indeed, let $K \subset \Omega$ *be a compact set and* $\varphi \in \mathscr{D}_K(\Omega)$*. Then cases:*
if $a \in K$*, then,* $|\varphi(a)| \leq \sup_{x\in K}|\varphi(x)|$*,*
If $a \notin K$*, we have* $|\varphi(a)| = 0 \leq \sup_{x\in K}|\varphi(x)|$*.*
Hence:

$$\begin{aligned} |\langle \delta_a, \varphi\rangle| &= |\varphi(a)| \\ &\leq \sup_{x\in K}|\varphi(x)| \\ &= P_{K,0}(\varphi). \end{aligned}$$

Example 2.4 : *Consider the distribution* T *defined for any point* $a \in \mathbb{R}^n$ *and for any* $\alpha \in \mathbb{N}^n$ *as follows:*

$$\forall \varphi \in \mathscr{D}(\Omega) : \langle T, \varphi\rangle = D^\alpha\varphi(a)$$

T *is a distribution of order* $m \leq |\alpha|$ *(it can be shown that* $m = |\alpha|$*).*

Indeed, consider $K \subset \Omega$ *as a compact set and* $\varphi \in \mathscr{D}_K(\Omega)$*. As before:*

$$\begin{aligned} |\langle T, \varphi\rangle| &= |D^\alpha\varphi(a)| \\ &\leq \sup_{x\in K}|D^\alpha\varphi(x)| \\ &\leq \sup_{x\in K}\sup_{|\beta|\leq|\alpha|}|D^\beta\varphi(x)| \\ &= P_{K,|\alpha|}(\varphi). \end{aligned}$$

Now, let $\psi \in \mathscr{D}(\Omega)$ *such that* $\psi(a) = 1$ *(the function* ψ *exists according to the Urysohn's lemma, see Theorem 1.6). We define* $\varphi_0(x) = (x-a)^\alpha\psi(x)$ *for all* $x \in \Omega$*. Then, for all* $\beta \in \mathbb{N}^n$ *such that* $|\beta| \leq |\alpha|$*, we have:*

$$\begin{aligned} D^\beta\varphi_0(x) &= \sum_{\gamma\leq\beta} C_\alpha^\beta D^\beta(x-a)^\alpha.D^{\beta-\gamma}\psi(x) \\ &= \sum_{\gamma\leq\beta} C_\alpha^\beta (x-a)^{\beta-\gamma}.D^{\beta-\gamma}\psi(x) \end{aligned}$$

Then: $D^\beta \varphi_0(a) = 0$ *pour* $|\beta| < |\alpha|$ *et* $D^\beta \varphi_0(a) = 1$.
We deduce that $|\langle T, \varphi_0 \rangle| = P_{K,|\alpha|}(\varphi_0)$, *hence:* T *is of order* $|\alpha|$.

Example 2.5 : *Let* $f \in L^1_{loc}(\Omega)$. *We can associate a distribution* T_f *defined as follows:*

$$\forall \varphi \in \mathscr{D}(\Omega) : \langle T_f, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx$$

The distribution T_f *is a Radon measure on* Ω. *Writing* $|\langle T_f, \varphi \rangle| = |\langle f, \varphi \rangle|$ *for* $\varphi \in \mathscr{D}(\Omega)$.

Indeed, consider $K \subset \Omega$ *as a compact set and* $\varphi \in \mathscr{D}_K(\Omega)$. *Then:*

$$\begin{aligned} |\langle T_f, \varphi \rangle| &= \left| \int_\Omega f(x)\varphi(x)dx \right| \\ &\leq \int_\Omega |f(x)\varphi(x)|dx \\ &= \int_K |f(x)\varphi(x)|dx \\ &\leq \sup_{x\in K} |\varphi(x)| \int_K |f(x)|dx \\ &= \|f\|_{L^1(K)}.P_{K,0}(\varphi). \end{aligned}$$

Example 2.6 : *The Heaviside function* H *defined by* (2.2) *belongs to* $L^1_{loc}(\mathbb{R})$, *and it defines a distribution on* $\mathscr{D}(\mathbb{R})$. *It is a Radon measure.*

Example 2.7 : *For* $f \in L^1_{loc}(\mathbb{R}^2)$, *we define:*

$$\forall \varphi \in \mathscr{D}(\mathbb{R}^3) : \langle T, \varphi \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y)\varphi(x,y,0)dxdy$$

T *is a Radon measure, known as the simple layer distribution with density* f.

Indeed, consider $K \subset \Omega$ *as a compact set and* $\varphi \in \mathscr{D}_K(\Omega)$. *Let* $K' = K \cap (\mathbb{R}^2 \times 0)$, *which is a compact set in* $\mathbb{R}^2$. *Then:*

$$\begin{aligned} |\langle T, \varphi \rangle| &= \left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y)\varphi(x,y,0)dxdy \right| \\ &\leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(x,y)\varphi(x,y,0)|dxdy \\ &= \int_{K'} |f(x,y)\varphi(x,y,0)|dxdy \\ &\leq \sup_{(x,y,z)\in K} |\varphi(x,y,z)| \int_{K'} |f(x,y)|dxdy \\ &= \|f\|_{L^1(K')}.P_{K,0}(\varphi). \end{aligned}$$

Example 2.8 : *For* $f \in L^1_{Loc}(\mathbb{R}^2)$, *we set:*

$$\forall \varphi \in \mathscr{D}(\mathbb{R}^3) : \langle T, \varphi \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y)\frac{\partial \varphi}{\partial z}(x,y,0)dxdy$$

T *is a distribution of order* 1, *called the double layer distribution with density* f.

Indeed, consider K as a compact set in Ω, and φ in $D_K(\Omega)$. Let $K' = K \cap (\mathbb{R}^2 \times \{0\})$, which is a compact set in $\mathbb{R}^2$. Then:

$$\begin{aligned}
|\langle T, \varphi\rangle| &= \left|\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\frac{\partial\varphi}{\partial z}(x,y,0)dxdy\right| \\
&\leq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(x,y)\frac{\partial\varphi}{\partial z}(x,y,0)|dxdy \\
&= \int_{K'} |f(x,y)\frac{\partial\varphi}{\partial z}(x,y,0)|dxdy \\
&\leq \sup_{(x,y,z)\in K} |\varphi(x,y,z)| \int_{K'} |f(x,y)|dxdy \\
&= \|f\|_{L^1(K')}.P_{K,1}(\varphi).
\end{aligned}$$

Now, let $\psi \in \mathscr{D}(\mathbb{R})$ be such that $\psi(x,y,z) = 1$ in a compact neighbourhood K_0 of $(0,0,0)$ (ψ exists according to Urysohn's lemma, see Theorem 1.6).
Setting: $\varphi_0(x,y,z) = z\psi(x,y,z)$. Then: $\dfrac{\partial\varphi_0}{\partial z}(x,y,0) = \psi(x,y,0) = 1$ on $\overset{0}{K_0}$.
We deduce that $P_{K_0,1}(\varphi) \geq 1$.
Thus, T is of order 1.

Example 2.9 : *The Cauchy principal value distribution $vp_{\frac{1}{x}}$ is a distribution of order 1 defined as follows:*

$$\forall\varphi \in \mathscr{D}_K(\mathbb{R}) : \langle vp_{\frac{1}{x}}, \varphi\rangle = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}dx.$$

Indeed, let $a > 0$, $K \subset [-a,a]$ be a compact set and $\varphi \in \mathscr{D}_K(\mathbb{R})$.
Setting: $\psi(x) = \displaystyle\int_0^1 \varphi'(tx)dt$. Therefore, $\psi(0) = \varphi'(0)$ et $\dfrac{\varphi(x)}{x} = \dfrac{\varphi(0)}{x} + \psi(x)$ for $x \neq 0$.

$$\begin{aligned}
\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}dx &= \int_{-a}^{-\varepsilon} \frac{\varphi(x)}{x}dx + \int_{\varepsilon}^{a} \frac{\varphi(x)}{x}dx \\
&= \int_{-a}^{-\varepsilon} \left[\frac{\varphi(0)}{x} + \psi(x)\right]dx + \int_{\varepsilon}^{a} \left[\frac{\varphi(0)}{x} + \psi(x)\right]dx \\
&= \left[-\frac{\varphi(0)}{x}\right]_{-a}^{-\varepsilon} + \left[-\frac{\varphi(0)}{x}\right]_{a}^{\varepsilon} + \int_{|x|>\varepsilon} \psi(x)dx \\
&= \int_{|x|>\varepsilon} \psi(x)dx.
\end{aligned}$$

Hence, $\langle vp_{\frac{1}{x}}, \varphi\rangle = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}dx = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \psi(x)dx = \int_{-a}^{a} \psi(x)dx.$

Noting that $\int_{-a}^{a} \psi(x), dx$ *exists because* ψ *is continuous, then*

$$\begin{aligned} |\langle vp_{\frac{1}{x}}, \varphi\rangle| &= \left|\int_{-a}^{a} \psi(x)dx\right| \\ &= \left|\int_{-a}^{a}\int_{0}^{1} \varphi'(tx)dtdx\right| \\ &\leq \int_{-a}^{a}\int_{0}^{1} |\varphi'(tx)|dtdx \\ &\leq 2a \sup_{x\in K} |\varphi'(x)| \\ &= 2aP_{K,1}(\varphi) \end{aligned}$$

So, $vp_{\frac{1}{x}}$ *is a distribution of order less than or equal to* 1.

To show that $vp_{\frac{1}{x}}$ *is of order* 1 *see exercise 2.2*

> If we can express a distribution T on Ω in the form $\int_{\Omega} f(x)\varphi(x), dx$, we say that T is a **regular** distribution, and f is the associated function to T. Otherwise, we say that T is a **singular** distribution.
> For example, the Heaviside function defines a regular distribution, and the Dirac measure is a singular distribution.

2.2 Properties and Results

Proposition 2.2 : *Let* (f_j) *a sequence in* $L^1(\mathbb{R}^n)$ *such that for all* $j \in \mathbb{N}$*:*

1. $f_j \geq 0$ *and* $\int_{\mathbb{R}^n} f_j(x)dx = 1,$
2. $\operatorname{supp} f_j \subset B(0, \varepsilon_j)$ *where* $\lim_{j\to+\infty} \varepsilon_j = 0.$ *for all* $j \in \mathbb{N}$.

Then: $f_j \to \delta$ *in* $\mathscr{D}'(\mathbb{R}^n)$.

Proof: Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$. since $\operatorname{supp} f_j \subset B(0, \varepsilon_j)$ we have:

$$\langle f_j, \varphi\rangle = \int_{\mathbb{R}^n} f_j(x)dx = \int_{B(0,\varepsilon_j)} f_j(x)dx = 1,$$

$$\langle f_j, \varphi\rangle = \int_{\mathbb{R}^n} f_j(x)\varphi(x)dx = \int_{B(0,\varepsilon_j)} f_j(x)\varphi(x)dx.$$

We deduce that:

$$\begin{aligned}
|\langle f_j,\varphi\rangle-\langle\delta,\varphi\rangle| &= \left|\int_{B(0,\varepsilon_j)} f_j(x)\varphi(x)dx-\varphi(0)\right|\\
&= \left|\int_{B(0,\varepsilon_j)} f_j(x)\varphi(x)dx-\int_{B(0,\varepsilon_j)} f_j(x)\varphi(0)\right|\\
&= \left|\int_{B(0,\varepsilon_j)} f_j(x)(\varphi(x)-\varphi(0))dx\right|\\
&\leq \int_{B(0,\varepsilon_j)} f_j(x)|\varphi(x)-\varphi(0)|dx\\
&\leq \sup_{x\in\overline{B}(0,\varepsilon_j)}|\varphi(x)-\varphi(0)|\int_{B(0,\varepsilon_j)}|f_j(x)dx\\
&= \sup_{x\in\overline{B}(0,\varepsilon_j)}|\varphi(x)-\varphi(0)|.
\end{aligned}$$

For all $j\in\mathbb{N}$, there exists $x_j\in B(0,\varepsilon_j)$ such that $\sup_{x\in\overline{B}(0,\varepsilon_j)}|\varphi(x)-\varphi(0)|=|\varphi(x_j)-\varphi(0)|$ (since $\overline{B}(0,\varepsilon_j)$ is a compact set and φ is continuous). Moreover, we have: $\lim_{j\to+\infty}\varphi(x_j)=\varphi(0)$. Then:

$$0\leq\lim_{j\to+\infty}|\langle f_j,\varphi\rangle-\langle\delta,\varphi\rangle|\leq\lim_{j\to+\infty}|\varphi(x_j)-\varphi(0)|=0.$$

Hence, the result. ■

Theorem 2.1 (***Lemma of Dubois-Reymond***) : *Let $f\in L^1_{loc}(\Omega)$, and T_f be the distribution defined as follows:*

$$\forall\varphi\in\mathscr{D}(\Omega):\langle T_f,\varphi\rangle=\int_\Omega f(x)\varphi(x)dx$$

The following two properties are equivalent:

i) *$T_f=0$ in $\mathscr{D}'(\Omega)$,*

ii) *$f=0$ a.e in Ω.*

Therefore, if $f,g\in L^1_{loc}(\Omega)$, then $f=g$ almost everywhere on Ω if and only if $T_f=T_g$.

Proof: The implication **ii)** $\Rightarrow$ **i)** is immediate. We will now prove the implication **i)** $\Rightarrow$ **ii)**.

First Method: Let $K\subset\Omega$ be a compact set. We define $\delta_K=d(K,C^{\Omega}_{\mathbb{R}^n})$. Choose $\varepsilon<\delta_K$, and let χ_K be the characteristic function of K.

Consider a sequence $\{\psi_j\}_{j\in\mathbb{N}^*}\subset\mathscr{D}(\Omega)$ where $0\leq\psi_j\leq1$, $\psi_j=1$ on K, and $\operatorname{supp}\psi_j\subset B(0,\frac{\varepsilon}{j})$. (The existence of such a sequence follows from the Urysohn's Lemma, see Theorem 1.6).

The sequence $\{\psi_j\}j\in\mathbb{N}$ converges pointwise to χ_K. Consequently, $\{f\cdot\psi_j\}_{j\in\mathbb{N}}$ converges pointwise to $f\cdot\chi_K$. Moreover, we have $|f\cdot\psi_j|\leq f\cdot\chi_K$.

By the Dominated Convergence Theorem (Theorem 1.13), we can write:

$$\int_K f(x)dx = \int_\Omega f(x).\chi_K(x)dx = \lim_{j\to+\infty}\int_\Omega f(x).\psi_j(x)dx = 0$$

Since K is arbitrary, we can conclude from Theorem 1.11 that $f = 0$ a.e. on Ω.

Second Method: Let $\{K_j\}_{j\in\mathbb{N}}$ be an exhaustive sequence of Ω (see Proposition 1.12). Define $O_j = \overset{\circ}{K_j}$. We will show that $f = 0$ on every open set $(O_j)_{j\in\mathbb{N}}$.
Let $f_j = f_{|}O_j$, then $f_j \in L^1(O_j)$ (because O_j is bounded). Due to the density of $\mathscr{D}(O_j)$ in $L^1(O_j)$, for $\varepsilon > 0$, there exists $\psi_\varepsilon^j \in \mathscr{D}(O_j)$ such that $\|\psi_\varepsilon^j - f_j\|_{L^1(O_j)} < \varepsilon$.

Now, let $\varphi \in \mathscr{D}(O_j)$. Since $\displaystyle\int_{O_j} f_j(x)\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx = 0$, we obtain:

$$\int_{O_j}\psi_\varepsilon^j(x)\varphi(x) = \int_{O_j}(\psi_\varepsilon^j(x) - f_j(x))\varphi(x)dx + \int_{O_j} f_j(x)\varphi(xdx) = \int_{O_j}(\psi_\varepsilon^j(x) - f_j(x))\varphi(x)dx.$$

Then:

$$\left|\int_{O_j}\psi_\varepsilon^j(x)\varphi(x)dx\right| \le \int_{O_j}|\psi_\varepsilon^j(x) - f_j(x)|.|\varphi(x)|dx \le \varepsilon \sup_{x\in K_j}|\varphi(x)|.$$

Let $\eta > 0$. Set: $\varphi = \dfrac{\psi_\varepsilon^j}{\sqrt{\eta^2 + (\psi_\varepsilon^j)^2}} \in \mathscr{D}(O_j)$. Then: $|\varphi| \le 1$ and $\psi_\varepsilon^j.\varphi = \dfrac{(\psi_\varepsilon^j)^2}{\sqrt{\eta^2 + (\psi_\varepsilon^j)^2}}$.

So, $\displaystyle\int_{O_j}\frac{(\psi_\varepsilon^j)^2}{\sqrt{\eta^2 + (\psi_\varepsilon^j)^2}} \le \varepsilon$.

As we let η tend to 0, we obtain, according to the Dominated Convergence Theorem (Theorems 1.13 and 1.4): $\displaystyle\int_{O_j}|\psi_\varepsilon^j| \le \varepsilon$. Hence:

$$\|f_j\|_{L^1(O_j)} \le \|f_j - \psi_\varepsilon^j\|_{L^1(O_j)} + \|\psi_\varepsilon^j\|_{L^1(O_j)} \le 2\varepsilon, \forall\varepsilon > 0.$$

This implies that $|f_j|_{L^1(O_j)} = 0$, i.e., $f = 0$ a.e. on O_j, and consequently, within K_j, for all $j \in \mathbb{N}$. Since $\{K_j\}$ is a covering of Ω, we can conclude that $f = 0$ a.e. in Ω. ■

Proposition 2.3 : *Let (f_j) be a sequence in $L^1(\Omega)$ converge a.e. to a function f.*
Assume that there exists a function $g \in L^1(\Omega)$ such that $f_j \le g$ a.e. for all $j \in \mathbb{N}$.
Then: $f \in L^1(\Omega)$ et $f_j \to f$ in $\mathscr{D}'(\Omega)$.

Proof: Let $\varphi \in \mathscr{D}(\Omega)$. We have: $\langle f_j, \varphi\rangle = \displaystyle\int_\Omega f_j(x)dx$ et $\langle f, \varphi\rangle = \displaystyle\int_\Omega f(x)dx$.
On sais d'après théorème de convergence dominée de Lebesgue (Théorème 1.13) que $f \in L^1(\Omega)$.
According to the Dominated Convergence Theorem of Lebesgue (Theorem 1.13), we know that $f \in L^1(\Omega)$.

Let's consider the function $h_j = f_j\varphi$. The sequence (h_j) is in $L^1(\Omega)$, converges almost everywhere to the function $h = f\varphi$, and since φ is bounded, there exists $M > 0$ such that $h_j \le Mg$ almost everywhere for all $j \in \mathbb{N}$. The function Mg belongs to $L^1(\Omega)$. By applying the Dominated Convergence Theorem of Lebesgue (Theorem 1.13), we get:

$$\lim_{j\to+\infty}\int_\Omega h_j(x)dx=\int_\Omega h(x)dx,\ \text{i.e}\ \lim_{j\to+\infty}\int_\Omega f_j(x)\varphi(x)dx=\int_\Omega f(x)\varphi(x)dx.$$

Then: $\lim\limits_{j\to+\infty}\langle f_j,\varphi\rangle=\langle f,\varphi\rangle$. Hence: $f_j\to f$ in $\mathscr{D}'(\Omega)$. ■

Remark 2.3 *: The theorem above remains valid if we consider a sequence in $L^1_{loc}(\Omega)$.*

2.3 Derivation

Before providing the definition of the derivative of a distribution, we present the following important result:

Proposition 2.4 *: Let T be a distribution on Ω, and let T_i $(1\le i\le n)$ be the linear functional on $\mathscr{D}(\Omega)$ defined as follows:*

$$\forall\varphi\in\mathscr{D}(\Omega):\langle T_i,\varphi\rangle=\left\langle T,\frac{\partial\varphi}{\partial x_i}\right\rangle.$$

Then, T_i is a distribution on Ω.

Proof: Let $K\subset\Omega$ be a compact set, and let $\varphi\in\mathscr{D}_K(\Omega)$. Then, $\psi=\frac{\partial\varphi}{\partial x_i}\in\mathscr{D}_K(\Omega)$. Therefore, there exist $M>0$ and $m\in\mathbb{N}$ such that:

$$|\langle T,\psi\rangle|\le M.P_{K,m}(\psi)=M.P_{K,m}\left(\frac{\partial\varphi}{\partial x_i}\right)\le M.P_{K,m+1}(\varphi)$$

. So, we have:

$$|\langle T_i,\varphi\rangle|=|\langle T,\psi\rangle|\le M.P_{K,m+1}(\varphi).$$

Therefore, T_i is a distribution on Ω. ■

Now, let's proceed with the next definition:

Definition 2.5 : *For $T\in\mathscr{D}'(\Omega)$, the derivative of T (with respect to x_i) is defined as follows:*

$$\forall\varphi\in\mathscr{D}(\Omega):\left\langle\frac{\partial T}{\partial x_i},\varphi\right\rangle=-\left\langle T,\frac{\partial\varphi}{\partial x_i}\right\rangle \tag{2.4}$$

Remark 2.4 : *According to Proposition 2.4 and Definition 2.5:*

1. *We can show by induction that every distribution is infinitely differentiable.*
2. *If T is of order m, then $\dfrac{\partial T}{\partial x_i}$ is of order at most $m+1$.*

Proposition 2.5 : *Let T be a distribution on Ω. Then:*

$$\forall\alpha\in\mathbb{N}^n,\forall\varphi\in\mathscr{D}(\Omega):\langle D^\alpha T,\varphi\rangle=(-1)^{|\alpha|}\langle T,D^\alpha\varphi\rangle$$

Proof: Let $\alpha \in \mathbb{N}^n$ and $\varphi \in \mathscr{D}(\Omega)$. Then:

$$\langle D^{\alpha_i}T, \varphi\rangle = -\langle D^{\alpha_i - 1}T, \frac{\partial \varphi}{\partial x_i}\rangle = \cdots (-1)^{\alpha_i}\langle T, D^{\alpha_i}\varphi\rangle.$$

Finally:
$\langle D^{\alpha}T, \varphi\rangle = \langle D^{\alpha_1}\cdots D^{\alpha_n}T, \varphi\rangle = (-1)^{\alpha_1}\cdots(-1)^{\alpha_n}\langle T, D^{\alpha_1}\cdots D^{\alpha_n}\varphi\rangle = (-1)^{\alpha}\langle T, D^{|\alpha|}\varphi\rangle.$ ■

Proposition 2.6 : *Let (T_j) be a sequence of distributions on Ω. If $T_j \to T$ in $\mathscr{D}'(\Omega)$, then for any multi-index α, we have $D^{\alpha}T_j \to D^{\alpha}T$ in $\mathscr{D}'(\Omega)$.*

We say that the differentiation operator is a continuous operator.

Proof: Let $\alpha \in \mathbb{N}^n$ and $\varphi \in \mathscr{D}(\Omega)$. Then: $D^{\alpha}\varphi \in \mathscr{D}(\Omega)$ and we have:

$$\begin{aligned}
|\langle D^{\alpha}T_j, \varphi\rangle - \langle D^{\alpha}T, \varphi\rangle| &= ||-1|^{\alpha}\langle T_j, D^{\alpha}\varphi\rangle - |-1|^{\alpha}\langle T, D^{\alpha}\varphi\rangle| \\
&= ||-1|^{\alpha}(\langle T_j, D^{\alpha}\varphi\rangle - \langle T, D^{\alpha}\varphi\rangle|)| \\
&= |\langle T_j, D^{\alpha}\varphi\rangle - \langle T, D^{\alpha}\varphi\rangle| \overset{j\to+\infty}{\longrightarrow} 0\ .
\end{aligned}$$

Therefore: $D^{\alpha}T_j \to D^{\alpha}T$ dans $\mathscr{D}'(\Omega)$. ■

Example 2.10 : *Let f be a differentiable function on $]a, b[$ such that $f' \in L^1_{loc}(]a, b[)$. For any $\varphi \in \mathscr{D}(]a, b[)$, there exist a_0, b_0 such that $\operatorname{supp}\varphi \subset [a_0, b_0] \subset]a, b[$. Then:*

$$\begin{aligned}
\langle (T_f)', \varphi\rangle &= -\langle T_f, \varphi'\rangle \\
&= -\int_a^b f(x)\varphi'(x)dx \\
&= -\int_{a_0}^{b_0} f(x)\varphi'(x)dx \\
&= -[f(x)\varphi(x)]_{a_0}^{b_0} + \int_{a_0}^{b_0} f'(x)\varphi(x)dx \\
&= f(a_0)\varphi(a_0) - f(b_0)\varphi(b_0) + \int_{a_0}^{b_0} f'(x)\varphi(x)dx \\
&= \int_{a_0}^{b_0} f'(x)\varphi(x)dx \\
&= \langle T_{f'}, \varphi\rangle.
\end{aligned}$$

Then: $(T_f)' = T_{f'}$.

Example 2.11 : *Let H be the Heaviside function defined in (2.2). For any $\varphi \in \mathscr{D}(\mathbb{R})$, there exists $a > 0$ such that $\operatorname{supp}\varphi \subset [-a, a]$. Then:*

$$\begin{aligned}
\langle H', \varphi\rangle &= -\langle H, \varphi'\rangle \\
&= -\int_{-\infty}^{+\infty} H(x)\varphi'(x)dx \\
&= -\int_0^a \varphi'(x)dx \\
&= -[\varphi(x)]_0^a = \varphi(0) - \varphi(a) = \varphi(0) = \langle \delta, \varphi\rangle.
\end{aligned}$$

Hence: $H' = \delta$.

Example 2.12 : *Let* $f \in \mathscr{C}^1(\mathbb{R}\backslash\{a\})$. *Assuming that* f *and* f' *have a type 1 discontinuity (i.e., the limits* $\lim\limits_{x \overset{>}{\to} a} f(x) = f(a^+)$, $\lim_{x \overset{<}{\to} a} f(x) = f(a^-)$, $\lim_{x \overset{>}{\to} a} f'(x) = f'(a^+)$, *and* $\lim_{x \overset{<}{\to} a} f'(x) = f'(a^-)$ *exist and are finite). Let* $\varphi \in \mathscr{D}(\mathbb{R})$, *there exists* $A > 0$ *such that* $\operatorname{supp}\varphi \subset [-a, a]$ *and* $-A < a < A$. *Then:*

$$\begin{aligned}
\langle (T_f)', \varphi\rangle &= -\langle T_f, \varphi'\rangle \\
&= -\int_{-\infty}^{+\infty} f(x)\varphi'(x)dx \\
&= -\int_{-A}^{A} f(x)\varphi'(x)dx \\
&= -\int_{-A}^{a} f(x)\varphi'(x)dx - \int_{a}^{A} f(x)\varphi'(x)dx \\
&= -[f(x)\varphi(x)]_{-A}^{a} + \int_{-A}^{a} f'(x)\varphi(x)dx - [f(x)\varphi(x)]_{a}^{A} + \int_{a}^{A} f'(x)\varphi(x)dx \\
&= -f(a^-)\varphi(a) + \int_{-A}^{a} f'(x)\varphi(x)dx + f(a^+)\varphi(a) + \int_{a}^{A} f'(x)\varphi(x)dx \\
&= (f(a^+) - f(a^-))\varphi(a) + \int_{-A}^{A} f'(x)\varphi(x)dx \\
&= \langle T_{f'}, \varphi\rangle + (f(a^+) - f(a^-))\langle \delta_a, \varphi\rangle.
\end{aligned}$$

Then: $(T_f)' = T_{f'} + (f(a^+) - f(a^-))\delta_a$.

The following lemma is important for proving the subsequent theorem:

Lemma 2.1 : *Let* (a, b) *be an open interval in* $\mathbb{R}$.

1. φ *has an antiderivative in* $\mathscr{D}(a, b)$ *if and only if* $\displaystyle\int_a^b \varphi(x)dx = 0$.
2. *If* φ *has an antiderivative in* $\mathscr{D}(a, b)$, *then this antiderivative is unique.*

Proof:

1. Suppose φ has an antiderivative $\psi \in \mathscr{D}(a, b)$. Then:

$$\int_a^b \varphi(x)dx = \int_a^b \psi'(x)dx = \psi(b) - \psi(a) = 0.$$

Conversely, assume that $\displaystyle\int_a^b \varphi(x)dx = 0$, and define $\psi(x) = \displaystyle\int_a^x \varphi(t)dt$. Then, $\psi' = \varphi$. We will show that $\operatorname{supp}\psi$ is compact. Since $\operatorname{supp}\varphi$ is compact, there exist a_0, b_0 such that $\operatorname{supp}\varphi \subset [a_0, b_0] \subset]a, b[$. This means $\varphi = 0$ on $]a, a_0[\cup]b_0, b[$. Take $x \in]a, a_0[$. Then: $\psi(x) = \displaystyle\int_a^x \varphi(t)dt = 0$. For $x \in]b_0, b[$, we have: $\psi(x) = \displaystyle\int_a^x \varphi(t)dt = \int_a^b \varphi(t)dt = 0$. Therefore, $\operatorname{supp}\psi \subset [a_0, b_0]$, so $\operatorname{supp}\psi$ is compact.

2. Let ψ_1 and ψ_2 be two antiderivatives of φ in $\mathscr{D}(a,b)$. Then, there exists $c \in \mathbb{R}$ such that $\psi_1 = c + \psi_2$. For $x \notin (\operatorname{supp}\psi_1 \cup \psi_2)$, we have: $0 = \psi_1(x) = c + \psi_2(x) = c$.
   Thus, $\psi_1 = \psi_2$.

■

Theorem 2.2 : *Let (a,b) be an open interval in $\mathbb{R}$.*

1. *The only distributions T on (a,b) such that $T' = 0$ are the constant distributions.*
2. *For any $T \in \mathscr{D}'(a,b)$, there exists $S \in \mathscr{D}'(a,b)$ such that $S' = T$ (every distribution has a primitive).*

Proof:

1. Let $\psi \in \mathscr{D}(a,b)$ be such that $\displaystyle\int_a^b \psi(x)dx = 1$. Set: $\langle T, \psi\rangle = c$.
   Let $\varphi \in \mathscr{D}(a,b)$. We define $\rho = \varphi - \psi.\displaystyle\int_a^b \varphi(x)dx$. Then, $\rho \in \mathscr{D}(a,b)$, and we have:
$$\int_a^b \rho(x)dx = \int_a^b \varphi(x)dx - \int_a^b \phi(x)dx.\int_a^b \varphi(x)dx = 0.$$
   Then: there exists $\theta \in \mathscr{D}(a,b)$ such that $\theta' = \rho$. So,
$$\begin{aligned}
\langle T, \varphi\rangle &= \langle T, \rho + \psi.\int_a^b \varphi(x)dx\rangle \\
&= \langle T, \theta' + \psi.\int_a^b \varphi(x)dx\rangle \\
&= \langle T, \theta'\rangle + \langle T, \psi\rangle.\int_a^b \varphi(x)dx \\
&= -\langle T', \theta\rangle + c.\int_a^b \varphi(x)dx \\
&= \int_a^b c.\varphi(x)dx \\
&= \langle c, \varphi\rangle.
\end{aligned}$$

2. Let T be a distribution on $]a,b[$. As before, we define $\theta' = \rho$, where $\rho = \varphi - \psi.\displaystyle\int_a^b \varphi(x)dx$ and $\displaystyle\int_a^b \psi(x)dx = 1$ (noting that θ is unique according to Lemma 2.1). We define: $\langle S, \varphi\rangle = -\langle T, \theta\rangle$. Then, for $K \in]a,b[$ and $\varphi \in \mathscr{D}_K(a,b)$, there exist $m \in \mathbb{N}$, $K' = K \cup \operatorname{supp}\theta$, and $M > 0$ such that:
$$\begin{aligned}
|\langle S, \varphi\rangle| &= |\langle T, \theta\rangle| \\
&\leq M_1 P_{K',m}(\theta) \\
&= M_1 \max\{\sup_{x\in K'}|\theta|, \sup_{x\in K', 1\leq k\leq m}|\theta^{(k)}|\}
\end{aligned}$$

Noting that:

$$\begin{aligned}
|\theta(x)| &= \int_a^x \rho(t)dt \\
&= \left|\int_a^x \varphi(t)dt + \int_a^x \psi(t)dt. \int_a^b \varphi(s)ds\right| \\
&= \int_a^x |\varphi(t)|dt + \int_a^x |\psi(t)|dt. \int_a^b |\varphi(s)|ds \\
&\leq \int_a^b |\varphi(t)|dt + \int_a^b |\psi(t)|dt. \int_a^b |\varphi(s)|ds \\
&= 2\int_a^b |\varphi(t)|dt \\
&\leq 2(b-a)\sup_{x\in K}|\varphi|,
\end{aligned}$$

and $\sup\limits_{x\in K',1\leq k\leq m} |\theta^{(k)}| \leq M_2 \sup\limits_{x\in K,0\leq k\leq m-1} |\varphi^{(k)}|$ (taking into account that $\varphi = 0$ outside of K).

We then get: $|\langle S,\varphi\rangle| \leq M.P_{K,m-1}(\varphi)$.

Now, let $\varphi \in \mathscr{D}(a,b)$. φ is a primitive of φ' in $\mathscr{D}(a,b)$. Then:

$\langle S',\varphi\rangle = -\langle S,\varphi'\rangle = \langle T,\varphi\rangle$.

Hence: S is a primitive of T.

■

Theorem 2.3 : *Let $T \in \mathscr{D}'(\mathbb{R}^n)$ such that for all $i = 1\cdots n$ we have: $\dfrac{\partial T}{\partial x_i} = 0$. Then: T is constant.*

Proof: We have $\dfrac{\partial T}{\partial x_i} = 0$, so T depends only on $x_2,\ldots,x_n$. Thus, step by step, we can prove that T is constant. ■

2.4 Operators on Distributions

Definition 2.6 (***restriction of distribution***) : *Let T be a distribution on Ω. For any open subset ω of Ω, we define the restriction T_ω of T as follows:*

$$\forall\varphi \in \mathscr{D}(\omega) : \langle T_\omega,\varphi\rangle = \langle T,\varphi\rangle.$$

The restriction of a distribution to ω is indeed a distribution on ω. This is because if we take $K \subset \omega$ as a compact set, we have $K \subset \Omega$, and for any $\varphi \in \mathscr{D}_K(\omega)$, we also have $\varphi \in \mathscr{D}_K(\Omega)$. Then, there exists $M > 0$ and $m \in \mathbb{N}$ such that $|\langle T,\varphi\rangle| \leq MP_{K,m}(\varphi)$.

Definition 2.7 (***translation of a distribution***) : *Let T be a distribution on $\mathbb{R}^n$, and let $a \in \mathbb{R}^n$. The translation $\tau_a T$ by the vector a is defined as follows:*

$$\forall\varphi \in \mathscr{D}(\mathbb{R}^n) : \langle \tau_a T,\varphi\rangle = \langle T,\tau_{-a}\varphi\rangle,$$

where $\tau_{-a}\varphi(x) = \varphi(x+a)$ for all $x \in \mathbb{R}^n$.

If we take a compact set $K \subset \mathbb{R}^n$ and $\varphi \in \mathscr{D}_K(\mathbb{R}^n)$, then $K_a = x + a, x \in K$ is also a compact set, and $\tau_{-a}\varphi \in \mathscr{D}_{K_a}(\mathbb{R}^n)$. Therefore, $\langle T, \tau_{-a}\varphi\rangle$ makes sense, and consequently, $\tau_a T$ is a distribution on $\mathbb{R}^n$.

Example 2.13 : *Let* $\varphi \in \mathscr{D}(\mathbb{R}^n)$. *Then:*
$\langle \tau_a\delta, \varphi\rangle = \langle \delta, \tau_{-a}\varphi\rangle = \varphi(a) = \langle \delta_a, \varphi\rangle$. *So,* $\tau_a\delta = \delta_a$.

Definition 2.8 (***dilatation of a distribution***) : *Let* T *be a distribution on* $\mathbb{R}^n$. *The dilation* T_λ *with scale factor* $\lambda \neq 0$ *is defined as follows:*

$$\forall \varphi \in \mathscr{D}(\mathbb{R}^n) : \langle T_\lambda, \varphi\rangle = |\lambda|^n \langle T, \varphi_{\frac{1}{\lambda}}\rangle,$$

where: $\varphi_{\frac{1}{\lambda}}(x) = \varphi(\lambda x), \forall x \in \mathbb{R}^n$

If we take a compact set $K \subset \mathbb{R}^n$ and $\varphi \in \mathscr{D}K(\mathbb{R}^n)$, then $K\lambda = \lambda x, x \in K$ is also a compact set, and $\varphi_{\frac{1}{\lambda}} \in \mathscr{D}K\lambda(\mathbb{R}^n)$. Therefore, $\langle T, \varphi_{\frac{1}{\lambda}}\rangle$ makes sense, and consequently, T_λ is a distribution on $\mathbb{R}^n$.

Example 2.14 : *Let* $\varphi \in \mathscr{D}(\mathbb{R}^n)$. *Then:*
$\langle \delta_\lambda, \varphi\rangle = |\lambda|^n\langle \delta, \varphi_{\frac{1}{\lambda}}\rangle = |\lambda|^n\varphi(0) = |\lambda|^n\langle \delta, \varphi\rangle$. *So,* $\delta_\lambda = |\lambda|^n\delta$.

Definition 2.9 : *We denote* $\check{\varphi}$ *as* φ_{-1}, *i.e.,* $\check{\varphi}(x) = \varphi(-x)$ *for all* $x \in \mathbb{R}^n$.
Let T *be a distribution on* $\mathbb{R}^n$. *The symmetry of* T *is the distribution* $\check{T}$ *defined as follows:*

$$\forall \varphi \in \mathscr{D}(\mathbb{R}^n) : \langle \check{T}, \varphi\rangle = \langle T, \check{\varphi}\rangle.$$

1. We say that T is even if $\check{T} = T$.
2. We say that T is odd if $\check{T} = -T$.
3. We say that T is homogeneous of order $m \in \mathbb{Z}$ if for every $\lambda > 0$, we have: $T_\lambda = \lambda^{-m}T$.

Example 2.15 :

1. *We have:* $\check{\delta} = \delta$, *i.e* δ *is even.*
2. *Since* $\delta_\lambda = |\lambda|^n\delta$, *we deduce that* δ *is homogeneous of order* n.

Definition 2.10 (**product of a distribution by a function**) *Let* T *be a distribution on* Ω, *and* $f \in \mathscr{C}^\infty(\Omega)$. *We define* $f.T$ *as follows:*

$$\forall \varphi \in \mathscr{D}(\Omega) : \langle f.T, \varphi\rangle = \langle T, f.\varphi\rangle.$$

If we take a compact set $K \subset \Omega$ and $\varphi \in \mathscr{D}_K(\Omega)$, then $f.\varphi \in \mathscr{D}_K(\Omega)$. Therefore, $\langle T, f\varphi\rangle$ makes sense, and consequently, $f.T$ is a distribution on Ω.

Remark 2.5 :

1. *If T is a distribution of order m, and $f \in \mathscr{C}^\infty$, then $f.T$ is a distribution of order less than or equal to m.*

2. *In general, we cannot define the product of two distributions (see exercise 2.7).*

Example 2.16 : *Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and $f \in \mathscr{C}^\infty(\mathbb{R}^n)$. We have:*
$\langle f.\delta, \varphi\rangle = \langle \delta, f.\varphi\rangle = f(0).\varphi(0) = f(0).\langle \delta, \varphi\rangle$. *Then:* $f.\delta = f(0).\delta$.

Proposition 2.7 : *Let $\{f_j\}_{j=1}^{+\infty} \subset \mathscr{C}^\infty(\Omega)$ and $\{f_j\}_{j=1}^{+\infty} \subset \mathscr{D}'(\Omega)$ be two sequences such that $f_j \to f$ in $\mathscr{C}^\infty(\Omega)$ and $T_j \to T$ in $\mathscr{D}'(\Omega)$. Then, $f_j.T_j \to f.T$ in $\mathscr{D}'(\Omega)$.*

Proof: : Let $\varphi \in \mathscr{D}(\Omega)$, and let $K \subset \Omega$ be a compact set such that $\operatorname{supp}\varphi \subset K$. Then, $\operatorname{supp}(f_j\varphi) \subset K$ and $\operatorname{supp}(f\varphi) \subset K$.

Since $f_j \to f$ in $\mathscr{C}^\infty(\Omega)$ and $f_j\varphi, f\varphi \in \mathscr{C}^\infty(\Omega)$, we have $\lim\limits_{j\to+\infty} P_{K,m}(f_j\varphi - f\varphi) = 0$ for all $m \in \mathbb{N}$. The convergence is in $\mathscr{D}(\Omega)$ since K is fixed.

Since $T_j \to T$ in $\mathscr{D}'(\Omega)$, $\langle T_j, f_j.\varphi\rangle$ tends to $\langle T, f.\varphi\rangle$. According to the Banach-Steinhaus theorem (Corollary 1.2), the convergence is in $\mathscr{D}'(\Omega)$. ■

2.5 Supports of distributions

Definition 2.11 : *The null open set set of $T \in \mathscr{D}'(\Omega)$ is the largest open set $O \subset \Omega$ such that:*

$$\forall\varphi \in \mathscr{D}(O) : \langle T, \varphi\rangle = 0$$

The support of T ($\operatorname{supp} T$) is $\Omega \setminus O$.

Suppose there exists a non-empty open set where $T = 0$, and consider a family $(O_i)_{i\in I}$ of open sets where $T = 0$. Let's define $O = \bigcup\limits_{i\in I} O_i$, which is an open set. Let $\varphi \in \mathscr{D}(O)$. Then, $K = \operatorname{supp}\varphi \subset O = \bigcup\limits_{i\in I} O_i$.

The family $(O_i)_{i\in I}$ is a covering for the compact set K, so we can extract a finite covering $(O_j)_{j=1}^N$. According to Theorem 1.7 (partition of unity), there exists a family $(\theta_j)_{j=1}^N$ where $\theta_j \in \mathscr{D}(O_j), 0 \le \theta_j \le 1$, and $\sum\limits_{j=1}^N \theta_j = 1$. Then, for any $x \in \Omega$, we have: $\varphi(x) = \sum\limits_{j=1}^N \theta_j(x)\varphi(x)$, and for all $1 \le j \le N$, $\theta_j.\varphi \in \mathscr{D}(O_j)$. Therefore:

$$\langle T, \varphi\rangle = \langle T, \sum_{j=1}^N \theta_j.\varphi\rangle = \sum_{j=1}^N \langle T, \theta_j.\varphi\rangle = 0.$$

Then O is the null open set of T and $\operatorname{supp} T = \Omega \setminus O$.

Example 2.17 :

1. *Let* $\varphi \in \mathscr{D}(\mathbb{R}^n)$ *such that* $\operatorname{supp}\varphi \subset (\mathbb{R}^n \setminus 0)$*. Then:* $\langle \delta, \varphi \rangle = \varphi(0) = 0$*. Thus, the null open set of* δ *is included in* $\mathbb{R}^n \setminus 0$*. This inclusion is strict because if we take* $\varphi_0 \in \mathscr{D}(\mathbb{R}^n)$ *such that* $\varphi = 1$ *in the neighbourhood of* $B(0,1)$*, we have* $\langle \delta, \varphi \rangle = \varphi(0) = 1 \neq 0$*. Therefore:* $\operatorname{supp}\delta = \{0\}$*.*

2. *Let* $\varphi \in \mathscr{D}(\mathbb{R})$ *such that* $\operatorname{supp}\varphi \subset]-\infty, 0[$*. On a:*
   $\langle H, \varphi \rangle = \displaystyle\int_0^{+\infty} \varphi(x)dx = 0$*. Then: the null open set of* H *est is included in* $]-\infty, 0[$*.*
   Let $(\varphi_j)_{j \in \mathbb{N}}$ *be a sequence of functions in* $\mathscr{D}(\mathbb{R})$ *such that* $\varphi \geq 0$ *et* $\varphi = 1$ *on* $\left[-\frac{1}{n}, \frac{1}{n}\right]$*.Then:*
   $\langle H, \varphi \rangle \geq \displaystyle\int_0^{\frac{1}{n}} \varphi(x)dx = \frac{1}{n} \neq 0.$
   We deduce that the null open set of H *is* $]-\infty, 0[$*. Then:* $\operatorname{supp} H = [0, +\infty[$*.*

Proposition 2.8 :

1. *Let* T *be a distribution with compact support on* Ω*, and let* $\varphi \in \mathscr{D}(\Omega)$ *such that* $\varphi = 0$ *in the neighbourhood of* $\operatorname{supp} T$ *(i.e.,* $\operatorname{supp}\varphi \cap \operatorname{supp} T = \emptyset$*). Then:* $\langle T, \varphi \rangle = 0$*.*

2. *The support of* T*, denoted* $\operatorname{supp} T$*, is the smallest closed set* F *such that: if* $\varphi \in \mathscr{D}(\Omega)$ *and* $\varphi = 0$ *in the neighbourhood of* F*, then:* $\langle T, \varphi \rangle = 0$*.*

Proof:

1. Since $\operatorname{supp}\varphi \cap \operatorname{supp} T = \emptyset$ we have: $\operatorname{supp}\varphi \subset (\mathbb{R}^n \setminus \operatorname{supp} T) = O$ the null open set of T. Then: $\varphi \in \mathscr{D}(O)$, which lead to: $\langle T, \varphi \rangle = 0$.

2. Let F_0 be the smallest closed set that satisfies the property: If $\varphi \in \mathscr{D}(\Omega)$, and $\varphi = 0$ in the vicinity of F_0, then $\langle T, \varphi \rangle = 0$.
   It is clear that $\operatorname{supp} T$ satisfies the property, and if F_1 and F_2 satisfy the property, then $F_1 \cap F_2$ and $F_1 \cup F_2$ also satisfy the property.
   Assume that $F_0 \subset \operatorname{supp} T$ with strict inclusion. Then, there exists $x_0 \in \operatorname{supp} T$ such that $x_. \notin F_0$. since F_0 is a closed set we have: $d(x_0, F_0) = 2r > 0$. Then: $B(x_0, r) \cap F_0 = \emptyset$ et $G_0 = \operatorname{supp} T \cap B(x_0, r) \neq \emptyset$.
   It results that $G_0 \cup F_0 \subset \operatorname{supp} T$ et $F_0 \subset (G_0 \cup F_0)$ with strict inclusion, which contradicts the fact that F_0 is the smallest closed set satisfying the property.

■

Remark 2.6 : *If we replace* $\varphi = 0$ *in the neighbourhood of* $\operatorname{supp} T$ *with* $\varphi = 0$ *on* $\operatorname{supp} T$*, the proposition above does not hold. For example, we have* $\operatorname{supp}\delta = 0$*, but if* $\psi \in \mathscr{D}(\mathbb{R})$ *such that* $\psi = 1$ *in the neighbourhood of* 0 *and* $\varphi(x) = x\psi(x)$*, then* $\varphi = 0$ *on* $\operatorname{supp}\delta$

Let T be a distribution with compact support on Ω. Then, T is of finite order m, and for any neighbourhood of a compact set $K \subset \Omega$, there exists a positive constant M such that for all $\varphi \in \mathscr{D}(\Omega)$, we have $|\langle T, \varphi\rangle| \leq M.P_{K,m}(\Omega)$.

Proof: Let K be a compact neighborhood of $\operatorname{supp} T$, and let $\chi \in \mathscr{D}(\Omega)$ such that $\operatorname{supp}\chi \subset K$ and $\chi = 1$ in a neighborhood of $\operatorname{supp} T$.

Now, consider $\varphi \in \mathscr{D}(\Omega)$. We have $\varphi - \chi \cdot \varphi = 0$ in a neighborhood of $\operatorname{supp} T$. Thus, $\langle T, \varphi - \chi\varphi\rangle = 0$, which implies $\langle T, \varphi\rangle = \langle T, \chi\varphi\rangle$.

There exists $M_0 > 0$ and $m \in \mathbb{N}$ such that for all $\psi \in \mathscr{D}_K(\Omega)$, we have $|\langle T, \psi\rangle| \leq M_0.P_{K,m}(\psi)$. Since $\chi \cdot \varphi \in \mathscr{D}_K(\Omega)$, we have:

$$|\langle T, \varphi\rangle| = |\langle T, \chi.\varphi\rangle| \leq M_0.P_{K,m}(\chi.\varphi) \leq M.P_{K,m}(\varphi).$$

Notably, m depends only on K, which is fixed (a neighbourhood of $\operatorname{supp} T$). Therefore, we conclude that T is of finite order. ■

We use this result to extend the duality bracket $\langle ., .\rangle_{\mathscr{D}',\mathscr{D}}$ as follows:

Definition 2.12 (***Duality Bracket*** $\langle ., .\rangle_{\mathscr{E}',\mathscr{E}}$) : *We denote by $\mathscr{E}(\Omega)$ the space of $\mathscr{C}^\infty(\Omega)$ functions, and by $\mathscr{E}'(\Omega)$ the space of distributions with compact support. For any $T \in \mathscr{E}'(\Omega)$ and $\varphi \in \mathscr{E}(\Omega)$, we define:*

$$\langle T, \varphi\rangle_{\xi',\xi} = \langle T, \chi\varphi\rangle_{\mathscr{D}',\mathscr{D}},$$

where $\chi \in \mathscr{D}(\Omega)$ with $\chi = 1$ in the neighbourhood of $\operatorname{supp} T$. *We write: $(\mathscr{C}^\infty(\Omega))' = \mathscr{E}'(\Omega)$.*

This result is independent of the choice of χ because if we take χ_1 and χ_2 in $\mathscr{D}(\Omega)$ such that $\chi_1 = \chi_2 = 1$ in the neighbourhood of $\operatorname{supp} T$, we have: $\chi_1.\varphi = \chi_2.\varphi = 0$ in the neighbourhood of $\operatorname{supp} T$. Therefore, we have: $\langle T, \chi_1.\varphi - \chi_1.\varphi\rangle = 0$, which implies: $\langle T, \chi_1.\varphi\rangle = \langle T, \chi_2.\varphi\rangle$.

Theorem 2.4 : *The canonical injection of $\mathscr{E}'(\Omega)$ into $\mathscr{D}'(\Omega)$ is continuous.*
We set: $\mathscr{E}'(\Omega) \hookrightarrow \mathscr{D}'(\Omega)$.

Proof: Let's denote by i the map from $\mathscr{E}'(\Omega)$ into $\mathscr{D}'(\Omega)$ defined as follows:

$$\forall T \in \mathscr{E}'(\Omega) : i(T) = T.$$

his map is linear, and if $T \in \mathscr{E}'(\Omega)$ such that $i(T) = 0$, then for all $\varphi \in \mathscr{D}(\Omega)$, we have $\langle T, \varphi\rangle = 0$. Therefore, for all $\psi \in \mathscr{E}(\Omega)$ and $\chi \in \mathscr{D}(\Omega)$, we have:

$$\langle T, \varphi\rangle_{\mathscr{E}',\mathscr{E}} = \langle T, \chi.\varphi\rangle = 0.$$

This implies that $T = 0$, i.e., i is injective.
Now, consider a sequence $\{T_j\}_{j\in\mathbb{N}} \subset \mathscr{E}'(\Omega)$ converging to 0 in $\mathscr{E}'(\Omega)$. Then, for all $\varphi \in \mathscr{E}'(\Omega)$, we have $\langle T_j, \varphi\rangle_{\mathscr{E}',\mathscr{E}}$ converging to 0.

However, we also have: $\langle T_j, \varphi\rangle_{\mathscr{E}',\mathscr{E}} = \langle T_j, \chi.\varphi\rangle_{\mathscr{D}',\mathscr{D}} = \langle T, \psi\rangle$ converging to 0 for all $\psi \in \mathscr{D}(\Omega)$. Therefore, i is continuous. ■

Exercises

Exercise 2.1 : *We define the following functional on* $\mathscr{D}(\mathbb{R})$*:*

$$\forall \varphi \in \mathscr{D}(\mathbb{R}) : \langle pf_{\frac{1}{x^2}}, \varphi\rangle = \lim_{\varepsilon\to 0}\left[\int_{|x|>\varepsilon} \frac{\varphi(x)}{x^2}dx - 2\frac{\varphi(0)}{\varepsilon}\right].$$

Show that $pf_{\frac{1}{x^2}}$ *defines a distribution on* $\mathbb{R}$*.*

Exercise 2.2 : *The purpose of this exercise is to show that the distribution* $vp_{\frac{1}{x}}$ *is of order* 1*. By contradiction, we assume that* $vp_{\frac{1}{x}}$ *is of order* 0 *(we know from example 2.9 that* $vp_{\frac{1}{x}}$ *is of order less than or equal to* 1*).*

Let $\varphi \in \mathscr{D}(\mathbb{R})$*, be an even function such that* $\varphi \geq 0, \varphi = 1$ *in the neighbourhood of* 0*. Let* $a > 0$ *such that* $\operatorname{supp}\varphi \subset K = [-a, a]$*. We know that* $\langle vp_{\frac{1}{x}}, \varphi\rangle \leq 2aP_{K,1}(\varphi)$*.*

Consider the sequence of functions $(\varphi_j)_{j\in\mathbb{N}}$*, defined as follows:* $\varphi_j(x) = \varphi(x)\arctan(jx)$*.*

1. *Show that there exists* $M > 0$ *such that for every* $j \in \mathbb{N}$ *we have:* $\sup_{x\in K} |\varphi_j(x)| < M$*.*
2. *Calculate* $\varphi_j'(0)$*. What can we conclude about* $PK, 1(\varphi_j)$*?*
3. *Deduce.*

Exercise 2.3 : *The goal of this exercise is to show the existence of distributions of infinite order. Consider the functional* T *defined on* $\mathscr{D}(\mathbb{R})$ *as follows:*

$$\forall \varphi \in \mathscr{D}(\mathbb{R}) : \langle T, \varphi\rangle = \sum_{k=0}^{\infty} \varphi^{(k)}(k).$$

1. *Show that* T *defines a distribution on* $\mathbb{R}$*.*
2. *Suppose that* T *is of finite order* m*. Let* $\psi_0 \in \mathscr{D}\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\right)$ *such that* $\psi_0 \geq 0$ *and* $\psi_0 = 1$ *on* $\left[-\frac{1}{4}, \frac{1}{4}\right]$*. Define* $\psi(x) = \dfrac{x^{m+1}}{(m+1)!}\psi_0(x)$*.*
   i) *For* $\lambda > 1$*, define:* $\varphi(x) = \psi(\lambda(x - (m+1)))$*.*
   Show that $\varphi \in \mathscr{C}^\infty(\mathbb{R})$ *and* $\operatorname{supp}\varphi \subset K_m = \left[m + \frac{1}{2}, m + \frac{3}{2}\right]$*.*
   ii) *Show that* $\langle T, \varphi\rangle = \lambda^{m+1}$*.*
   iii) *Show that there exists* $M_K > 0$ *such that:* $\lambda^{m+1} \leq M_K \sum_{k=0}^{m} \lambda^k. \sup|\psi^{(k)}|$*.*
   iv) *Show that* λ *is finite.*
   v) *Conclude that* T *is of infinite order.*

Exercise 2.4 : *Let* $\{T_j\}_{j\in\mathbb{N}}$ *be the sequence of distributions associated with the locally integrable functions* $\dfrac{\sin(jx)}{\pi x}$.

Show that T_j *converges to* δ *as* j *tends to infinity (Note:* $\displaystyle\int_0^{+\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$*).*

Exercise 2.5 : *Let* T *be the functional defined on* $\mathscr{D}(\mathbb{R})$ *as follows:*

$$\forall\varphi \in \mathscr{D}(\mathbb{R}) : \langle T, \varphi\rangle = \int_{|x|>\varepsilon} \ln|x|.\varphi(x)dx.$$

Consider the sequence of functions $\{f_j\}_{j\in\mathbb{N}}$ *defined as:* $f_j(x) = \begin{cases} \ln|x| & : \ |x| \geq \frac{1}{j} \\ -\ln(j) & : \ |x| < \frac{1}{j} \end{cases}$

1. *Show that* T *defines a distribution on* $\mathbb{R}$*, denoted as* $\ln|x|$.
2. *Show that* $f_j \in L^1_{loc}(\mathbb{R})$ *for all* $j \in \mathbb{N}$.
3. *Show that* $f_j \to \ln|x|$ *in* $\mathscr{D}'(\mathbb{R})$.
4. *Show that* $(\ln|x|)' = vp_{\frac{1}{x}}$.

Exercise 2.6 :

1. *Calculate* $x.vp_{\frac{1}{x}}$ *and* $x.\delta$.
2. *Calculate* $(x\ln x)'$, $x.\delta^{(k)}$ *for* $k \geq 1$.
3. *Solve the equation* $xT = 0$ *in* $\mathscr{D}'(\mathbb{R})$.

Exercise 2.7 : *Consider the sequence of functions* $\{f_j\}_{j\in\mathbb{N}}$ *defined as follows:*

$$f_j(x) = \begin{cases} j & : \ x \in \left[0, \frac{1}{j}\right] \\ 0 & : \ x \notin \left[0, \frac{1}{j}\right] \end{cases}$$

Let $T_j = f_j$ *be the corresponding distributions.*

1. *Show that* $T_j \to \delta$ *as* $j \to +\infty$.
2. *Find the expression for* $\langle f_j.T_j, \varphi\rangle$ *for all* φ *in* $\mathscr{D}'(\mathbb{R})$.
3. *Show that* $\langle f_j.T_j, \varphi\rangle \to +\infty$.
   (take $\varphi \in \mathscr{D}(\mathbb{R})$ *such that* $\varphi = 1$ *in the neighbourhood of* 0*).*

Exercise 2.8 :

1. *Calculate* $(f.T)', (fT)''$*, where* $f \in \mathscr{C}^\infty(\mathbb{R})$ *and* $T \in \mathscr{D}'(\mathbb{R})$.
2. *Calculate* $\langle T_1 + T_2, \varphi\rangle$*, for* $T_1, T_2 \in \mathscr{D}'(\mathbb{R}^n)$ *and* $\varphi \in \mathscr{D}(\Omega)$ *où* $\Omega = \mathbb{R}^n\backslash(\operatorname{supp} T_1 \cup \operatorname{supp} T_2)$.

Solutions of exercises

Solution 2.1 : $\forall \varphi \in \mathscr{D}(\mathbb{R}) : \langle pf_{\frac{1}{x^2}}, \varphi\rangle = \lim_{\varepsilon\to 0}\left[\int_{|x|>\varepsilon}\frac{\varphi(x)}{x^2}dx - 2\frac{\varphi(0)}{\varepsilon}\right].$

Let $\varphi \in \mathscr{D}(\mathbb{R})$. On a:

$$\begin{aligned}\int_{|x|>\varepsilon}\frac{\varphi(x)}{x^2}dx &= \int_{-\infty}^{-\varepsilon}\frac{\varphi(x)}{x^2}dx + \int_{\varepsilon}^{-\infty}\frac{\varphi(x)}{x^2}dx\\ &= -\left[\frac{\varphi(x)}{x}\right]_{-\infty}^{-\varepsilon} - \left[\frac{\varphi(x)}{x}\right]_{\varepsilon}^{+\infty} + \int_{-\infty}^{-\varepsilon}\frac{\varphi'(x)}{x}dx + \int_{\varepsilon}^{-\infty}\frac{\varphi'(x)}{x}dx\\ &= \frac{\varphi(-\varepsilon)}{\varepsilon} + \frac{\varphi(-\varepsilon)}{\varepsilon} + \int_{|x|>\varepsilon}\frac{\varphi'(x)}{x}dx\end{aligned}$$

Then:

$$\begin{aligned}\langle pf_{\frac{1}{x^2}}, \varphi\rangle &= \lim_{\varepsilon\to 0}\left[\frac{\varphi(-\varepsilon)}{\varepsilon} + \frac{\varphi(-\varepsilon)}{\varepsilon} + \int_{|x|>\varepsilon}\frac{\varphi'(x)}{x}dx - 2\frac{\varphi(0)}{\varepsilon}\right]\\ &= \lim_{\varepsilon\to 0}\left[-\frac{\varphi(-\varepsilon)-\varphi(0)}{-\varepsilon} + \frac{\varphi(-\varepsilon)-\varphi(0)}{\varepsilon} + \int_{|x|>\varepsilon}\frac{\varphi'(x)}{x}dx\right]\\ &= -\varphi'(0) + \varphi'(0) + \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\varphi'(x)}{x}dx\\ &= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\varphi'(x)}{x}dx\\ &= \langle vp_{\frac{1}{x}}, \varphi'\rangle\end{aligned}$$

As $vp_{\frac{1}{x}}$ defines a distribution on $\mathbb{R}$ and since $\varphi' \in \mathscr{D}(\mathbb{R})$, we deduce that $pf_{\frac{1}{x^2}}$ defines a distribution on $\mathbb{R}$.

Solution 2.2 : *$\varphi \in \mathscr{D}(\mathbb{R})$ is an even function such that $\varphi \geq 0$, $\varphi = 1$ in the neighbourhood of 0, and $a > 0$ with $\operatorname{supp}\varphi \subset K = [-a, a]$.*

$\varphi_j(x) = \varphi(x)\arctan(jx)$ *for* $j \in \mathbb{N}$.

1. *We have:* $\sup_{x\in K}|\varphi_j(x)| = \sup_{x\in K}|\arctan(jx).\varphi(x)| = \frac{\pi}{2}\sup_{x\in K}|\varphi(x)| < M.$
2. *We have:* $\varphi_j'(x) = \frac{j\varphi(x)}{1+j^2x^2} + \varphi'(x)\arctan(jx)$. *Then:* $\varphi_j'(0) = j\varphi(0) = j$.
   There exists $j_0 \in \mathbb{N}$ *such that* $\sup_{x\in K}|\varphi_j'(x)| = j_0 \geq M > \sup_{x\in K}|\varphi_j(x)|$.
   Then: $P_{K,1}(\varphi_j) = j$ *pour* $j \geq j_0$.
3. *Since* $P_{K,1}(\varphi_j) = \sup_{x\in K}|\varphi_j'(x)|$, *we deduce that the order of* $vp_{\frac{1}{x}}$ *is* 1.

Solution 2.3 : $\forall \varphi \in \mathscr{D}(\mathbb{R}) : \langle T, \varphi\rangle = \sum_{k=0}^{\infty}\varphi^{(k)}(k).$

1. *Let $K \subset \mathbb{R}$ be a compact. There exists $j \in \mathbb{N}$ such that $K \subset [-j, j]$. Let $\varphi \in \mathscr{D}_K(\mathbb{R})$. Then:* $\operatorname{supp}\varphi^{(k)} \subset [-j, j]$ *for all* $k \in \mathbb{N}$, *i.e* $\varphi^{(k)}(k) = 0$ *pour tout* $k > j$, *which lead to:*

$$|\langle T, \varphi\rangle| = \left|\sum_{k=0}^{j}\varphi^{(k)}(k)\right| \leq (j+1)\sup_{x\in K, k\leq j}|\varphi^{(k)}(x)| = (j+1)P_{K,j}(\varphi).$$

Then: T defines a distribution on $\mathbb{R}$.

2. *Suppose that T est d'ordre m.*
   $\psi_0 \in \mathscr{D}\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right), \psi_0 \geq 0, \psi_0 = 1$ *on* $\left[-\frac{1}{4},\frac{1}{4}\right].\psi(x) = \frac{x^{m+1}}{(m+1)!}\psi_0(x)$.

   i) $\lambda > 1, \varphi(x) = \psi(\lambda(x-(m+1)))$.
   since $\psi \in \mathscr{C}^\infty(\mathbb{R})$ *Then:* $\varphi \in \mathscr{C}^\infty(\mathbb{R})$.
   From the definition of $\varphi, x \in \operatorname{supp}\varphi$*, it implies that* $\lambda(x-(m+1)) \in \left]-\frac{1}{2},\frac{1}{2}\right[$.
   Then: $(x-(m+1)) \in \left]-\frac{1}{2\lambda},\frac{1}{2\lambda}\right[\subset \left]-\frac{1}{2},\frac{1}{2}\right[$ (*since* $\lambda > 1$.)
   Hence: $x \in \left[m+\frac{1}{2}, m+\frac{3}{2}\right]$. *So,* $\operatorname{supp}\varphi \subset K_m = \left[m+\frac{1}{2}, m+\frac{3}{2}\right]$.

   ii) *Since* $\operatorname{supp}\varphi \subset K_m = \left[m+\frac{1}{2}, m+\frac{3}{2}\right]$, *we deduce that*

   $$\langle T, \varphi\rangle = \varphi^{(m+1)}(m+1) = \lambda^{m+1}\psi^{(m+1)}(0) = \lambda^{m+1}.$$

   iii) *We assumed that T est d'ordre m, then: there exists $M_K > 0$ such that:*
   $\lambda^{m+1} = |\langle T,\varphi\rangle| \leq M_K \sum_{k=0}^{m} \sup|\varphi^{(k)}| = M_K \sum_{k=0}^{m} \lambda^k. \sup|\psi^{(k)}|$.

   iv) *Since* $\lambda > 0$, *for all* $k \leq m$ *we have:* $\lambda^k \leq \lambda^m$. *Then:* $\lambda^{m+1} \leq M_K.\lambda^m \sum_{k=0}^{m} \sup|\psi^{(k)}|$.
   hence: $\lambda \leq M_K. \sum_{k=0}^{m} \sup|\psi^{(k)}|$. *So, λ is finite.*

   v) *If we let λ tend to infinity, we have a contradiction with λ being finite. Therefore, T is of infinite order.*

Solution 2.4 : $\forall \varphi \in \mathbb{R} : \langle T_j, \varphi\rangle = \int_{-\infty}^{+\infty} \frac{sin(jx)}{\pi x}\varphi(x)dx, \; j \in \mathbb{N}$.
We have: $\int_{-\infty}^{+\infty} \frac{\sin x}{x}dx = 2\int_0^{+\infty}\frac{\sin x}{x}dt = \pi$. *Then:* $\varphi(0) = \int_{-\infty}^{+\infty}\frac{\sin x}{\pi x}\varphi(0)dx$.
We change the variable $t = jx$, we obtain:
$\int_{-\infty}^{+\infty}\frac{\sin(jx)}{\pi x}\varphi(x)dx = \int_{-\infty}^{+\infty}\frac{\sin t}{\pi t}\varphi\left(\frac{t}{j}\right)dt = \int_{-\infty}^{+\infty}\frac{\sin x}{\pi x}\varphi\left(\frac{x}{j}\right)dx$. *Then:*

$$\begin{aligned}
|\langle T_j,\varphi\rangle - \langle\delta,\varphi\rangle| &= \left|\int_{-\infty}^{+\infty}\frac{\sin x}{\pi x}\varphi\left(\frac{x}{j}\right)dx - \int_{-\infty}^{+\infty}\frac{\sin x}{\pi x}\varphi(0)dx\right| \\
&\leq \left|\int_{-\infty}^{+\infty}\frac{\sin x}{\pi x}\left(\varphi\left(\frac{x}{j}\right) - \varphi(0)\right)dx\right| \\
&\leq \int_{-\infty}^{+\infty}\frac{\sin x}{\pi x}\left|\varphi\left(\frac{x}{j}\right) - \varphi(0)\right|dx \\
&\leq \sup\left|\varphi\left(\frac{x}{j}\right) - \varphi(0)\right|\int_{-\infty}^{+\infty}\frac{\sin x}{\pi x}dx \\
&= \sup\left|\varphi\left(\frac{x}{j}\right) - \varphi(0)\right|.
\end{aligned}$$

Since $\lim\limits_{j\to+\infty}\sup\left|\varphi\left(\frac{x}{j}\right)-\varphi(0)\right|=0$, *we deduce that* T_j *converge to* δ *where* $j\longrightarrow+\infty$.

Solution 2.5 : $\forall\varphi\in\mathscr{D}(\mathbb{R}):\langle T,\varphi\rangle=\displaystyle\int_{|x|>\varepsilon}\ln|x|.\varphi(x)dx,\ f_j(x)=\begin{cases}\ln|x| & : \ |x|\geq\frac{1}{j}\\ -\ln(j) & : \ |x|<\frac{1}{j}\end{cases}$

1. *Let* $\varepsilon>0$ *be small, and* $K\subset\mathbb{R}$ *be a compact . there exists* $a>1$ *such that* $K\subset[-a,a]$. *Let* $\varphi\in\mathscr{D}_K(\mathbb{R})$, *Then:*

$$\begin{aligned}\langle T,\varphi\rangle &= \int_{|x|>\varepsilon}\ln|x|.\varphi(x)dx\\ &= \int_{\varepsilon<|x|\leq a}\ln|x|.\varphi(x)dx\\ &= \int_{\varepsilon<|x|<1}\ln|x|.\varphi(x)dx+\int_{1\leq|x|\leq a}\ln|x|.\varphi(x).\end{aligned}$$

On one hand: $\left|\displaystyle\int_{1\leq|x|\leq a}\ln|x|.\varphi(x)\right|\leq\sup\limits_{1\leq|x|\leq a}|\varphi(x)|.\displaystyle\int_{1\leq|x|\leq a}\ln|x|dx\leq M_1P_{K,0}(\varphi)$.
On the other hand, according to the mean value theorem, there exists $\varepsilon<|x|<|x_\varepsilon|<1$ *such that* $|\ln|x||=-\ln|x|=\ln 1-\ln|x|\leq\dfrac{1-x}{|x_\varepsilon|}\leq\dfrac{1-x}{|x|}$.
then: $\displaystyle\int_{\varepsilon<|x|<1}\ln|x|.\varphi(x)dx\leq\int_{\varepsilon<|x|<1}\frac{1-x}{|x|}.\varphi(x)dx,$
and this last term can be treated as $vp_{\frac{1}{x}}$. .
Then: T *defines a distribution on* $\mathbb{R}$, *on la note par* $\ln|x|$.

2. *The function* f_j *is continuous for all* $j\in\mathbb{N}$. *Then:* $f_j\in L^1_{Loc}(\mathbb{R})$ *for all* $j\in\mathbb{N}$.

3. *We can write:* $\forall\varphi\in\mathscr{D}(\mathbb{R}):<\ln|x|,\varphi>=\displaystyle\int_{|x|\geq\frac{1}{j}}\ln|x|.\varphi(x)dx$, *then:*

$|\langle f_j,\varphi\rangle-\langle\ln|x|,\varphi\rangle|=\displaystyle\int_{|x|<\frac{1}{j}}-\ln j.\varphi(x)dx\leq\ln j\int_{-\frac{1}{j}}^{\frac{1}{j}}|\varphi(x)|dx.$
Applying the mean value theorem, there exists $x_j\in[-\frac{1}{j},\frac{1}{j}]$ *such that*
$\displaystyle\int_{-\frac{1}{j}}^{\frac{1}{j}}|\varphi(x)|dx=\frac{2}{j}|\varphi(x_j)|$. *Then:* $|\langle f_j,\varphi\rangle-\langle\ln|x|,\varphi\rangle|\leq\dfrac{2\ln j}{j}|\varphi(x_j)|$.
Since φ *is continuous:* $\lim\limits_{j\to+\infty}|\varphi(x_j)|=|\varphi(0)|$. *Then:*
$0\leq\lim\limits_{j\to+\infty}|\langle f_j,\varphi\rangle-\langle\ln|x|,\varphi\rangle|\leq\lim\limits_{j\to+\infty}\dfrac{2\ln j}{j}|\varphi(x_j)|=0.$
Hence: $f_j\to\ln|x|$ *in* $\mathscr{D}'(\mathbb{R})$.

4. *Let* $\varphi\in\mathscr{D}(\mathbb{R})$. *We have:* $\langle(\ln|x|)',\varphi\rangle=-\langle\ln|x|,\varphi'\rangle=-\displaystyle\int_{|x|>\varepsilon}\ln|x|.\varphi'(x)dx.$

There exists $a > \varepsilon$ *such that* $\operatorname{supp}\varphi \subset [-a, a]$. *Then:*

$$\begin{aligned} -\int_{|x|>\varepsilon} \ln|x|.\varphi'(x)dx &= -\int_{-a}^{-\varepsilon} \ln|x|.\varphi'(x)dx - \int_{\varepsilon}^{a} \ln|x|.\varphi'(x)dx \\ &= [-\ln|x|.\varphi(x)]_{-a}^{-\varepsilon} + \int_{-a}^{-\varepsilon} \frac{\varphi(x)}{x}dx + [-\ln|x|.\varphi(x)]_{\varepsilon}^{a} + \int_{\varepsilon}^{a} \frac{\varphi(x)}{x}dx \\ &= \ln\varepsilon(\varphi(\varepsilon) - \varphi(-\varepsilon)) + \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}dx. \end{aligned}$$

We have: $\varphi(\varepsilon) - \varphi(-\varepsilon) = 2\varepsilon\varphi'(0) + 0(\varepsilon)$, *then:* $\lim_{\varepsilon\to 0} \ln\varepsilon(\varphi(\varepsilon) - \varphi(-\varepsilon)) = \lim_{\varepsilon\to 0} 2\epsilon\varepsilon\varphi'(0) = 0$. *So,*
$\langle(\ln|x|)', \varphi\rangle = \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}dx = \langle vp_{\frac{1}{x}}, \varphi\rangle.$
which lead to: $(\ln|x|)' = vp_{\frac{1}{x}}$.

Solution 2.6 :

1. *Let* $\varphi \in \mathscr{D}(\mathbb{R})$. *Then:*

$$\begin{aligned} *)\ \langle x.v_p\tfrac{1}{x}, \varphi\rangle &= \langle v_p\tfrac{1}{x}, x.\varphi\rangle \\ &= \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} x.\frac{\varphi(x)}{x}dx \\ &= \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \varphi(x)dx \\ &= \int_{-\infty}^{+\infty} \varphi(x)dx \\ &= \langle 1, \varphi\rangle. \end{aligned}$$

Then: $x.v_p\dfrac{1}{x} = 1$.
$*)\ \langle x.\delta, \varphi\rangle = \langle \delta, x.\varphi\rangle = 0$.
Then: $x.\delta = 0$.

2. *Let* $\varphi \in \mathscr{D}(\mathbb{R})$. *Then:*

$$\begin{aligned} *)\ \langle (x.\ln x)', \varphi\rangle &= -\langle x.\ln x, \varphi'\rangle \\ &= -\langle \ln x, x.\varphi'\rangle \\ &= -\lim_{\varepsilon\to 0} \int_{\varepsilon}^{+\infty} x.\ln x.\varphi'(x)dx \\ &= \lim_{\varepsilon\to 0}[(1+\ln x)\varphi(x)]_{\varepsilon}^{+\infty} + \lim_{\varepsilon\to 0} \int_{\varepsilon}^{+\infty} (1+\ln x)\varphi(x)dx \\ &= \int_{0}^{+\infty} \varphi(x)dx + \int_{\varepsilon}^{+\infty} \ln x.\varphi(x)dx \\ &= \langle H, \varphi\rangle + \langle \ln x, \varphi\rangle. \end{aligned}$$

Hence: $(x\ln x)' = H + \ln x$.

$$\begin{aligned} *)\ \langle x.\delta^{(k)}, \varphi\rangle &= \langle \delta^{(k)}, x.\varphi\rangle \\ &= (-1)^k \langle \ln x, (x.\varphi)^{(k)}\rangle \\ &= (-1)^k \sum_{i=0}^{k} C_k^i (x)^{(k-i)} \varphi^{(k)}|_{x=0} \\ &= (-1)^k C_k^1 \varphi'(0) \\ &= (-1)^k.k\langle \delta', \varphi\rangle. \end{aligned}$$

Then: $x.\delta^{(k)} = (-1)^k.k\delta'$.

3. *From the first question, if* $T = \delta$ *then:* $xT = 0$.
   So, if $T = c.\delta$ $(c \in \mathbb{R})$ *then:* $xT = 0$.
   Now, assume that $xT = 0$*, then:*
   for all $\varphi \in \mathscr{D}(\mathbb{R})$ *we have:* $\langle x.T, \varphi\rangle = \langle T, x.\varphi\rangle = 0$.
   Let $(\varphi_j) \in \mathscr{D}(\mathbb{R})$ *such that* $\operatorname{supp}\varphi_j \subset \left]-\frac{1}{j}, \frac{1}{j}\right[$ *et* $\varphi_j = 1$ *on* $\left[-\frac{1}{2j}, \frac{1}{2j}\right]$.
   From Proposition 2.8, we obtain $\operatorname{supp} T = \{0\}$. *Then: there exists* $k \in \mathbb{N}$ *et* $c \in \mathbb{R}$ *such that* $T = c.\delta^{(k)}$.
   However, according to the second question, $x\delta^{(k)}$ *is different from* 0 *when* $k \geq 1$. *Therefore, the solutions to the equation* $xT = 0$ *in* $\mathscr{D}'(\mathbb{R})$ *are distributions of the form* $c.\delta$ *(where* $c \in \mathbb{R}$*).*

Solution 2.7 : $f_j(x) = \begin{cases} j & : \ x \in \left[0, \frac{1}{j}\right] \\ 0 & : \ x \notin \left[0, \frac{1}{j}\right] \end{cases}$ $, j \in \mathbb{N}.\ T_j = f_j.$

1. *Let* $\varphi \in \mathscr{D}(\mathbb{R})$. *Then:*

$$|\langle T_j, \varphi\rangle - \langle \delta, \varphi\rangle| = \left| j \int_0^{\frac{1}{j}} \varphi(x)dx - \varphi(0)\right|$$

   Applying the mean value theorem, there exists $x_j \in [0, 1j]$ *such that:*

$$j \int_0^{\frac{1}{j}} = j.\frac{1}{j}\varphi(x_j)$$

   Then:

$$|\langle T_j, \varphi\rangle - \langle \delta, \varphi\rangle| = |\varphi(x_j) - \varphi(0)| \overset{j\to+\infty}{\longrightarrow} 0\ .$$

   Hence: $T_j \to \delta$ *when* $j \to +\infty$.

2. *Let* φ *dans* $\mathscr{D}'(\mathbb{R})$. *We have:* $\langle f_j.T_j, \varphi\rangle = \langle T_j, f_j.\varphi\rangle = j^2 \int_0^{\frac{1}{j}} \varphi(x)dx.$

3. *Let* $\varphi \in \mathscr{D}(\mathbb{R})$ *such that* $\varphi \geq 0$ *et* $\varphi = 1$ *on* $\left[-\frac{1}{j}, \frac{1}{j}\right]$*.Then:*

$$\langle f_j.T_j, \varphi\rangle = j^2 \int_0^{\frac{1}{j}} \varphi(x)dx \geq j^2 \int_0^{\frac{1}{j}} dx = j \overset{j\to+\infty}{\longrightarrow} +\infty\ .$$

Hence: $\langle f_j.T_j, \varphi\rangle \to +\infty$.

Solution 2.8 :

1. *Let* $\varphi \in \mathscr{D}(\mathbb{R})$, *then:* $\langle (f.T)', \varphi\rangle = -\langle T, f.\varphi'\rangle$.
   We have: $(f.\varphi)' = f.\varphi' + f'.\varphi$. *Then:*

$$\begin{aligned}
\langle (f.T)', \varphi\rangle &= -\langle f.T, \varphi'\rangle \\
&= -\langle T, f.\varphi'\rangle \\
&= -\langle T, (f.\varphi)' - f'.\varphi\rangle \\
&= -\langle T, (f.\varphi)'\rangle + \langle T, f'.\varphi\rangle \\
&= \langle T', f.\varphi\rangle + \langle f'.T, \varphi\rangle \\
&= \langle f.T' + f'.T, \varphi\rangle.
\end{aligned}$$

   which lead to: $(f.T)' = f.T' + f'.T$.
   Then:

$$(f.T)'' = (f.T' + f'.T)' = f.T'' + 2f'.T' + f'.T.$$

2. $\langle T_1 + T_2, \varphi\rangle = \langle T_1, \varphi\rangle + \langle T_2, \varphi\rangle = 0$.

CHAPTER 3

CONVOLUTION PRODUCT

The role of the convolution product is to regularize certain functions with bad behaviour. In this chapter, we will generalize the convolution product, which is defined on functions. We begin with a brief motivation.

Let P and Q be two polynomials of degree p and q, taking values in $\mathbb{R}$ or $\mathbb{C}$, given by:

$$P(x) = \sum_{j=0}^{p} a_j x^j, \qquad Q(x) = \sum_{j=0}^{q} b_j x^j.$$

The product of P and Q is given by:

$$(P.Q)(x) = \sum_{j=0}^{p+q} \sum_{k=0}^{j} a_k.b_{j-k} x^j$$

We extend the two sequences (a_j) and (b_j) by adding zeros towards $\mathbb{Z}$ and still denote them as (a_j) and (b_j). The two polynomials P and Q define two formal power series:

$$P(x) = \sum_{j \in \mathbb{Z}} a_j x^j, \qquad Q(x) = \sum_{j \in \mathbb{Z}} b_j x^j.$$

Thus, the product $P * Q$ is given by the formal power series $\sum_{j \in \mathbb{Z}} c_j x^j$, where

$$c_j = \sum_{k=0}^{j} a_k.b_{j-k} x^j.$$

The series $\sum_{j \in \mathbb{Z}} c_j x^j$ is called the product of the series $\sum_{j \in \mathbb{Z}} a_j x^j$ and $\sum_{j \in \mathbb{Z}} b_j x^j$.

We can now consider two arbitrary absolutely convergent power series $\sum_{j \in \mathbb{Z}} a_j x^j$ and

$\sum_{j\in\mathbb{Z}} b_j x^j$. Then, the product of these series is also absolutely convergent.

A similar result is obtained when replacing (a_j) and (b_j) with integrable functions f and g, resulting in the quantity

$$\int f(y).g(x-y)dy,$$

which called the convolution product of f and g.

3.1 Convolution of functions

Definition 3.1 *: Let $f, g \in L^1_{loc}(\mathbb{R}^n)$. The convolution product de de f et g, denoted as $f*g$ is the function defined by:*

$$(f*g)(x) = \int_{\mathbb{R}^n} f(y).g(x-y)dy, \;\; x \in \mathbb{R}^n. \tag{3.1}$$

Proposition 3.1 : *If $f*g$ exists we have:*

1. $f*g = g*f$
2. $\operatorname{supp}(f*g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}$.

Proof: Suppose that $f*g$ exists.

1. Let $x \in \mathbb{R}^n$, then:
$(f*g)(x) = \int_{\mathbb{R}^n} f(y).g(x-y)dy = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(y).g(x-y)dy_1 \cdots dy_n$.
Making the change of variable $z = x - y$, we obtain:
$(f*g)(x) = \int_{+\infty}^{-\infty} \cdots \int_{+\infty}^{-\infty} f(x-z).g(z)(-dz_1)\cdots(-dz_n)$.
Then: $(f*g)(x) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x-z).g(z)dz_1 \cdots dz_n = \int_{\mathbb{R}^n} f(x-z).g(z)dz$.
Hence: $(f*g)(x) = (g*f)(x)$.

2. Let $x \notin (\operatorname{supp} f + \operatorname{supp} g)$. Thus: fur all $y \in \operatorname{supp} f$ we have: $x - y \notin \operatorname{supp} g$, i.e $(f*g)(x) = 0$.
So: the null open set of $f*g$ contains $\overbrace{C_{\mathbb{R}^n}^{\operatorname{supp} f + \operatorname{supp} g}}^{0}$.
Then: $\operatorname{supp}(f*g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}$.

■

Proposition 3.2 : *Let $f \in L^p(\mathbb{R}^n), g \in L^{p'}(\mathbb{R}^n)$ $(1 \le p \le +\infty, \frac{1}{p}+\frac{1}{p'} = 1$ ou $p' = +\infty$ si $p = 1)$, then: $f*g$ is defined everywhere, bounded, Moreover: $\|f*g\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)}.\|g\|_{L^{p'}(\mathbb{R}^n)}$.*

Proof: Suppose that $f \in L^p(\mathbb{R}^n), g \in L^{p'}(\mathbb{R}^n)$. Si $1 < p < +\infty$, By the Hölder's inequality:

$$\begin{aligned} |(f*g)(x)| &\leq \left(\int_{\mathbb{R}^n} |f(y)|^p dy\right)^{\frac{1}{p}} \cdot \left(\int_{\mathbb{R}^n} |g(x-y)|^{p'} dy\right)^{\frac{1}{p'}} \\ &= \left(\int_{\mathbb{R}^n} |f(y)|^p dy\right)^{\frac{1}{p}} \cdot \left(\int_{\mathbb{R}^n} |g(y)|^{p'} dy\right)^{\frac{1}{p'}}. \end{aligned}$$

So, $|(f*g)(x)| \leq \|f\|_{L^p(\mathbb{R}^n)}.\|g\|_{L^{p'}(\mathbb{R}^n)}$.
If $p = 1$ then: $p' = +\infty$ and we have:

$$|(f*g)(x)| \leq \|g\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f(y)|dy = \|f\|_{L^1(\mathbb{R}^n)}.\|g\|_{L^\infty(\mathbb{R}^n)}.$$

Then: $f*g$ is defined everywhere, is bounded, and we have: $\|f*g\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}.\|g\|_{L^{p'}(\mathbb{R}^n)}$.
■

Remark 3.1 :

i) *If* $p \in]1, +\infty[$ *then:* $f*g$ *est continue.*

ii) *Si* $p, q \in [1, +\infty], f \in L^p(\mathbb{R}^n)$*, with a compact support,* $g \in L^q_{loc}(\mathbb{R}^n)$*, then:* $f*g$ *is continuous.*

Proposition 3.3 : *Let* $f, g \in L^1(\mathbb{R}^n)$*, then:* $f*g \in L^1(\mathbb{R}^n)$ *et on a:*

$$\int_{\mathbb{R}^n} (f*g)(x)dx = \left(\int_{\mathbb{R}^n} f(x)dx\right) \cdot \left(\int_{\mathbb{R}^n} g(y)dy.\right)$$

Proof: We have:

$$\int_{\mathbb{R}^n} (f*g)(x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)g(y)dxdy.$$

By applying the Fubini's theorem, we obtain:

$$\int_{\mathbb{R}^n} (f*g)(x)dx = \int_{\mathbb{R}^n} f(x-y)dx \int_{\mathbb{R}^n} g(y)dy.$$

However, we have: $\int_{\mathbb{R}^n} f(x-y)dx = \int_{\mathbb{R}^n} f(x)dx$. So,

$$\int_{\mathbb{R}^n} (f*g)(x)dx = \int_{\mathbb{R}^n} f(x)dx. \int_{\mathbb{R}^n} g(y)dy.$$

■

Proposition 3.4 : *Let* $p, q \in [1, +\infty]$ *such that* $\frac{1}{p} + \frac{1}{q} \geq 1$*, then:* $f*g$ *defined a.e. in* $\mathbb{R}^n$*. Moreover, if* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$*, then:* $(f*g) \in L^r(\mathbb{R}^n)$ *and we have:*

$$\|f*g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}.\|g\|_{L^q(\mathbb{R}^n)}.$$

Proof: By writing: $|f(y)g(x-y)| = (|f(y)|^p|g(x-y)|^q)^{\frac{1}{r}}.(|f(y)|^p)^{\frac{1}{p}-\frac{1}{r}}.(|g(x-y)|^q)^{\frac{1}{q}-\frac{1}{r}}$.
Now, since $|f|^p \in L^1(\mathbb{R}^n), |g|^q \in L^1(\mathbb{R}^n)$ on a: $|f|^p * |g|^q \in L^1(\mathbb{R}^n)$.
By applying the generalized Hölder's inequality, taking into account the relationship: $\frac{1}{r} + \left(\frac{1}{p} - \frac{1}{r}\right) + \left(\frac{1}{q} - \frac{1}{r}\right) = 1$, we obtain:

$$\int_{\mathbb{R}^n} |f(x)g(x-y)|dy \leq \left(\int_{\mathbb{R}^n} |f(x)|^p|g(x-y)|^q dy\right)^{\frac{1}{r}} . \left(\int_{\mathbb{R}^n} |f(y)|^p dy\right)^{1-\frac{p}{r}} . \left(\int_{\mathbb{R}^n} |f(y)|^q dy\right)^{1-\frac{q}{r}}$$

Then:

$$|(f*g)(x)| \leq |(|f|^p * |g|^q)(x)|^{\frac{1}{r}} \|f\|^{1-\frac{p}{r}}_{L^p(\mathbb{R}^n)}.\|g\|^{1-\frac{q}{r}}_{L^q(\mathbb{R}^n)}.$$

Integrating with respect to x, we obtain:

$$\int_{\mathbb{R}^n} |(f*g)(x)|^r dx \leq \||f|^p * |g|^q\|_{L^1(\mathbb{R}^n)} \|f\|^{r-p}_{L^p(\mathbb{R}^n)}.\|g\|^{r-q}_{L^q(\mathbb{R}^n)}.$$

But: $\||f|^p * |g|^q\|_{L^1(\mathbb{R}^n)} \leq \||f|^p\|_{L^1(\mathbb{R}^n)}.\||g|^q\|_{L^1(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}.\|g\|_{L^q(\mathbb{R}^n)}$.
So, $\|f*g\|^r_{L^r(\mathbb{R}^n)} \leq \|f\|^r_{L^p(\mathbb{R}^n)}.\|g\|^r_{L^q(\mathbb{R}^n)}$. ■

Proposition 3.5 : *Let $k \in \mathbb{N}, f \in L^1_{Loc}(\mathbb{R}^n), g \in \mathscr{C}^k(\mathbb{R}^n)$, $\operatorname{supp} f$ ou $\operatorname{supp} f$ be compact, then: $f*g \in \mathscr{C}^k(\mathbb{R}^n)$, and for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq k$ we have: $D^\alpha(f*g) = f*D^\alpha g$.*

Proof:

i) Case where $\operatorname{supp} f$ is compact:

Since $g \in \mathscr{C}^k(\mathbb{R}^n)$, for all $|\alpha| \leq k$, the function $D^\alpha g$ is bounded on any compact set (locally bounded). It follows that $f * D^\alpha g$ is continuous.

The function $x \mapsto D^\alpha g(x-y)$ is dominated, and by applying the theorem of differentiation under the integral sign, we obtain the result.

ii) Case where $\operatorname{supp} g$ is compact:

Since $f \in L^1_{loc}$ and $D^\alpha g$ is bounded, then $f * D^\alpha g$ is well-defined and continuous.

For the equality $D^\alpha(f*g) = f*D^\alpha g$, we refer to [13], tome 1, p. 122.

■

Proposition 3.6 : *Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a regularization sequence (see definition 1.24) and $f \in L^1_{loc}(\mathbb{R}^n)$. The sequence $\{f_j\} = \{\varphi_j * f\}$ is called a regularized sequence, and it satisfies:*

1. *For all $j \in \mathbb{N}$ we have: $f_j \in \mathscr{C}^\infty(\mathbb{R}^n)$.*
2. *If $f \in \mathscr{C}^\infty(\mathbb{R}^n)$, then $f_j \to f$ in $\mathscr{C}^\infty(\mathbb{R}^n)$.*
3. *If $f \in \mathscr{D}(\mathbb{R}^n)$, then $f_j \to f$ in $\mathscr{D}(\mathbb{R}^n)$.*
4. *If $f \in L^p(\mathbb{R}^n)$, then $f_j \to f$ in $L^p(\mathbb{R}^n)$.*

Proof: According to Definition 1.24, $\varphi_j \in \mathscr{D}(\mathbb{R}^n)$, and there exists $0 < \varepsilon_j \longrightarrow 0$ such that:

$$\varphi_j \geq 0, \qquad \operatorname{supp}\varphi_j \subset B(0,\varepsilon_j), \qquad \int_{B(0,\varepsilon_j)} \varphi_j(x)dx = \int_{\mathbb{R}^n} \varphi_j(x)dx = 1.$$

$$f_j(x) = (\varphi_j * f)(x) = \int_{\mathbb{R}^n} f(y)\varphi_j(x-y)dy.$$

1. For any $k \in \mathbb{N}$, we can consider $\varphi_j \in \mathscr{C}^k(\mathbb{R}^n)$ with compact support. Then, according to Proposition 3.5, $f_j \in \mathscr{C}^k(\mathbb{R}^n)$, and since k is arbitrary, it follows that $f_j \in \mathscr{C}^\infty(\mathbb{R}^n)$.

2. Let $K \subset \mathbb{R}^n$ be a compact set and $m \in \mathbb{R}^n$. For $|\alpha| \leq m$, there exists $j_0 \in \mathbb{R}^n$ such that for all $j \geq j_0$, we have $B(0,\varepsilon_j) \subset K$. Then:

$$\begin{aligned}
|D^\alpha(f_j)(x) - D^\alpha f(x)| &= |D^\alpha(f * \varphi_j)(x) - D^\alpha f(x)| \\
&= |(D^\alpha f * \varphi_j)(x) - D^\alpha f(x)| \\
&= \left| \int_{\mathbb{R}^n} f(x-y).\varphi_j(y)dy - \int_{\mathbb{R}^n} f(x).\varphi_j(y)dy \right| \\
&= \left| \int_{\mathbb{R}^n} (D^\alpha f(x-y) - D^\alpha f(x)).\varphi_j(y)dy \right| \\
&= \int_{\mathbb{R}^n} |D^\alpha f(x-y) - D^\alpha f(x)|.\varphi_j(y)dy \\
&= \int_{B(0,\varepsilon_j)} |D^\alpha f(x-y) - D^\alpha f(x)|.\varphi_j(y)dy \\
&\leq \sup_{y \in B(0,\varepsilon_j)} |D^\alpha f(x-y) - D^\alpha f(x)| \int_{B(0,\varepsilon_j)} \varphi_j(y)dy \\
&\leq \sup_{y \in B(0,\varepsilon_j)} |D^\alpha f(x-y) - D^\alpha f(x)|.
\end{aligned}$$

   Therefore, $\sup_{x\in K} |D^\alpha(f_j)(x) - D^\alpha f(x)| \leq \sup_{x \in K} \sup_{y \in B(0,\varepsilon_j)} |D^\alpha f(x-y) - D^\alpha f(x)|$.

   Due to the continuity of D^α, we have $\lim_{\varepsilon \to 0} \sup_{y \in B(0,\varepsilon_j)} |D^\alpha f(x-y) - D^\alpha f(x)| = 0$.

   Therefore, $\lim_{\varepsilon \to 0} \sup_{x \in K} |D^\alpha(f_j)(x) - D^\alpha f(x)| = 0$, which implies that $f_j \to f$ in $\mathscr{C}^\infty(\mathbb{R}^n)$.

3. Since $f \in \mathscr{D}(\mathbb{R}^n)$, there exists $K \subset \mathbb{R}^n$ such that $\operatorname{supp} f \subset K$, and $\operatorname{supp}\varphi_j \subset K$ for all $j \in \mathbb{R}^n$ (this is possible because $\varepsilon \longrightarrow 0$). Since $f_j \to f$ in $\mathscr{C}^\infty(\mathbb{R}^n)$, we have: $f_j \to f$ in $\mathscr{D}(\mathbb{R}^n)$,

4. Let $\varepsilon > 0$. From the density of $\mathscr{K}(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ (see theorem IV.12 in [5]) there exists a fixed $g \in \mathscr{K}(\mathbb{R}^n)$ such that $\|f - g\|_{L^p(\mathbb{R}^n)} < \varepsilon$.
   Using similar arguments as in 2., the sequence $(\varphi_j * g)$ converges to g in $\mathscr{K}(\mathbb{R}^n)$, i.e $(\varphi_j * g)$ converges to g uniformly on any compact set.
   We have: $\operatorname{supp}(\varphi_j * g) \subset \operatorname{supp}\varphi_j + \operatorname{supp} g \subset K$, where K is a fixed compact. Then:

$$\|\varphi_j * g - g\|^p_{L^p(\mathbb{R}^n)} = \int_K |(\varphi_j * g)(x) - g(x)|^p dx \leq mes(K).\sup_{x \in K} |(\varphi_j * g)(x) - g(x)|^p \overset{j \longrightarrow +\infty}{\longrightarrow} 0$$

By writing: $\varphi_j * f - f = (\varphi_j * (f-g)) + (\varphi_j * g - g) + (f-g)$, it follows:

$$\begin{aligned}
\|\varphi_j * f - f\|_{L^p(\mathbb{R}^n)} &\leq \|\varphi_j * (f-g)\|_{L^p(\mathbb{R}^n)} + \|\varphi_j * g - g\|_{L^p(\mathbb{R}^n)} + \|f-g\|^p_{L^p(\mathbb{R}^n)} \\
&\leq \|\varphi_j\|_{L^1(\mathbb{R}^n)}.\|f-g\|_{L^p(\mathbb{R}^n)} + \|\varphi_j * g - g\|_{L^p(\mathbb{R}^n)} + \|f-g\|_{L^p(\mathbb{R}^n)} \\
&= 2\|f-g\|_{L^p(\mathbb{R}^n)} + \|\varphi_j * g - g\|_{L^p(\mathbb{R}^n)} \\
&\leq 3\varepsilon
\end{aligned}$$

Therefore, $\varphi_j * f$ tends to f in $L^p(\mathbb{R}^n)$.

This completes the proof. ■

3.2 Main results

These results play an important role in the definitions related to the convolution product between distributions.

Definition 3.2 (***convolution-compatible family***) :

i) *Two closed sets F and $G \subset \mathbb{R}^n$ are said to be convolution-compatible if:*

$$\forall R > 0, \exists \rho > 0 : (x \in F \wedge y \in G \wedge |x+y| < R) \Rightarrow (|x| < \rho \wedge |y| < \rho).$$

ii) *A finite family of closed sets $(F_j)_{j \in J} \subset \mathbb{R}^n$ is said to be convolution-compatible if:*

$$\forall I \subset J, \forall R > 0, \exists \rho > 0 : ((x_j)_{i \in I} \in F_i \wedge \left|\sum_{i \in I} x_i\right| < R) \Rightarrow (|x_i| < \rho, i \in I).$$

Example 3.1 : *Suppose that A is closed, and B is compact. Since B is compact, there exists $r > 0$ such that for all $y \in B$, we have $|y| \leq r$. Let $R > 0$, $x \in A$, and $y \in B$ such that $|x+y| < R$. We have:*
$|x| = |x+y-y| \leq |x+y| + |y| < r + R$ *and* $|y| \leq r < r + R$.
Therefore, A and B are convolution-compatible.

Example 3.2 : *Consider the finite family $\{[a_i, +\infty[\}_{i \in I}$ in $\mathbb{R}$.*
Let $R > 0$ and let $x_i \in [a_i, +\infty[$ such that $\left|\sum_{i \in I} x_i\right| < R$. We have:
$0 \leq \sum_{i \in I}(x_i - a_i) \leq \left|\sum_{i \in I} x_i\right| + \left|\sum_{i \in I} a_i\right| < R + \left|\sum_{i \in I} a_i\right|.$
Then: $a_i \leq x_i \leq R + \left|\sum_{i \in I} a_i\right|$. There exists ρ_i such that $|x_i| \leq \rho_i$.
We can set: $\rho = \max_{i \in I} \rho_i$ and find that : $|x_i| \leq \rho$ pour tout i.
Therefore, the family $([a_i, +\infty[)_{i \in I}$ is convolution-compatible.

Example 3.3 : *Consider the two intervals $[a,+\infty[$ and $]-\infty,b]$. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $n > a$ and $-n < b$. Therefore, $n+(-n) = 0 < R$ for any $R > 0$, but n tends to $+\infty$, so it is unbounded. Hence, $[a,+\infty[$ and $]-\infty,b]$ are not convolution-compatible for any $a, b \in \mathbb{R}$.*

Proposition 3.7 : *If F and G are two convolution-compatible closed sets, then $F+G$ is closed.*

Proof: : Let $x_j + y_{j_{j\in\mathbb{R}^n}}$ be a sequence in $F+G$ that converges to $z \in \mathbb{R}^n$. This means that the sequence $\{x_j + y_j\}$ is bounded, i.e., there exists $R > 0$ such that $x_j + y_j < R$ for all $j \in \mathbb{N}$.

Since F and G are convolution-compatible, there exists $\rho > 0$ such that $x_j < \rho$ and $y_j < \rho$ for all $j \in \mathbb{N}$. Both sequences are bounded, so we can extract two sequences $\{x_{jk}\}$ and $\{y_{jk}\}$ such that $\{x_{jk}\}$ converges to x and $\{y_{jk}\}$ converges to y. Therefore, $\{x_{jk} + y_{jk}\}$ converges to $x+y$, and by the uniqueness of limits, we deduce that $z = x + y \in F + G$.

Hence, $F+G$ is a closed set in $\mathbb{R}^n$. ■

Theorem 3.1 (***Distributional Derivative Under the Bracket***) : *Let $\varphi \in \xi(\mathbb{R}^{p+q}), T \in \xi'(\mathbb{R}^p)$. The function:*

$$\begin{array}{rccl} f: & \mathbb{R}^q & \longrightarrow & \mathbb{R} \\ & y & \longmapsto & f(y) = \langle T(x), \varphi(x,y)\rangle \end{array}$$

is of class $\mathscr{C}^\infty(\mathbb{R}^q)$ and for all $\alpha \in \mathbb{N}^n$ we have: $D^\alpha f(y) = \langle T(x), D_y^\alpha \varphi(x,y)\rangle$.

Proof: Let $x \in \mathbb{R}^q$ et $y_0 \in \mathbb{R}^q$. According to the Taylor expansion formula, for $h \in \mathbb{R}^q$:

$$\varphi(x, y_0 + h) = \varphi(x, y_0) + \sum_{i=0}^{q} \frac{\partial \varphi}{\partial y_i}.h_i + R(x, y_0, h),$$

where $R(x, y_0, h) = 2 \displaystyle\sum_{|\alpha|\leq 2} \frac{h^\alpha}{\alpha!} \int_0^1 (1-t) D_y^\alpha \varphi(x, y_0 + th)dt.$

The function $y \longmapsto R(x, y_0, h)$ is of class $\mathscr{C}^\infty$, and T is distribution with compact support. Therefore, there exist $M > 0$ and $m \in \mathbb{N}$ such that in the compact neighbourhood K of $\operatorname{supp} T$:

$$|\langle T, R(x, y_0, h)\rangle| \leq M P_{K,m}(R) = M. \sup_{|\beta|\leq m} \sup_{(x,y)\in K} |D_y^\beta R(x, y_0, h)|.$$

For $|h|$ sufficiently small:

$$|D_y^\beta R(x, y_0, h)| \leq C_1.|h|^2 \sup_{|\alpha|\leq 2} |D_y^\alpha a D_y^\beta \varphi(x, y_0)| \leq C P_{K,m+2}(\varphi)$$

We deduce that $|\langle T, R(x, y_0, h)\rangle| = o(|h|^2)$.Then:

$$\langle T, \varphi(x, y_0 + h)\rangle = \langle T, \varphi(x, y_0)\rangle + \sum_{i=0}^{q} \langle T, \frac{\partial \varphi}{\partial y_i}\rangle.h_i + o(|h|^2),$$

so, $f(y_0+h) = f(y_0) + \sum_{i=0}^{q} \langle T, \frac{\partial \varphi}{\partial y_i}.h_i + o(|h|^2)$.
Therefore: f is differentiable at the point y_0 and we have: $\frac{\partial f}{\partial y_i} = \langle T, \frac{\partial \varphi}{\partial y_i}$.
Since this holds for all i, we conclude that $f \in \mathscr{C}^1$. The result is then obtained by induction.
■

Theorem 3.2 (***Distributional integration under the bracket***) : *Let $\varphi \in \xi(\mathbb{R}^{p+q}), T \in \xi'(\mathbb{R}^p)$. Let P be a compact slab of $\mathbb{R}^q$. Then:*

$$\left\langle T(x), \int_P \varphi(.,y)dy \right\rangle = \int_P \langle T(x), \varphi(.,y)\rangle \, dy.$$

Proof: Writing $P = [a_1, b_1] \times \cdots \times [a_q, b_q]$, we obtain:

$$\int_P \varphi(.,y)dy = \int_{a_1}^{b_1} \cdots \int_{a_q}^{b_q} \varphi(.,y)dy_1 \cdots dy_q.$$

Let $F_i(y_i) = \left\langle T(x), \int_{a_i}^{y_i} \varphi(., y_1, \cdots s, \cdots, y_q)ds \right\rangle$ $(1 \leq i \leq q)$.

Applying the previous theorem, we get:

$$F_i'(y_i) = \langle T(x), \varphi(., y_1, \cdots y_i, \cdots, y_q)\rangle .$$

Hence: $F_i(y_i) = \int_{a_i}^{y_i} \langle T(x), \varphi(., y_1, \cdots s, \cdots, y_q)\rangle \, ds.$
Thus, by integrating q times, we obtain the result. ■

Remark 3.2 **:** *The compact slab P can be replaced by another measurable set.*

3.3 Convolution of a function with a distribution

We can express the convolution product of a function $f \in L^1(\mathbb{R}^n)$ and a function $\varphi \in \mathscr{D}(\mathbb{R}^n)$ as follows:

$$(f * \varphi)(x) = \int_{\mathbb{R}^n} \varphi(x-y)f(y)dy = \langle f(y), \varphi(x-y)\rangle = \langle f, \tau_x \check{\varphi}\rangle$$

where $\tau_x \check{\varphi}(y) = \varphi(x-y)$.

This brings us back to the following definitions:

Definition 3.3 : *Let $T \in \mathscr{D}'(\mathbb{R}^n), \varphi \in \mathscr{D}(\mathbb{R}^n)$. The convolution product $T * \varphi$ is defined as follows:*

$$\forall x \in \mathbb{R}^n : (T * \varphi)(x) = \langle T, \tau_x \check{\varphi}\rangle . \tag{3.2}$$

We can extend the previous result in the case where T has compact support and $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$. We have the following definition:

Definition 3.4 : *Let $T \in \mathscr{E}'(\mathbb{R}^n)$ and $\varphi \in \mathscr{E}(\mathbb{R}^n)$. The convolution product $T * \varphi$ is defined as follows: For all $x \in \mathbb{R}^n$, $(T * \varphi)(x) = \langle T, \tau_x \check{\varphi} \rangle_{\mathscr{E}', \mathscr{E}}$.*

Theorem 3.3 : *Let $T \in \mathscr{D}'(\mathbb{R}^n), \varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ be such that* $\operatorname{supp} \varphi$ *where* $\operatorname{supp} T$ *be a compact. Then:*

1. $T * \varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$.
2. *For all $\alpha \in \mathbb{N}^n$ we have:* $D^\alpha(T * \varphi) = T * D^\alpha \varphi = (D^\alpha T) * \varphi$.
3. $\operatorname{supp}(T * \varphi) \subset \operatorname{supp} T + \operatorname{supp} \varphi$.

Proof: : From the definition: $T * \varphi$ is a function defined on $\mathbb{R}^n$.

1. Since $\tau_x \check{\varphi} \in \mathscr{C}^\infty(\mathbb{R}^n)$, we deduce from Theorem 3.1 that $T * \varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$.
2. Let $\forall \alpha \in \mathbb{N}^n$, then:

$$\begin{aligned} D^\alpha(T * \varphi)(x) &= D^\alpha \langle T, \tau_x \check{\varphi} \rangle \\ &= \langle T, D^\alpha \tau_x \check{\varphi} \rangle \\ &= |-1|^{|\alpha|} \langle T, \tau_x D^\alpha \check{\varphi} \rangle \\ &= \langle T, \tau_x \check{D}^\alpha \varphi \rangle \\ &= (T * D^\alpha \varphi)(x) \end{aligned}$$

On the other hand:

$$\begin{aligned} D^\alpha(T * \varphi)(x) &= D^\alpha \langle T, \tau_x \check{\varphi} \rangle \\ &= \langle T, D^\alpha \tau_x \check{\varphi} \rangle \\ &= |-1|^{|\alpha|} \langle T, \tau_x D^\alpha \check{\varphi} \rangle \\ &= \langle D^\alpha T, \tau_x \check{\varphi} \rangle \\ &= (D^\alpha T * \varphi)(x) \end{aligned}$$

hence: $D^\alpha(T * \varphi) = T * D^\alpha \varphi = (D^\alpha T) * \varphi$.

3. Using arguments analogous to the proof of Proposition 3.1, part 2.

■

Theorem 3.4 : *Let $T \in \mathscr{D}'(\mathbb{R}^n)$, φ, and $\psi \in \mathscr{C}^\infty(\mathbb{R}^n)$ such that either* $\operatorname{supp} T$ *is compact, or* $\operatorname{supp} \varphi$ *and* $\operatorname{supp} \psi$ *are both compact. Then:*

1. $(T * \varphi) * \psi = T * (\varphi * \psi)$
2. $\langle T * \varphi, \psi \rangle = \langle T, \varphi * \psi \rangle$

Proof: We have:

1. On one hand:

$$\begin{aligned}(T*\varphi)*\psi(x) &= \int_{\mathbb{R}^n}(T*\varphi)(y)\psi(x-y)dy\\ &= \int_{\mathbb{R}^n}\langle T(z),\varphi(y-z)\rangle\psi(x-y)dy\\ &= \left\langle T(z),\int_{\mathbb{R}^n}\varphi(y-z)\psi(x-y)dy)\right\rangle.\end{aligned}$$

   On the other hand:

$$\begin{aligned}T*(\varphi*\psi)(x) &= \langle T(z),(\varphi*\psi)(x-z)\rangle\\ &= \left\langle T(z),\int_{\mathbb{R}^n}\varphi(x-z-t)\psi(t)dt\right\rangle.\end{aligned}$$

   Using the change of variable $t=x-y$, we get:

$$\int_{\mathbb{R}^n}\varphi(x-z-t)\psi(t)dt=\int_{\mathbb{R}^n}\varphi(y-z)\psi(x-y)dy.$$

   Therefore: $T*(\varphi*\psi)(x)=\int_{\mathbb{R}^n}\varphi(y-z)\psi(x-y)dy.$
   which leads to: $(T*\varphi)*\psi=T*(\varphi*\psi)$.

2.

$$\begin{aligned}\langle T*\varphi,\psi\rangle &= \int_{\mathbb{R}^n}(T*\varphi)(x)\psi(x)dx\\ &= \int_{\mathbb{R}^n}\langle T(y),\varphi(x-y)\rangle\psi(x)dx\\ &= \left\langle T(y),\int_{\mathbb{R}^n}\varphi(x-y)\psi(x)dx)\right\rangle.\\ &= \langle T(y),(\varphi*\psi)(y)\rangle.\end{aligned}$$

   Then: $\langle T*\varphi,\psi\rangle=\langle T,\varphi*\psi\rangle$.

■

Proposition 3.8 : *Let* $T\in\xi'(\mathbb{R}^n),\varphi\in\mathscr{D}(\mathbb{R}^n)$. *Then:*

1. $\forall a\in\mathbb{R}^n:\tau_aT*\varphi=T*\tau_a\varphi=\tau_a(T*\varphi)$

2. $\langle T,\varphi\rangle=(T*\check{\varphi})(0)$

Proof: Let $T\in\xi'(\mathbb{R}^n),\varphi\in\mathscr{D}(\mathbb{R}^n)$.

1. Let $a\in\mathbb{R}^n$. On one hand:

$$\begin{aligned}\tau_aT*\varphi(x) &= \langle\tau_aT,\varphi(x-y)\rangle\\ &= \langle T,\varphi(a+x-y)\rangle\\ &= \langle T,\tau_x\check{\varphi}(-a+y)\rangle\\ &= \langle T,\tau_x\check{\tau_a\varphi}\rangle\\ &= (T*\tau_a\varphi)(x).\end{aligned}$$

On the other hand:

$$\begin{aligned}\tau_a(T*\varphi)(x) &= (T*\varphi)(x-a)\\ &= \langle T,\varphi(x-a-y)\rangle\\ &= \langle T,\check{\varphi}(a+y-x)\rangle\\ &= \langle \tau_a T,\check{\varphi}(y-x)\rangle\\ &= \langle \tau_a T,\tau_x\check{\varphi}\rangle\\ &= (\tau_a T*\varphi)(x).\end{aligned}$$

Then: $\tau_a T*\varphi = T*\tau_a\varphi = \tau_a(T*\varphi)$

2. $\langle T,\varphi\rangle = \langle T,\tau_0\varphi\rangle = \langle T,\tau_0\check{\check{\varphi}}\rangle = (T*\check{\varphi})(0)$.

■

No, we will extend the convolution product de which defined on $\mathscr{E}'(\mathbb{R}^n)\times\mathscr{E}(\mathbb{R}^n)$ to $\mathscr{D}'(\mathbb{R}^n)\times\mathscr{E}(\mathbb{R}^n)$ as follows:

Definition 3.5 : *Let $T\in\mathscr{D}'(\mathbb{R}^n),\psi\in\mathscr{E}(\mathbb{R}^n)$ be such that $\operatorname{supp}T,\operatorname{supp}\psi$ are convolutes. Let $(\psi_j)\subset\mathscr{D}(\mathbb{R}^n)$ be a regularization sequence. For any $\varphi\in\mathscr{D}(\mathbb{R}^n)$, we put:*

$$\langle T*\psi,\varphi\rangle = \lim_{j\to+\infty}\langle T*\psi_j,\varphi\rangle.$$

The following result becomes a consequence of the convolution product:

Theorem 3.5 : *The space $\mathscr{D}(\Omega)$ is dense in $\mathscr{D}'(\Omega)$.*

3.4 Tensor product

Let $U\subset\mathbb{R}^n, V\subset\mathbb{R}^m$ be two open sets.

Definition 3.6 (***Tensor product of functions***) : *Let $f:U\longrightarrow\mathbb{R}, g:V\longrightarrow\mathbb{R}$. The tensor product $f\otimes g$ of f and g is the function defined on $U\times V$ as follows:*

$$\forall(x,y)\in U\times V: f\otimes g(x,y) = f(x).g(y).$$

The following properties directly follow from the definition:

Proposition 3.9 :

1. *If $f\in\xi(U), g\in\xi(V)$, then: $f\otimes g\in\xi(U\times V)$.*
2. $\operatorname{supp} f\otimes g = \operatorname{supp} f\times\operatorname{supp} g$.

The following result is important for what follows:

$\mathscr{D}(U)\times\mathscr{D}(V)$ is dense in $\mathscr{D}(U\times V)$.

Note that if $f \in L^1_{loc}(U), g \in L^1_{loc}(V)$ et $\Phi \in \mathscr{D}(U \times V)$then from Fubini's theorem we have:

$$\begin{aligned}
\iint_{U\times V} (f\otimes g)(x,y)\Phi(x,y)dxdy &= \iint_{U\times V} f(x)g(y)\Phi(x,y)dxdy \\
&= \int_U f(x)dx \int_V g(y)\Phi(x,y)dy \\
&= \int_V g(y)dy \int_U f(x)\Phi(x,y)dx
\end{aligned}$$

Then:

$$\langle f\otimes g, \varphi\rangle = \langle f, \langle g(y), \Phi(.,y)\rangle\rangle = \langle g, \langle f(x), \Phi(x,.)\rangle\rangle.$$

If we have: $\Phi(x,y) = (\varphi\otimes\psi)(x,y) = \varphi(x).\psi(y)$, then:

$$\langle f\otimes g, \varphi\otimes\psi\rangle = \langle f,\varphi\rangle.\langle g,\psi\rangle.$$

We have a result similar to the result above concerning the distributions:

Theorem 3.6 : *Let $T \in \mathscr{D}'(U), S \in \mathscr{D}'(V)$. There exists a unique $W \in \mathscr{D}'(U\times V)$ such that for all $\varphi \in \mathscr{D}(U)$ and for all $\varphi \in \mathscr{D}(V)$ we have:*

$$\langle T\otimes S, \varphi\otimes\psi\rangle = \langle T,\varphi\rangle.\langle S,\psi\rangle.$$

Moreover, we have for any $\Phi \in \mathscr{D}(U\times V)$:

$$\langle W,\Phi\rangle = \langle T, \langle S(y), \Phi(.,y)\rangle\rangle = \langle S, \langle T(x), \Phi(x,.)\rangle\rangle.$$

Proof: Setting: $F(x) = \langle S(y), \Phi(.,y)\rangle$. From Theorem 3.1, we have: $F \in \mathscr{C}^\infty(U)$ and more preciously $F \in \mathscr{D}(U)$.Consider $K = G\times H \subset U\times V$ a compact from $\mathbb{R}^n\times\mathbb{R}^m$. Suppose that $\Phi \in \mathscr{D}_K(U\times V)$. Set $\langle W,\Phi\rangle = \langle T,F\rangle$. Then, there exists $M_1 > 0, M_2 > 0, m_1 \in \mathbb{N}$ et $m_2 \in \mathbb{N}$ such that:

$$\begin{aligned}
|\langle W,\Phi\rangle| &= |\langle T,F\rangle| \\
&\leq M_1. \sup_{|\alpha|\leq m_1, x\in K_1} |F(x)| \\
&= M_1. \sup_{|\alpha|\leq m_1, x\in K_1} |\langle S(y), \Phi(.,y)\rangle| \\
&\leq M_1.m_2 \sup_{|\alpha|\leq m_1, x\in K_1} \sup_{|\beta|\leq m_2, x\in K_2} |\Phi(x,y)| \\
&\leq M.P_{K,m_1+m_2}(\Phi).
\end{aligned}$$

Then: W defines a unique distribution (by definition).
The second formula becomes according to the density of $\mathscr{D}(U)\times\mathscr{D}(V)$ in $\mathscr{D}(U\times V)$. ■

Here's the translation of the provided the following definition:

Definition 3.7 (***tensor product of distributions***) : *Let $T \in \mathscr{D}'(U), S \in \mathscr{D}'(V)$. The tensor product of T and S is the distribution noted by $T\otimes S \in \mathscr{D}'(U\times V)$, and defined as* *follows:*

$$\forall\varphi \in \mathscr{D}(U), \forall\varphi \in \mathscr{D}(V) : \langle T \otimes S, \varphi \otimes \psi\rangle = \langle T, \varphi\rangle.\langle S, \psi\rangle.$$

The general formula is given as follows:

$$\forall\Phi \in \mathscr{D}(U \times V) : \langle T \otimes S, \Phi\rangle = \langle T, \langle S, \Phi(., y)\rangle\rangle = \langle S, \langle T, \Phi(x, .)\rangle\rangle$$

Remark 3.3 *: The tensor product remains valid for distributions with compact support by using the bracket* $\langle ., .\rangle_{\mathscr{E}',\mathscr{E}}$.

Example 3.4 : *Let* $a \in \mathbb{R}^n$ *and* $b \in \mathbb{R}^m$*. For all* $\Phi \in \mathscr{D}(\mathbb{R}^{n+m})$ *we have:*

$$\begin{aligned}\langle \delta \otimes \delta_b, \Phi\rangle &= \langle \delta_a, \langle \delta_b, \Phi(., y)\rangle\rangle \\ &= \langle \delta_a, \Phi(., b)\rangle \\ &= \Phi(a, b) \\ &= \langle \delta_{(a,b)}, \Phi\rangle.\end{aligned}$$

Then: $\delta_a \otimes \delta_b = \delta_{(a,b)}$.

Example 3.5 : *Let* $\Phi \in \mathscr{D}(\mathbb{R}^2)$*. Then:*

$$\begin{aligned}\langle \delta \otimes H, \Phi\rangle &= \langle \delta, \langle H, \Phi(., y)\rangle\rangle \\ &= \langle \delta, \int_0^{+\infty} \Phi(., y)dy\rangle \\ &= \int_0^{+\infty} \Phi(0, y)dy.\end{aligned}$$

Example 3.6 : *Let* $\Phi \in \mathscr{D}(\mathbb{R}^2)$*. Then:*

$$\begin{aligned}\langle H \otimes H, \Phi\rangle &= \langle H, \langle H, \Phi(., y)\rangle\rangle \\ &= \langle H, \int_0^{+\infty} \Phi(., y)dy\rangle \\ &= \int_0^{+\infty}\int_0^{+\infty} \Phi(x, y)dxdy \\ &= \langle \chi_{\mathbb{R}^2_+}, \Phi\rangle.\end{aligned}$$

Therefore: $H \otimes H = \chi_{\mathbb{R}^2_+}$.

Proposition 3.10 : *Let* $T \in \mathscr{E}'(U), S \in \mathscr{E}'(V), f \in \mathscr{E}(U)$ *and* $g \in \mathscr{E}(V)$*. Then:*

1. $\operatorname{supp}(T \otimes S) = \operatorname{supp} T \times \operatorname{supp} S$.
2. $D_x^\alpha D_y^\beta (T \otimes D) = D_x^\alpha T \otimes D_y^\beta S$.
3. $(f \otimes g)(T \otimes S) = (f.T) \otimes (g.S)$.
4. *The tensor product is associative.*
5. *The tensor product is not commutative in general.*

3.5 Convolution of two distributions

Now, let's generalize the convolution product of functions using another approach, based on the following distributional formula:

Let $f, g \in L^1(\mathbb{R}^n)$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$. We know that $(f * g) \in L^1(\mathbb{R}^n)$ and we have:

$$\begin{aligned}\langle f * g, \varphi\rangle &= \int_{\mathbb{R}^n} (f * g)(y)dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(z-y)g(y)\varphi(z)dzdy.\end{aligned}$$

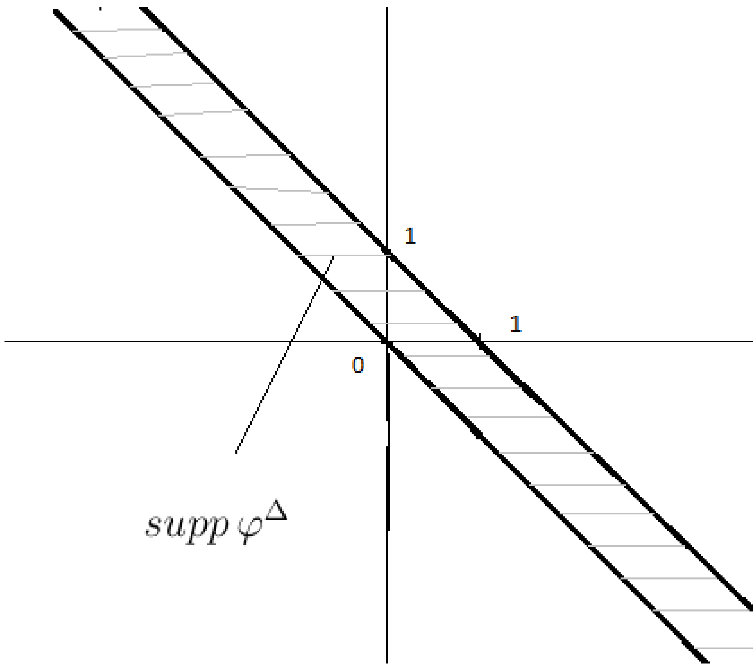
Using the change of variables $x = z - y$, we get:

$$\begin{aligned}\langle f * g, \varphi\rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(y)\varphi(x+y)dxdy \\ &= \int_{\mathbb{R}^n} f(x)dx \int_{\mathbb{R}^n} g(y)\varphi(x+y)dy\end{aligned}$$

Noting $\varphi^\Delta(x, y) = \varphi(x + y)$, we obtain:

$$\begin{aligned}\langle f * g, \varphi\rangle &= \int_{\mathbb{R}^n} f(x)dx \int_{\mathbb{R}^n} g(y)\varphi^\Delta(x,y)dy \\ &= \langle f(x), \langle g(y), \varphi^\Delta(., y)\rangle\rangle \\ &= \langle f \otimes g, \varphi^\Delta\rangle.\end{aligned}$$

To generalize this notion to distributions, we need to make sense of the bracket $\langle . \otimes ., \varphi^\Delta\rangle$. This is not immediate because φ^Δ does not necessarily belong to $\mathscr{D}(\mathbb{R}^{2n})$. For example, if we take $\varphi \in \mathscr{D}(\mathbb{R})$ such that $\operatorname{supp}\varphi$ is in $[0, 1]$, then: $\operatorname{supp}\varphi^\Delta = \{(x, y) \in \mathbb{R}^2 : 0 \leq x + y \leq 1\}$ is not compact.



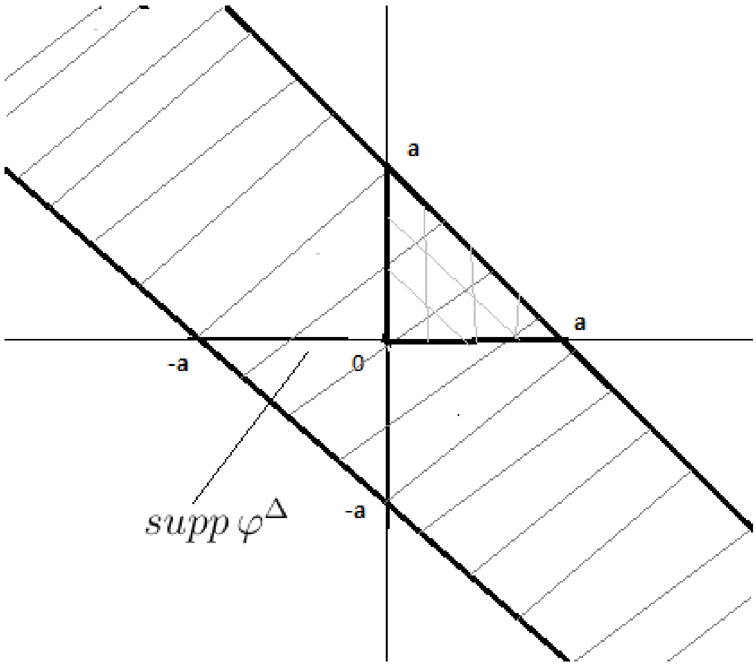
The bracket $\langle S \otimes T, \varphi^\Delta\rangle$ makes sense if $\operatorname{supp} S$ and $\operatorname{supp} T$ are convolutives in the sense of Definition **??**, in this case, we give the following definition:

Definition 3.8 (***convolution product of distributions***) : *Let* $S, T \in \mathscr{D}'(\mathbb{R}^n)$ *such that* $sipp\, S, \operatorname{supp} T$ *are convolutives. We define the convolution product of* $S * T$ *as follows:*

$$\forall \varphi \in \mathscr{D}(\mathbb{R}^n) : \langle S * T, \varphi \rangle_{\mathscr{D}', \mathscr{D}} = \langle S \otimes T, \varphi^{\Delta} \rangle_{\xi', \xi}$$

ou $\varphi^{\Delta}(x, y) = \varphi(x + y)$

Example 3.7 : *Let* $\varphi \in \mathscr{D}(\mathbb{R})$. *There exists* $a > 0$ *such that* $\operatorname{supp} \varphi \subset [-a, a]$,
$i.e \operatorname{supp} \varphi^{\Delta} = \{(x, y) \in \mathbb{R}^2 : -a \leq x + y \leq a\}$. *Then:*



$$\begin{aligned} \langle H * H, \varphi \rangle &= \langle H \otimes H, \varphi^{\Delta} \rangle \\ &= \langle \chi_{\mathbb{R}^2_+}, \varphi^{\Delta} \rangle \\ &= \int_0^{+\infty} \int_0^{+\infty} \varphi(x + y) dx dy \\ &= \int_0^{+\infty} dy \int_y^{+\infty} \varphi(z) dz \\ &= \int_0^{a} dy \int_y^{a} \varphi(z) dz. \end{aligned}$$

Proposition 3.11 : *let* $a \in \mathbb{R}^n, \alpha \in \mathbb{N}^n, S, T \in \mathscr{D}'(\mathbb{R}^n)$ *such that* $sipp\, S, \operatorname{supp} T$ *are convolutives. Then:*

1. $\delta_a * T = \tau_a T$. *In particular* $\delta * T = \tau_0 T = T$.
2. $\tau_a(T * S) = \tau_a T * S = T * \tau_a S$.
3. $D^{\alpha}(T * S) = D^{\alpha} T * S = T * D^{\alpha} S$. *In particular:* $D^{\alpha} \delta * T = D^{\alpha} T$.

Proof: Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Then:

1. For $a \in \mathbb{R}^n$:
$$\begin{aligned}\langle \delta_a * T, \varphi \rangle &= \langle \delta_a \otimes T, \varphi^{\Delta} \rangle \\ &= \langle T, \langle \delta_a, \varphi^{\Delta} \rangle \rangle \\ &= \langle T, \varphi(a+x) \rangle \\ &= \langle T, \tau_{-a}\varphi \rangle \\ &= \langle \tau_a T, \varphi \rangle.\end{aligned}$$
Then: $\delta_a * T = \tau_a T$, in particular $\delta * T = \tau_0 T = T$.

2. On one hand:
$$\begin{aligned}\langle \tau_a(T * S), \varphi \rangle &= \langle (T * S), \tau_{-a}\varphi \rangle \\ &= \langle T \otimes S, (\tau_{-a}\varphi)^{\Delta} \rangle \\ &= \langle T, \langle S, \varphi(a+x+y) \rangle \rangle \\ &= \langle T, \langle S, \tau_{-a}\varphi^{\Delta} \rangle \\ &= \langle T, \langle \tau_a S, \varphi \rangle \rangle \\ &= \langle T \otimes \tau_a S, \varphi^{\Delta} \rangle \\ &= \langle T * \tau_a S, \varphi \rangle.\end{aligned}$$
On the other hand:
$$\begin{aligned}\langle \tau_a(T * S), \varphi \rangle &= \langle (T * S), \tau_{-a}\varphi \rangle \\ &= \langle T \otimes S, (\tau_{-a}\varphi)^{\Delta} \rangle \\ &= \langle S, \langle T, \varphi(a+x+y) \rangle \rangle \\ &= \langle S, \langle T, \tau_{-a}\varphi^{\Delta} \rangle \\ &= \langle S, \langle \tau_a T, \varphi \rangle \rangle \\ &= \langle \tau_a T \otimes S, \varphi^{\Delta} \rangle \\ &= \langle \tau_a T * S, \varphi \rangle.\end{aligned}$$
Then: $\tau_a(T * S) = \tau_a T * S = T * \tau_a S$.

3. On one hand:
$$\begin{aligned}\langle D^{\alpha}(T * S), \varphi \rangle &= |-1|^{\alpha} \langle (T * S), D^{\alpha}\varphi \rangle \\ &= |-1|^{\alpha} \langle T \otimes S, (D^{\alpha}\varphi)^{\Delta} \rangle \\ &= \langle T, |-1|^{\alpha} \langle S, D^{\alpha}\varphi(x+y) \rangle \rangle \\ &= \langle T, |-1|^{\alpha} \langle S, D^{\alpha}\varphi^{\Delta} \rangle \\ &= \langle T, \langle D^{\alpha} S, \varphi \rangle \rangle \\ &= \langle T \otimes D^{\alpha} S, \varphi^{\Delta} \rangle \\ &= \langle T * D^{\alpha} S, \varphi \rangle.\end{aligned}$$

On the other hand:

$$\begin{aligned}
\langle D^\alpha(T*S),\varphi\rangle &= |-1|^\alpha\langle (T*S), D^\alpha\varphi\rangle\\
&= |-1|^\alpha\langle T\otimes S, (D^\alpha\varphi)^\Delta\rangle\\
&= \langle S, |-1|^\alpha\langle T, D^\alpha\varphi(x+y)\rangle\rangle\\
&= \langle S, |-1|^\alpha\langle T, D^\alpha\varphi^\Delta\rangle\\
&= \langle S, \langle D^\alpha T,\varphi\rangle\rangle\\
&= \langle D^\alpha T\otimes S, \varphi^\Delta\rangle\\
&= \langle D^\alpha T * S,\varphi\rangle.
\end{aligned}$$

Then: $D^\alpha(S*T) = D^\alpha S*T = S*D^\alpha T$. In particular: $D^\alpha\delta * T = D^\alpha(\delta*T) = D^\alpha T$.

■

3.6 Convolution equations:

Definition 3.9 : *A convolution equation is defined as any equation of the form $A*U=T$, where A and T are known distributions, and U is the unknown.*

Example 3.8 : *Consider the partial differential equation:*

$$\sum_{|\alpha|\le m} a_\alpha D^\alpha U = f,$$

where a_α are real constants, and f is a locally integrable function.
According to Theorem 3.3, we can write: $D^\alpha U = D^\alpha(\delta*U) = D^\alpha\delta*U$.
Then the equation can be written in the form $A*U=f$, where $A = \sum_{|\alpha|\le m} a_\alpha D^\alpha\delta$.

Definition 3.10 (***Elementary solution***) : *Let $A\in\mathscr{E}'(\mathbb{R}^n)$. We say that a distribution U_A is an elementary solution of A if we have $A*U_A=\delta$.*

Remark 3.4 :

1. *The elementary solution doesn't always exist.*
2. *If U_0 and U_1 are two elementary solutions of A, then: $U_1 = U_0+V$ where V is a general solution of the equation $A*V=0$. Indeed, if we set: $V=U_1-U_0$, we find:*

$$A*V = A*(U_1-U_0) = A*U_1 - A*U_0 = 0.$$

We assume the following theorem:

Theorem 3.7 (**Malgrange – Ehrenpreis**) : *Every partial differential equation with constant coefficients admits an elementary solution.*

Theorem 3.8 : *Let $A \in \mathscr{E}'(\mathbb{R}^n)$. Suppose that A has an elementary solution u_A. Then:*

1. *For any $T \in \mathscr{E}'(\mathbb{R}^n)$, there exists $U \in \mathscr{D}'(\mathbb{R}^n)$ such that $A * U = T$.*
2. *Let $T \in \mathscr{E}'(\mathbb{R}^n)$. If there exists $U \in \mathscr{E}'(\mathbb{R}^n)$ as a solution of the equation $A * U = T$, it is unique, and we have $U = U_A * f$.*

Proof: : Suppose that $A \in \mathscr{E}'(\mathbb{R}^n)$ has an elementary solution u_A.

1. For any $T \in \mathscr{E}'(\mathbb{R}^n)$, we set: $U = U_A * T$, then:

$$A * U = A * (U_A * T) = (A * U_A) * T = \delta * T = T.$$

2. Suppose that there exists $U \in \mathscr{E}'(\mathbb{R}^n)$ as a solution of the equation $A * U = T$, then:

$$U = \delta * U = (U_A * A) * U = U_A * (A * U) = U_A * T,$$

which shows uniqueness.

■

Example 3.9 : *The function w, defied by: $w(x) = \dfrac{|x|}{2}$, is a solution of the equation $u'' = \delta$ dans $\mathbb{R}$. Indeed, Let $\varphi \in \mathscr{D}(\mathbb{R})$. Then:*

$$\begin{aligned}
\langle w'', \varphi\rangle &= \langle w, \varphi''\rangle \\
&= \int_{-\infty}^{+\infty} \frac{|x|}{2}\varphi''(x)dx \\
&= \int_{-\infty}^{0} -\frac{x}{2}\varphi''(x)dx + \int_{0}^{+\infty} \frac{x}{2}\varphi''(x)dx \\
&= -\left[\frac{x}{2}\varphi'(x)\right]_{-\infty}^{0} + \int_{-\infty}^{0} \frac{\varphi'(x)}{2}dx + \left[\frac{x}{2}\varphi'(x)\right]_{0}^{+\infty} - \int_{0}^{+\infty} \frac{\varphi'(x)}{2}dx \\
&= \varphi(0) \\
&= \langle \delta, \varphi\rangle.
\end{aligned}$$

Example 3.10 : *The function w_2, defined as: $w_2(x) = \dfrac{\ln|x|}{2\pi}$, is an elementary solution of the Laplace operator Δ in $\mathbb{R}^2$.*
Indeed, let $\varphi \in \mathscr{D}(\mathbb{R}^2)$. Then:

$$\begin{aligned}
\langle \Delta w_2, \varphi\rangle &= \langle w_2, \Delta\varphi\rangle \\
&= \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{|x|>\varepsilon} \ln|x|.\Delta\varphi(x)dx.
\end{aligned}$$

Using the Green's formula, we get:

$$\int_{|x|>\varepsilon} \ln|x|\Delta\varphi(x)dx = \int_{|x|>\varepsilon} \Delta \ln|x|.\varphi(x)dx + \int_{|x|=\varepsilon} \ln|x|.\frac{\partial\varphi}{\partial\nu}d\sigma(x) - \int_{|x|=\varepsilon} \frac{\partial \ln|x|}{\partial\nu}\varphi d\sigma(x).$$

where ν *is the outward normal vector of the set* $\{|x| > \varepsilon\}$*, i.e., the inward normal vector of* $B(0, \varepsilon)$*, so* $\nu(x_1, x_2) = -(x_1, x_2)$.

$*)$ $\Delta \ln |x| = 0$ *on* $\{|x| > \varepsilon\}$*, so:* $\displaystyle\int_{|x|>\varepsilon} \Delta \ln |x|.\varphi(x)dx = 0$.

$**)$ $\dfrac{\partial \ln |x|}{\partial \nu} = -\dfrac{x_1}{x_1^2 + x_2^2}.x_1 - \dfrac{x_2}{x_1^2 + x_2^2}.x_2 = -1$.

Therefore: $\displaystyle -\int_{|x|=\varepsilon} \frac{\partial \ln |x|}{\partial \nu}\varphi d\sigma(x) = \int_{|x|=\varepsilon} \varphi d\sigma(x) = \int_0^{2\pi} \varphi(\varepsilon \cos\theta, \varepsilon \sin\theta)d\theta$.

According to the mean value theorem, there exists x_ε *such that* $|x_\varepsilon| = \varepsilon$*, and we have:*

$\displaystyle\int_0^{2\pi} \varphi(\varepsilon \cos\theta, \varepsilon \sin\theta)d\theta = 2\pi\varphi(x_\varepsilon)$.

Therefore: $\displaystyle\lim_{\varepsilon\to 0}\left[-\int_{|x|=\varepsilon} \frac{\partial \ln |x|}{\partial \nu}\varphi d\sigma(x)\right] = 2\pi\varphi(0)$.

$***)$ $\displaystyle\int_{|x|=\varepsilon} \ln |x|\frac{\partial \varphi}{\partial \nu}d\sigma(x) = -\ln\varepsilon \int_{|x|=\varepsilon}\left[x_1\frac{\partial \varphi}{\partial x_1} + x_2\frac{\partial \varphi}{\partial x_2}\right]d\sigma(x)$*. Then:* $\displaystyle\left|\int_{|x|=\varepsilon} \ln |x|\frac{\partial \varphi}{\partial \nu}d\sigma(x)\right| \leq$ $\displaystyle\varepsilon \ln \varepsilon \int_{|x|=\varepsilon}\left[\left|\frac{\partial \varphi}{\partial x_1}\right| + \left|\frac{\partial \varphi}{\partial x_2}\right|\right]d\sigma(x) \leq M(\varphi).\varepsilon \ln \varepsilon$.

Therefore: $\displaystyle\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon} \ln |x|\frac{\partial \varphi}{\partial \nu}d\sigma(x) = 0$.

Finally, we obtain:

$$\begin{aligned} \langle \Delta w_2, \varphi\rangle &= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{|x|>\varepsilon} \ln |x|.\Delta\varphi(x)dx \\ &= \varphi(0) \\ &= \langle \delta, \varphi\rangle. \end{aligned}$$

Example 3.11 : *sing the method above, we can show that the function* w_n*, defined as:* $w_n(x) = -\dfrac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}(n-2)}\dfrac{1}{|x|^{n-2}}$*, is an elementary solution of the Laplace operator* Δ *in* $\mathbb{R}^n$ *for* $(n \geq 3)$*, and* $\displaystyle\Gamma\left(\frac{n}{2}\right) = \int_0^{+\infty} t^{\frac{n-2}{2}}e^{-t}dt$.

The following results will be given without demonstration:

Theorem 3.9 : *Let* w *be the elementary solution of the operator* Δ*. For* $f \in \mathscr{E}'(\mathbb{R}^n)$*, we define* $u = w * f$*. Then:*

1. *For all* $n \geq 2, u$ *is a solution of the equation* $\Delta u = f$*, and we have:* $u \in \mathscr{E}'(C_{\mathbb{R}^n}^{\operatorname{supp} f})$.
2. *For all* $n \geq 3$*, we have:* $\displaystyle\lim_{|x|\to+\infty} u(x) = 0$.

Corollary 3.1 : *Let* $\Omega \subset \mathbb{R}^n$ *be an open set, and let* $u \in \mathscr{D}'(\Omega)$ *such that* $\Delta u = 0$*. Then:* $u \in \mathscr{C}^\infty(\Omega)$.

In other words:

Harmonic distributions are harmonic functions.

Theorem 3.10 : *Let P be a differential operator with constant coefficients in $\mathbb{R}^n$. Suppose that P has an elementary solution $w \in \mathscr{C}^\infty(\mathbb{R}^n)$. Then: for any $f \in \mathscr{E}'(\mathbb{R}^n)$, the distribution $w * f$ is a solution of the equation $Pu = f$, and we have $u \in \mathscr{E}'(C_{\mathbb{R}^n}^{\operatorname{supp} f})$.*

Theorem 3.11 : *Let the differential operator with constant coefficients be denoted as P_m, defined in $\mathscr{D}'(\mathbb{R}^+)$ as: $P_m U = U^{(m)} + c_{m-1}U^{(m-1)} + c_{m-2}U^{(m-2)} + \ldots + c_1 U' + c_0 U$.*
The operator P_m has a unique elementary solution $w \in \mathscr{D}'(\mathbb{R}^+)$, and we have $w = H.w_0$, where H is the Heaviside function, and w_0 is the unique solution of the initial value problem:

$$\begin{cases} Pw_0 = 0, \\ w_0^{(k)}(0) = 0, k = 0, \ldots, m-2, \\ w_0^{(m-1)}(0) = 1. \end{cases}$$

Exercises

Exercise 3.1 : *Find $f * g$ for the follows functions:*

1. $f(x) = e^{ax}, g(x) = H(x), a \in \mathbb{R}$.
2. $f(x) = \sin x, g(x) = e^{-|x|}$.
3. $f(x) = \chi_{[0,1]}(x), g(x) = x^2$.
4. $f(x) = g(x) = e^{-x^2}$.

Exercise 3.2 : *Consider the function θ defined on $\mathbb{R}$ by:*

$$\forall x \in \mathbb{R} : \theta(x) = \begin{cases} 0 & : & |x| > 1 \\ 1 & : & |x| \leq 1. \end{cases}.$$

1. *Show that $\theta \in L^1(\mathbb{R})$.*
2. *Calculate $\theta * \theta$.*
3. *Calculate $\theta * H$, where H is the Heaviside function.*

Exercise 3.3 : *Let $F, G \subseteq \mathbb{R}^n$ be two closed cone sets, i.e.,*

$$\forall \lambda > 0, \forall x \in F, \forall y \in G : \lambda x \in F, \lambda y \in G.$$

Suppose that (F, G) are convolution-compatible. Prove that $F \cap (-G) = \{0\}$.

Exercise 3.4 : *Let H be the Heaviside function. Determine the distributions:*

$$\nabla(H \otimes H), \qquad \Delta(H \otimes H), \qquad (xH \otimes y^2 H).$$

Exercise 3.5 : *Let H be the Heaviside function. Determine the distributions:*

$$(H * H)'', \quad (xH * x^2 H), \; (\delta'' * H) \quad (\delta' * vp_{\frac{1}{x}})$$

Exercise 3.6 : *Solve the following differential equation in $\mathscr{D}'(\mathbb{R})$:*

$$U'' = H.$$

Exercise 3.7 : *Consider the heat operator in $\mathbb{R}_+ \times \mathbb{R}$:*

$$D = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}.$$

Verify that the distribution associated with the function:

$$E(t, x) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

is an elementary solution of the operator D.

Exercise 3.8 : *Consider the wave operator in $\mathbb{R}^2$:*

$$D = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}.$$

Verify that the distribution associated with the function:

$$E(t, x) = \begin{cases} \dfrac{1}{2} & : \; t - |x| > 0 \\ 0 & : \; t - |x| \leq 0, \end{cases}$$

is an elementary solution of the operator D.

Solutions of exercises

Solution 3.1 : *Let $x \in \mathbb{R}$. We will calculate $(f * g)(x)$ in the following cases:*

1. $f(x) = e^{ax}, g(x) = H(x), a \in \mathbb{R}$.

$$\begin{aligned} (f * g)(x) &= \int_{-\infty}^{+\infty} f(x-y).g(y)dy \\ &= \int_{-\infty}^{+\infty} e^{a(x-y)} H(y)dy \\ &= \int_{0}^{+\infty} e^{a(x-y)} dy \end{aligned}$$

Then:

$$(f * g)(x) = \begin{cases} \dfrac{e^{ax}}{a} & : \; a > 0, \\ +\infty & : \; a < 0. \end{cases}$$

2. $f(x) = \sin x, g(x) = e^{-|x|}$.

$$\begin{aligned}
(f * g)(x) &= \int_{-\infty}^{+\infty} f(x-y).g(y)dy \\
&= \int_{-\infty}^{+\infty} \sin(x-y)e^{-|y|}dy \\
&= \int_{-\infty}^{0} \sin(x-y)e^{y}dy + \int_{0}^{+\infty} \sin(x-y)e^{-y}dy \\
&= [\sin(x-y)e^{y}]_{-\infty}^{0} + \int_{-\infty}^{0} \cos(x-y)e^{y}dy \\
&\quad -[\sin(x-y)e^{y}]_{0}^{+\infty} - \int_{0}^{+\infty} \cos(x-y)e^{-y}dy \\
&= \int_{-\infty}^{0} \cos(x-y)e^{y}dy - \int_{0}^{+\infty} \cos(x-y)e^{-y}dy \\
&= [\cos(x-y)e^{y}]_{-\infty}^{0} - \int_{-\infty}^{0} \sin(x-y)e^{y}dy \\
&\quad +[\cos(x-y)e^{y}]_{0}^{+\infty} - \int_{0}^{+\infty} \sin(x-y)e^{-y}dy \\
&= 2\cos x - \int_{-\infty}^{+\infty} \sin(x-y)e^{-|y|}dy \\
&= 2\cos x - (f * g)(x).
\end{aligned}$$

Then: $(f * g)(x) = \cos x$.

3. $f(x) = \chi_{[0,1]}(x), g(x) = x^2$.

$$\begin{aligned}
(f * g)(x) &= \int_{-\infty}^{+\infty} f(y).g(x-y)dy \\
&= \int_{0}^{1} (x-y)^2 dy \\
&= \left[-\frac{(x-y)^3}{3}\right]_{0}^{1} \\
&= \frac{x^3 - (x-1)^3}{3}.
\end{aligned}$$

4. $f(x) = g(x) = e^{-x^2}$.

$$\begin{aligned}
(f * g)(x) &= \int_{-\infty}^{+\infty} f(x-y).g(y)dy \\
&= \int_{-\infty}^{+\infty} e^{-(x-y)^2} e^{-y^2} dy \\
&= \int_{-\infty}^{+\infty} e^{-((x-y)^2+y^2)} dy \\
&= e^{-\frac{x^2}{2}} \int_{-\infty}^{+\infty} e^{-2(y-\frac{x}{2})^2} dy \\
&= \sqrt{\frac{\pi}{2}} e^{-\frac{x^2}{2}}.
\end{aligned}$$

Solution 3.2 : $\forall x \in \mathbb{R} : \theta(x) = \begin{cases} 0 & : \quad |x| > 1 \\ 1 & : \quad |x| \leq 1. \end{cases}$.

1. *We have:* $\displaystyle\int_{-\infty}^{+\infty} |\theta(x)|dx = \int_{-1}^{1} dx = 2 < +\infty.$
   Then: $\theta \in L^1(\mathbb{R})$.

2. *Let* $x \in \mathbb{R}$.
   $\displaystyle(\theta * \theta)(x) = \int_{-\infty}^{+\infty} \theta(x-y)\theta(y)dy = \int_{-1}^{1} \theta(x-y)dy.$
   Using the change of variable $t = x - y$, *we obtain:* $\displaystyle(\theta * \theta)(x) = \int_{x-1}^{x+1} \theta(t)dt.$
   $*)$ *If* $x + 1 \leq -1$ *ou* $x - 1 \geq 1$ *then:* $(\theta * \theta)(x) = 0.$
   $*)$ *If* $x - 1 < -1 \leq x + 1 \leq 1$ *then:* $\displaystyle(\theta * \theta)(x) = \int_{-1}^{x+1} dt = x + 2.$
   $*)$ *If* $-1 \leq x - 1 \leq 1 < x + 1$ *then:* $\displaystyle(\theta * \theta)(x) = \int_{x-1}^{1} dt = 2 - x.$
   $*)$ *If* $-1 \leq x - 1 \leq x + 1 \leq 1$ *then:* $\displaystyle(\theta * \theta)(x) = \int_{x-1}^{x+1} dt = 2.$

3. *Let* $x \in \mathbb{R}$.
   $\displaystyle(\theta * H)(x) = \int_{-\infty}^{+\infty} \theta(x-y)H(y)dy = \int_{0}^{+\infty} \theta(x-y)dy.$
   Using the change of variable $t = x - y$, *we get:* $\displaystyle(\theta * H)(x) = \int_{-\infty}^{x} \theta(t)dt.$
   $*)$ *If* $x \leq -1$ *then:* $(\theta * H)(x) = 0.$
   $*)$ *If* $-1 < x < 1$ *then:* $\displaystyle(\theta * H)(x) = \int_{-1}^{x} dt = x + 1.$
   $*)$ *If* $x \geq 1$ *then:* $\displaystyle(\theta * H)(x) = \int_{-1}^{1} dt = 2.$

Solution 3.3 : *Suppose that* (F, G) *are convolutive and let's prove that* $F \cap (-G) = \{0\}$.
From the definition, we have: $\{0\} \subset F \cap (-G)$.
Now, let $x \in F \cap (-G)$. *This implies that* $x \in F$ *and* $-x \in G$.
Let $R > 0$. *Then,* $|0| = |x + (-x)| < R$.
There exists $r > 0$ *such that:* $|x| < r$ *and* $|-x| < r$, *i.e.,* $|x| < r$.
As R *is arbitrary,* r *is arbitrary as well, so we have* $x = 0$.
Therefore, $F \cap (-G) = \{0\}$.

Solution 3.4 :

$$\begin{aligned} \nabla(H \otimes H) &= \left(\frac{\partial}{\partial x}(H \otimes H), \frac{\partial}{\partial y}(H \otimes H)\right) \\ &= \left(\frac{\partial H(x)}{\partial x} \otimes H(y), H(y) \otimes \frac{\partial H}{\partial y}\right) \\ &= (\delta_x \otimes H(y), H(x) \otimes \delta_y) \end{aligned}$$

For $\Phi \in \mathscr{D}(\mathbb{R}^2)$, *we obtain:*

$$\langle \nabla(H \otimes H), \Phi \rangle = \left(\int_0^{+\infty} \Phi(0, y)dy, \int_0^{+\infty} \Phi(x, 0)dx\right).$$

$$\begin{aligned}\Delta(H\otimes H) &= \frac{\partial^2}{\partial x^2}(H\otimes H)+\frac{\partial^2}{\partial y^2}(H\otimes H)\\ &= \frac{\partial^2 H(x)}{\partial x^2}\otimes H(y)+H(x)\otimes\frac{\partial^2 H(y)}{\partial y^2}\\ &= \frac{\partial\delta(x)}{\partial x}\otimes H(y)+H(x)\otimes\frac{\partial\delta(y)}{\partial y}\end{aligned}$$

For $\Phi\in\mathscr{D}(\mathbb{R}^2)$*, we get:*

$$\langle\Delta(H\otimes H),\Phi\rangle = \int_0^{+\infty}\frac{\partial}{\partial x}\Phi(0,y)dy+\int_0^{+\infty}\frac{\partial}{\partial y}\Phi(x,0)dx.$$

Let $\Phi\in\mathscr{D}(\mathbb{R}^2)$*. Then:*

$$\begin{aligned}\langle xH\otimes y^2H,\Phi\rangle &= \langle xH,\langle y^2H,\Phi(.,y)\rangle\rangle\\ &= \langle xH,\int_0^{+\infty}y^2\Phi(.,y)dy\rangle\\ &= \int_0^{+\infty}\int_0^{+\infty}xy^2\Phi(x,y)dxdy.\end{aligned}$$

Therefore: $xH\otimes y^2H = xy^2\chi_{\mathbb{R}^2_+}$.

Solution 3.5 :

$*)$ $(H*H)''=((H*H)')'=(H'*H)'=(\delta*H)'=H'=\delta.$

$**)$ *Let* $\varphi\in\mathscr{D}(\mathbb{R})$*. Then:*

$$\begin{aligned}\langle xH*y^2H,\varphi\rangle &= \langle xH\otimes y^2H,\varphi^{\Delta}\rangle\\ &= \langle xy^2\chi_{\mathbb{R}^2_+},\varphi^{\Delta}\rangle\\ &= \int_0^{+\infty}\int_0^{+\infty}xy^2\varphi(x+y)dxdy.\end{aligned}$$

$***)$ $\delta''*H=(\delta*H)''=H''=\delta'.$

$****)$ $\delta'*vp_{\frac{1}{x}}=(\delta*vp_{\frac{1}{x}})'=(vp_{\frac{1}{x}})'.$

Let $\varphi\in\mathscr{D}(\mathbb{R})$*. Then:*

$$\begin{aligned}\langle(vp_{\frac{1}{x}})',\varphi\rangle &= -\langle vp_{\frac{1}{x}},\varphi'\rangle\\ &= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}-\frac{\varphi'(x)}{x}dx\end{aligned}$$

We have:

$$\begin{aligned}\int_{|x|>\varepsilon}-\frac{\varphi'(x)}{x}dx &= \left[-\frac{\varphi(x)}{x}\right]_{-\infty}^{-\varepsilon}+\left[-\frac{\varphi(x)}{x}\right]_{\varepsilon}^{+\infty}-\int_{|x|>\varepsilon}\frac{\varphi(x)}{x^2}dx\\ &= \frac{\varphi(-\varepsilon)}{\varepsilon}+\frac{\varphi(\varepsilon)}{\varepsilon}-\int_{|x|>\varepsilon}\frac{\varphi(x)}{x^2}dx\\ &= -\frac{\varphi(-\varepsilon)-\varphi(0)}{-\varepsilon}+\frac{\varphi(\varepsilon)-\varphi(0)}{\varepsilon}-\left[\int_{|x|>\varepsilon}\frac{\varphi(x)}{x^2}dx-2\frac{\varphi(0)}{\varepsilon}\right]\end{aligned}$$

Then:

$$\begin{aligned}
\langle (vp_{\frac{1}{x}})', \varphi\rangle &= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} -\frac{\varphi'(x)}{x}dx\\
&= \lim_{\varepsilon\to 0}\left(-\frac{\varphi(-\varepsilon)-\varphi(0)}{-\varepsilon}+\frac{\varphi(\varepsilon)-\varphi(0)}{\varepsilon}-\left[\int_{|x|>\varepsilon}\frac{\varphi(x)}{x^2}dx-2\frac{\varphi(0)}{\varepsilon}\right]\right)\\
&= -\varphi'(0)+\varphi'(0)-\lim_{\varepsilon\to 0}\left[\int_{|x|>\varepsilon}\frac{\varphi(x)}{x^2}dx-2\frac{\varphi(0)}{\varepsilon}\right]\\
&= -\langle (pf_{\frac{1}{x^2}}), \varphi\rangle.
\end{aligned}$$

Hence: $\delta' * vp_{\frac{1}{x}} = -pf_{\frac{1}{x^2}}$.

Solution 3.6 : *According to Example 3.9, the function w defined by $w_0(x) = \dfrac{|x|}{2}$ is an elementary solution of the equation $U'' = \delta$. The general solution of the equation $U'' = \delta$ is $W(x) = \dfrac{|x|}{2} + ax + b$, where a and b are real numbers, as the function $x \longmapsto ax + b$ is the general solution of the equation $U'' = 0$ (see Remark 3.4 and Corollary 3.1).*
Therefore, the general solution of the differential equation $U'' = H$ is $W * H$.
Let $\varphi \in \mathscr{D}(\mathbb{R})$. Then:

$$\begin{aligned}
\langle W * H, \varphi\rangle &= \langle W(x)\otimes H(y), \varphi^{\Delta}\rangle\\
&= \int_{-\infty}^{+\infty}\int_0^{+\infty}\left(\frac{|x|}{2}+ax+b\right)\varphi(x+y)dxdy.\\
&= \int_{-\infty}^{0}\int_0^{+\infty}\frac{(2a-1)x+2b}{2}\varphi(x+y)dxdy\\
&\quad + \int_{0}^{+\infty}\int_0^{+\infty}\frac{(2a+1)x+2b}{2}\varphi(x+y)dxdy.
\end{aligned}$$

Solution 3.7 : $D = \dfrac{\partial}{\partial t} - \dfrac{\partial^2}{\partial x^2}$ $\quad E(t,x) = \dfrac{1}{2\sqrt{\pi t}}\exp\left(-\dfrac{x^2}{4t}\right)$, $t > 0, x \in \mathbb{R}$.
We have:

$$\begin{aligned}
\langle DE, \varphi\rangle &= -\left\langle E, \frac{\partial\varphi}{\partial t}+\frac{\partial^2\varphi}{\partial x^2}\right\rangle\\
&= -\lim_{\varepsilon\to 0}\int_{\varepsilon}^{+\infty}\int_{-\infty}^{+\infty}E(t,x)\left(\frac{\partial\varphi}{\partial t}+\frac{\partial^2\varphi}{\partial x^2}\right)dxdt.\\
&= -\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{+\infty}\int_{\varepsilon}^{+\infty}E(t,x)\frac{\partial\varphi}{\partial t}dtdx+\int_{\varepsilon}^{+\infty}\int_{-\infty}^{+\infty}E(t,x)\frac{\partial^2\varphi}{\partial x^2}dxdt\right).
\end{aligned}$$

Note that $\dfrac{\partial E}{\partial t} = \dfrac{\partial^2 E}{\partial x^2}$, *then:*

$$\begin{aligned}
\int_{-\infty}^{+\infty}\int_{\varepsilon}^{+\infty}E(t,x)\frac{\partial\varphi}{\partial t}dtdx &= \int_{-\infty}^{+\infty}[E(t,x)\varphi(t,x)]_{\varepsilon}^{+\infty}dx-\int_{-\infty}^{+\infty}\int_{\varepsilon}^{+\infty}\frac{\partial E}{\partial t}\varphi(t,x)dtdx\\
&= -\int_{-\infty}^{+\infty}E(\varepsilon,x)\varphi(\varepsilon,x)dx-\int_{-\infty}^{+\infty}\int_{\varepsilon}^{+\infty}\frac{\partial E}{\partial t}\varphi(t,x)dtdx\\
&= -\int_{-\infty}^{+\infty}E(\varepsilon,x)\varphi(\varepsilon,x)dx-\int_{\varepsilon}^{+\infty}\int_{-\infty}^{+\infty}\frac{\partial^2 E}{\partial x^2}\varphi(t,x)dxdt
\end{aligned}$$

Therefore: $\int_{-\infty}^{+\infty}\int_{\varepsilon}^{+\infty} E(t,x)\frac{\partial\varphi}{\partial t}dtdx+\int_{\varepsilon}^{+\infty}\int_{-\infty}^{+\infty}\frac{\partial^2 E}{\partial x^2}\varphi(t,x)dxdt = -\int_{-\infty}^{+\infty} E(\varepsilon,x)\varphi(\varepsilon,x)dx.$
We obtain:

$$\langle DE,\varphi\rangle = \lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} E(\varepsilon,x)\varphi(\varepsilon,x)dx = \lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty}\frac{1}{2\sqrt{\pi\varepsilon}}\exp\left(-\frac{x^2}{4\varepsilon}\right)\varphi(\varepsilon,x)dx$$

Using the change of variable $x = 2y\sqrt{\varepsilon}$*, we get:*

$$\langle DE,\varphi\rangle = \lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-y^2}\varphi(\varepsilon,2y\sqrt{\varepsilon})dy.$$

Knowing that $\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-y^2}dy = 1$*, we can write:*

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-y^2}\varphi(\varepsilon,2y\sqrt{\varepsilon})dy - \varphi(0,0) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-y^2}[\varphi(\varepsilon,2y\sqrt{\varepsilon})-\varphi(0,0)]dy$$

Lebesgue's dominated convergence theorem (Theorem 1.13 and Remark 1.4) shows that:

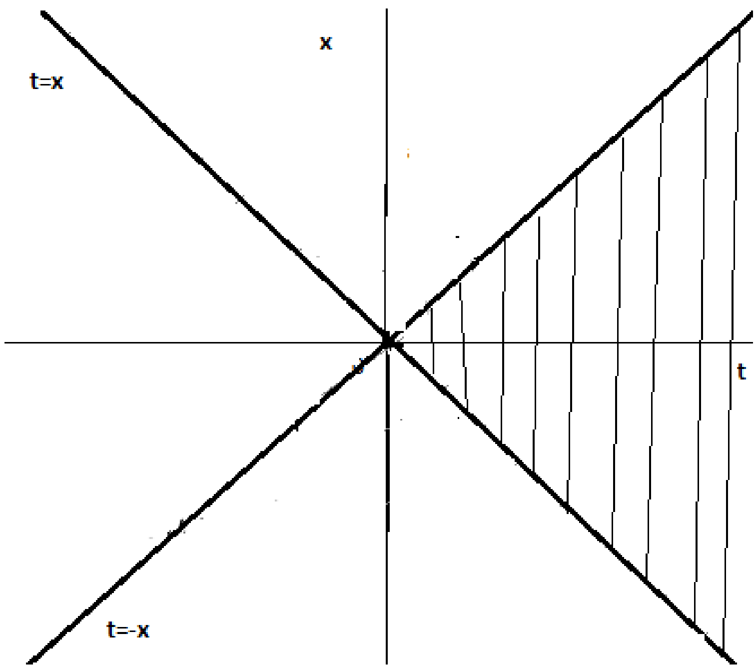
$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-y^2}[\varphi(\varepsilon,2y\sqrt{\varepsilon})-\varphi(0,0)]dy = 0$$

Then:

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-y^2}\varphi(\varepsilon,2y\sqrt{\varepsilon})dy = \varphi(0,0) = \langle\delta,\varphi\rangle.$$

Hence: $DE = \delta$.
So: the distribution associated with the function E *is an elementary solution of* D.

Solution 3.8 : $D = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$. $\quad E(t,x) = \begin{cases} \frac{1}{2} & : \ t-|x|>0 \\ 0 & : \ t-|x|\le 0, \end{cases}$



We have:

$$\begin{aligned}
\langle DE,\varphi\rangle &= \left\langle E,\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}\right\rangle\\
&= \left\langle E,\frac{\partial^2\varphi}{\partial t^2}\right\rangle-\left\langle E,\frac{\partial^2\varphi}{\partial x^2}\right\rangle\\
&= \frac{1}{2}\left(\int_{-\infty}^{0}\int_{-x}^{+\infty}\frac{\partial^2\varphi}{\partial t^2}dtdx+\int_{0}^{+\infty}\int_{x}^{+\infty}\frac{\partial^2\varphi}{\partial t^2}dtdx-\int_{0}^{+\infty}\int_{-t}^{t}\frac{\partial^2\varphi}{\partial x^2}dxdt\right)\\
&= -\frac{1}{2}\left(\int_{-\infty}^{0}\frac{\partial\varphi}{\partial t}(-x,x)dx+\int_{0}^{+\infty}\frac{\partial\varphi}{\partial t}(x,x)dx+\int_{0}^{+\infty}\frac{\partial\varphi}{\partial x}(t,t)dt-\int_{0}^{+\infty}\frac{\partial\varphi}{\partial x}(t,-t)dt\right)\\
&= -\frac{1}{2}\left(\int_{0}^{+\infty}\frac{\partial\varphi}{\partial t}(s,-s)ds+\int_{0}^{+\infty}\frac{\partial\varphi}{\partial t}(s,s)ds+\int_{0}^{+\infty}\frac{\partial\varphi}{\partial x}(s,s)ds-\int_{0}^{+\infty}\frac{\partial\varphi}{\partial x}(s,-s)ds\right)\\
&= -\frac{1}{2}\int_{0}^{+\infty}\left(\frac{\partial\varphi}{\partial t}(s,s)+\frac{\partial\varphi}{\partial x}(s,s)\right)ds-\frac{1}{2}\int_{0}^{+\infty}\left(\frac{\partial\varphi}{\partial t}(s,-s)-\frac{\partial\varphi}{\partial x}(s,-s)\right)ds.
\end{aligned}$$

Set: $\varphi_1(s)=\varphi(s,s)$ *et* $\varphi_2(s)=\varphi(s,-s)$, *we get:*
$\varphi_1'(s)=\frac{\partial\varphi}{\partial t}(s,s)+\frac{\partial\varphi}{\partial x}(s,s)$ *et* $\varphi_2'(s)=\frac{\partial\varphi}{\partial t}(s,-s)-\frac{\partial\varphi}{\partial x}(s,-s)$. *Then:*

$$\begin{aligned}
\langle DE,\varphi\rangle &= -\frac{1}{2}\int_{0}^{+\infty}\varphi_1'(s)ds-\frac{1}{2}\int_{0}^{+\infty}\varphi_2'(s)ds\\
&= \frac{1}{2}\varphi_1(0)+\frac{1}{2}\varphi_2(0)\\
&= \varphi(0,0)\\
&= \langle\delta,\varphi\rangle
\end{aligned}$$

So: distribution associated with the function E *is an elementary solution of the operator* D.

CHAPTER 4

FOURIER TRANSFORM

Among the various tools for the study of partial differential equations, we have the Fourier transform, which is a fundamental tool that generalizes Fourier series from the periodic case.

Let E be a $\mathbb{C}$-vector space, L a linear operator from E to E, and $T > 0$. Consider the real variable Cauchy problem, with vector-valued solutions in E:

$$(C) \quad \begin{cases} y'(t) = Ly(t), \\ y(0) = y_0. \end{cases}, \ t \in [0, T[, \ y_0 \in E.$$

I) If E is of finite dimension and y_0 is an eigenvector of L associated with the eigenvalue λ_0, then the function y defined as: $y(x) = e^{\lambda t}.y_0$. is a solution to problem (C).

Thus, if y_0 is a linear combination of eigenvectors $e_1, e_2, \dots e_k$ of L, associated with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$, i.e., $y_0 = \sum_{j=1}^{k} a_j.e_j$, then:

$$y(x) = \sum_{j=1}^{k} a_j e_j.e^{\lambda_j t},$$

is a solution of the problem (C).

Therefore, if we can determine the eigenvalues of L, it is easy to find explicit solutions to problem (C).

II) If E is of infinite dimension, for example, a space of functions on $[0, T[\times\mathbb{R}$ into $\mathbb{C}$, we obtain the problem:

$$(\mathcal{C}) \quad \begin{cases} y'(t, x) = Ly(t, x), \\ y(0, x) = y_0(x). \end{cases}, \ t \in [0, T[, \ x \in \mathbb{R}.$$

We look for eigenvectors of the operator L, i.e., functions y satisfying for certain eigenvalues λ the equation: $y'(t, x) = \lambda y(t, x)$.

The theory of Fourier series allows us to use the family $e_j = e^{\frac{2\pi ijx}{T}}\ j \in \mathbb{Z}$ as a Hilbert basis of the space $L^2_T(\mathbb{R})$ of functions in $L^2_{loc}(\mathbb{R})$ and T-periodic functions of $\mathbb{R}$ into $\mathbb{C}$, equipped with the norm:

$$\|f\|_{L^2_T(\mathbb{R})} = \frac{1}{T}\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(x)dx\right)^{\frac{1}{2}}.$$

So, for all $f \in L^2_{2\pi}(\mathbb{R})$ we have: $f = \sum_{j\in\mathbb{Z}} a_j(f).e_j$, where

$$a_j(f) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{\frac{-2\pi ijs}{T}} f(y)dy.$$

III) Now suppose that E is the space of functions defined on $[0,+\infty[\times\mathbb{R}$, and consider the same previous problem in $[0,+\infty[\times\mathbb{R}$. In the case of non-periodic functions, we let T tend to $+\infty$ in the previous problem.

Formally, for $T > 0$:

$$f(x) = \frac{1}{2\pi}\sum_{j\in\mathbb{Z}}\frac{2\pi}{T}\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{\frac{-2\pi ijy}{T}} f(y)dy\right)e^{\frac{2\pi ijx}{T}}.$$

If T tends to infinity, we obtain:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} e^{-iy.\xi}f(y)dy\right)e^{ix.\xi}d\xi.$$

The quantity $\displaystyle\int_{-\infty}^{+\infty} e^{-is.\xi}f(s)ds$ if it makes sense, is called the Fourier transform of f.

We can extend this to functions defined on $\mathbb{R}^n$. In the following, we will study the Fourier transform and its various properties.

4.1 Fourier transformation for functions

Definition 4.1 : *Let $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f, a complex-valued function denoted by $\widehat{f}$ or $\mathcal{F}(f)$, is defined for all $\xi \in \mathbb{R}^n$ as:*

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix.\xi}dx, \tag{4.1}$$

where $x.\xi = \sum_{i=1}^{n} x_i\xi_i$ (dot product).

Remark 4.1 *: The Fourier transform in $L^1(\mathbb{R}^n)$ is well-defined, linear, and there exists $c > 0$ such that: $\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)} \le c\|f\|_{L^1(\mathbb{R}^n)}$.*

Indeed, let $f \in L^1(\mathbb{R}^n)$. For all $\xi \in \mathbb{R}^n$, we have:

$$\begin{aligned}
|\widehat{f}(\xi)| &= \left|\int_{\mathbb{R}^n} f(x)e^{-ix.\xi}dx\right| \\
&\leq \int_{\mathbb{R}^n} |f(x)|e^{-ix.\xi}dx \\
&\leq 2\int_{\mathbb{R}^n} |f(x)|dx \\
&= 2|f|_{L^1(\mathbb{R}^n)} < +\infty.
\end{aligned}$$

Therefore, $\mathcal{F}$ is well-defined, and we have: $|\widehat{f}|_{L^\infty(\mathbb{R}^n)} \leq 2|f|_{L^1(\mathbb{R}^n)}$. It is said that $\mathcal{F}$ is a continuous map from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

Remark 4.2 : *If f is separable in variables, i.e., $f(x) = \prod_{i=1}^{n} f_i(x_i)$, then:*

$$\widehat{f}(x) = \prod_{i=1}^{n} \widehat{f}_i(x_i).$$

Definition 4.2 : *We define the conjugate Fourier transform in the same way for $f \in L^1(\mathbb{R}^n)$:*

$$\overline{\mathcal{F}}(f)(x) = \int_{\mathbb{R}^n} f(\xi)e^{ix.\xi}d\xi. \tag{4.2}$$

Example 4.1 : *Let $[a,b]$ be an interval. Then, we have:*

$$\begin{aligned}
\mathcal{F}f(\xi) &= \int_{-\infty}^{+\infty} f(x)e^{-ix.\xi}dx \\
&\leq \int_a^b e^{-ix.\xi}dx
\end{aligned}$$

Finally:

$$\mathcal{F}(\chi_{[a,b]})(x) = \begin{cases} \dfrac{e^{-ia\xi} - e^{-ib\xi}}{i\xi} & : \ \xi \neq 0 \\ b-a & : \ \xi = 0 \end{cases}$$

Proposition 4.1 : *We have the following properties:*

1. *If f is an even function, then $\widehat{f}$ is an even function.*
2. *If f is an odd function, then $\widehat{f}$ is an odd function.*
3. *If f is a real function, then: $\widehat{f}(-\xi) = \overline{\widehat{f}(\xi)}$.*
4. *If $f(-x) = \overline{f(x)}$ for all $x \in \mathbb{R}^n$, then $\widehat{f}$ is a real function.*
5. ***Translation:** For any $a \in \mathbb{R}^n$, we have:*

$$\mathcal{F}(\tau_a f) = e^{-ia.\xi}\widehat{f} \qquad \mathcal{F}(e^{ia.x}f) = \tau_a\widehat{f}$$

6. ***Dilatation:** $\mathcal{F}\left(f\left(\frac{x}{\lambda}\right)\right) = |\lambda|^n\widehat{f}(\lambda\xi)$ for a nonzero real λ.*

Proof: Let $f \in L^1(\mathbb{R})$.

1. Suppose that f is even and let $\xi \in \mathbb{R}^n$, on thena:

$$\mathcal{F}f(-\xi) = \int_{\mathbb{R}^n} f(x)e^{ix.\xi}dx$$

   Let's make the change of variable $y = -x$, we obtain:

$$\begin{aligned}\mathcal{F}f(-\xi) &= \int_{\mathbb{R}^n} f(-y)e^{-iy.\xi}dy \\ &= \int_{\mathbb{R}^n} f(y)e^{-iyx.\xi}dy \\ &= \mathcal{F}f(\xi).\end{aligned}$$

   Thus, $\widehat{f}$ is an even function.

2. Similarly, we can prove that if f is an odd function, then $\widehat{f}$ is an odd function.

3. Suppose that f is a real function, then:

$$\begin{aligned}\widehat{f}(-\xi) &= \int_{\mathbb{R}^n} f(x)e^{ix.\xi}dx \\ &= \int_{\mathbb{R}^n} f(x)\overline{e^{-ix.\xi}}dx \\ &= \int_{\mathbb{R}^n} \overline{f(x)e^{-ix.\xi}}dx \\ &= \overline{\int_{\mathbb{R}^n} f(x)e^{-ix.\xi}dx} \\ &= \overline{\widehat{f}(\xi)}\end{aligned}$$

4. Suppose that $f(-x) = \overline{f(x)}$ pour tout $s \in \mathbb{R}^n$, then:

$$\begin{aligned}\overline{\widehat{f}(\xi)} &= \overline{\int_{\mathbb{R}^n} f(x)e^{-ix.\xi}dx} \\ &= \int_{\mathbb{R}^n} \overline{f(x)}e^{ix.\xi}dx \\ &= \int_{\mathbb{R}^n} f(-x)e^{ix.\xi}dx \\ &= \int_{\mathbb{R}^n} f(y)e^{-iy.\xi}dy \\ &= \widehat{f}(\xi).\end{aligned}$$

   Therefore: $\widehat{f}$ is a real function.

5. Let $a \in \mathbb{R}^n$, then:

$$\begin{aligned}
\mathcal{F}(\tau_a f)(\xi) &= \int_{\mathbb{R}^n} \tau_a f(x) e^{-ix.\xi} dx \\
&= \int_{\mathbb{R}^n} f(x-a) e^{-ix.\xi} dx \\
&= \int_{\mathbb{R}^n} f(y) e^{-i(y+a).\xi} dy \\
&= e^{-ia} \int_{\mathbb{R}^n} f(y) e^{-iy.\xi} dy \\
&= e^{-ia} \widehat{f}(\xi).
\end{aligned}$$

So, $\mathcal{F}(\tau_a f) = e^{-ia.\xi} \widehat{f}$.

$$\begin{aligned}
\mathcal{F}(e^{ia.x} f)(\xi) &= \int_{\mathbb{R}^n} e^{ia.x} f(x) e^{-ix.\xi} dx \\
&= \int_{\mathbb{R}^n} f(x) e^{-ix.(\xi-a)} dx \\
&= \widehat{f}(\xi - a) \\
&= \tau_a \widehat{f}(\xi).
\end{aligned}$$

Hence:$\mathcal{F}(e^{ia.x} f) = \tau_a \widehat{f}$.

6. Let $\lambda \in \mathbb{R}^*$. Then:

$$\mathcal{F}\left(f\left(\frac{x}{\lambda}\right)\right) = \int_{\mathbb{R}^n} f\left(\frac{x}{\lambda}\right) e^{-ix.\xi} dx.$$

Let's take the change of variable $y = \dfrac{x}{\lambda}$, we obtain:

$*)$ For $\lambda > 0$: $\displaystyle\int_{\mathbb{R}^n} f\left(\frac{x}{\lambda}\right) e^{-ix.\xi} dx = \lambda^n \int_{\mathbb{R}^n} f(y) e^{-iy.\lambda\xi} dy$.

$*)$ For $\lambda < 0$: $\displaystyle\int_{\mathbb{R}^n} f\left(\frac{x}{\lambda}\right) e^{-ix.\xi} dx = (-\lambda)^n \int_{\mathbb{R}^n} f(y) e^{-iy.\lambda\xi} dy$.

So,

$$\mathcal{F}\left(f\left(\frac{x}{\lambda}\right)\right) = |\lambda|^n \int_{\mathbb{R}^n} f(y) e^{-iy.\lambda\xi} dy = |\lambda|^n \widehat{f}(\lambda\xi).$$

■

Theorem 4.1 ***(Riemann-Lebesgue Lemma):*** *Let $f \in L^1(\mathbb{R}^n)$. Then: $\widehat{f}$ is a continuous function and tends to 0 as $|\xi|$ tends to infinity.*

Proof: Let $f \in L^1(\mathbb{R}^n)$.

$*)$ Let $a \in \mathbb{R}^n$. Then:

$$\begin{aligned}
|\widehat{f}(\xi + a) - \widehat{f}(\xi)| &= |\widehat{e^{-iax} f}(\xi) - f(\xi)| \\
&= \left| \int_{\mathbb{R}^n} e^{-ia.x} f(x) e^{-ix.\xi} dx - \int_{\mathbb{R}^n} f(x) e^{-ix.\xi} dx \right| \\
&= \left| \int_{\mathbb{R}^n} (e^{-ia.x} - 1) f(x) e^{-ix.\xi} dx \right|.
\end{aligned}$$

The family of functions $x \mapsto (e^{-ia.x} - 1) f(x) e^{-ix.\xi}$ is a measurable family, converges to 0 as $|a|$ tends to 0, and we have $|(e^{-ia.x} - 1) f(x) e^{-ix.\xi}| \leq 6|f(x)|$ for all $a, \xi \in \mathbb{R}^n$, and $f \in L^1(\mathbb{R}^n)$.

According to the Dominated Convergence Theorem (Theorem 1.13 and Remark 1.4), we obtain: $\lim\limits_{|a|\to 0} |\widehat{f}(\xi+a)-\widehat{f}(\xi)| = 0$.

Thus, $\widehat{f}$ is continuous.

**) Let $\xi \in \mathbb{R}^n$ such that $\|\xi\|$ is sufficiently large. Then: there exists $1 \leq i \leq n$ such that $|\xi_i|$ sufficiently large. From the density of $\mathscr{D}(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$, we deduce that for $\varepsilon > 0$ there exists $\varphi \in \mathscr{D}(\mathbb{R}^n)$ such that $\|f-\varphi\|_{L^1(\mathbb{R}^n)} < \varepsilon$. Then:

$$\begin{aligned} |\widehat{f}(\xi)| &= \left| \int_{\mathbb{R}^n} e^{-ix.\xi} f(x)dx \right| \\ &\leq \left| \int_{\mathbb{R}^n} e^{-ix.\xi}(f(x)-\varphi(x))dx \right| + \left| \int_{\mathbb{R}^n} e^{-ix.\xi}\varphi(x)dx \right|. \end{aligned}$$

On one hand:

$$\begin{aligned} \left| \int_{\mathbb{R}^n} e^{-ix.\xi}(f(x)-\varphi(x))dx \right| &\leq \int_{\mathbb{R}^n} e^{-ix.\xi}|f(x)-\varphi(x)|dx \\ &\leq 2\int_{\mathbb{R}^n} |f(x)-\varphi(x)|dx \\ &= 2\|f-\varphi\|_{L^1(\mathbb{R}^n)} \\ &\leq 2\varepsilon. \end{aligned}$$

On the other hand:

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-ix.\xi}\varphi(x)dx &= \left[-\int_{\mathbb{R}^{n-1}} \frac{e^{-ix.\xi}}{\xi_i}\varphi(x)dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_n \right]_{-\infty}^{-\infty} + \int_{\mathbb{R}^n} \frac{e^{-ix.\xi}}{\xi_i}\frac{\partial\varphi}{\partial x_i}(x)dx \\ &= \frac{1}{\xi_i}\mathcal{F}\left(\frac{\partial\varphi}{\partial x_i}\right). \end{aligned}$$

Then:

$$|\widehat{f}(\xi)| \leq 2\varepsilon + \frac{1}{|\xi_i|}\left|\mathcal{F}\left(\frac{\partial\varphi}{\partial x_i}\right)\right|.$$

Let ε tends to 0, we obtain:

$$|\widehat{f}(\xi)| \leq \frac{1}{|\xi_i|}\left|\mathcal{F}\left(\frac{\partial\varphi}{\partial x_i}\right)\right| \overset{\|\xi\|\to+\infty}{\longrightarrow} 0 .$$

■

Theorem 4.2 : *Let $\alpha \in \mathbb{N}^n$, and let $f \in L^1(\mathbb{R}^n)$ such that $x^\alpha f \in L^1(\mathbb{R}^n)$. Then:*

$$D^\alpha \widehat{f} = \mathcal{F}((-i)^{|\alpha|}x^\alpha f).$$

Proof: The Dominated Convergence Theorem of Lebesgue (Theorem 1.13 and Remark 1.4) allows us to write:

$$\begin{aligned}
D^\alpha \widehat{f}(\xi) &= D^\alpha \int_{\mathbb{R}^n} f(x)e^{-ix.\xi}dx \\
&= \int_{\mathbb{R}^n} f(x)D^\alpha e^{-ix.\xi}dx \\
&= \int_{\mathbb{R}^n} f(x)\partial_{\xi_1}^{\alpha_1} e^{-ix_1.\xi_1}\dots\partial_{\xi_n}^{\alpha_n}\dots e^{-ix_n.\xi_n}dx \\
&= \int_{\mathbb{R}^n} f(x)(-i)^{\alpha_1+\dots\alpha_n}x_1^{\alpha_1}\dots x_n^{\alpha_n}e^{-ix.\xi}dx \\
&= \int_{\mathbb{R}^n} (-i)^{|\alpha|}x^\alpha f(x)e^{-ix.\xi}dx \\
&= \mathcal{F}((-i)^{|\alpha|}x^\alpha f).
\end{aligned}$$

■

Theorem 4.3 : *Let* $\alpha \in \mathbb{N}^n$, *and let* $f \in L^1(\mathbb{R}^n)$ *telle que* $D^\alpha f \in L^1(\mathbb{R}^n)$. *Alors:*

$$\mathcal{F}(D^\alpha f) = i^{|\alpha|}\xi^\alpha \widehat{f}.$$

Proof: Let $1 \le i \le n$. Then:

$$\begin{aligned}
\mathcal{F}(\partial_i f)(\xi) &= \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}(x)e^{-ix.\xi}dx \\
&= \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}(x)e^{-ix.\xi}dx_1\dots dx_i\dots dx_n \\
&= \int_{\mathbb{R}^{n-1}} [f(x)e^{-ix.\xi}]_{-\infty}^{+\infty}dx_1\dots dx_{i-1}dx_{i+1}\dots dx_n + i\xi_i\int_{\mathbb{R}^n} f(x)e^{-ix.\xi}dx.
\end{aligned}$$

Since $f \in L^1(\mathbb{R}^n)$, it approaches zero as it goes to infinity, so:

$$\mathcal{F}(\partial_i f)(\xi) = i\xi_i\widehat{f}.$$

Then:

$$\begin{aligned}
\mathcal{F}(D^\alpha f)(\xi) &= \mathcal{F}(\partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n} f)(\xi) \\
&= (i\xi_1)^{\alpha_1}\dots(i\xi_n)^{\alpha_n}\widehat{f}(\xi) \\
&= i^{|\alpha|}\xi^\alpha\widehat{f}(\xi).
\end{aligned}$$

■

Theorem 4.4 (***convolution***) : *Soit* $f, g \in L^1(\mathbb{R}^n)$. *Alors:* $\widehat{f * g} = \widehat{f}.\widehat{g}$.

Proof: Let $f, g \in L^1(\mathbb{R}^n)$. Then:

$$\begin{aligned}
\widehat{f*g}(\xi) &= \int_{\mathbb{R}^n} (f*g)(x)e^{-ix.\xi}dx \\
&= \int_{\mathbb{R}^n} e^{-ix.\xi}\int_{\mathbb{R}^n} f(y)g(x-y)dydx \\
&= \int_{\mathbb{R}^n} e^{-i(x-y).\xi}.e^{-iy.\xi}\int_{\mathbb{R}^n} f(y)g(x-y)dydx \\
&= \int_{\mathbb{R}^n} f(y)e^{-iy.\xi}dy\int_{\mathbb{R}^n} g(x-y)e^{-i(x-y).\xi}dx \\
&= \int_{\mathbb{R}^n} f(y)e^{-iy.\xi}dy\int_{\mathbb{R}^n} g(z)e^{-iz.\xi}dz \\
&= \widehat{f}(\xi).\widehat{g}(\xi).
\end{aligned}$$

Therefore: $\widehat{f*g} = \widehat{f}.\widehat{g}$. ■

Remark 4.3 *: Let $f, g \in L^2(\mathbb{R}^n)$. then: $f.g \in L^1(\mathbb{R}^n)$. Moreover, we have: $\widehat{f.g} = \widehat{f} * \widehat{g}$.*

Theorem 4.5 (***inversion***) : *Let $f \in L^1(\mathbb{R}^n)$ such that $\widehat{f} \in L^1(\mathbb{R}^n)$. then: $f = \dfrac{1}{(2\pi)^n}\overline{\mathcal{F}}(\widehat{f})$*

Proof: Let $x \in \mathbb{R}^n$. then:

$$\begin{aligned}
\frac{1}{(2\pi)^n}\overline{\mathcal{F}}(\widehat{f})(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix.\xi}d\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix.\xi}\int_{\mathbb{R}^n} f(y)e^{-iy.\xi}dyd\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)e^{i(x-y).\xi}dyd\xi.
\end{aligned}$$

The function $(y,\xi) \longmapsto f(y)e^{i(x-y).\xi}$ may not necessarily be integrable, so we cannot apply the Fubini's Theorem. However, we can consider, for $\varepsilon > 0$:

$$I_\varepsilon(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)e^{i(x-y).\xi}e^{-\frac{\varepsilon^2\|\xi\|^2}{4}}dyd\xi.$$

We have:

$$\begin{aligned}
I_\varepsilon(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix.\xi}e^{-\frac{\varepsilon^2\|\xi\|^2}{4}}\int_{\mathbb{R}^n} f(y)e^{-iy.\xi}dyd\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix.\xi}e^{-\frac{\varepsilon^2\|\xi\|^2}{4}}\widehat{f}(\xi)d\xi.
\end{aligned}$$

Set: $G_\varepsilon(\xi) = \dfrac{1}{(2\pi)^n}e^{ix.\xi}e^{-\frac{\varepsilon^2\|\xi\|^2}{4}}\widehat{f}(\xi)$.
$(G_\varepsilon)_{\varepsilon>0}$ is a sequence of integrable functions, which converges a.e to the function G_0 where $G_0(\xi) = \dfrac{1}{(2\pi)^n}e^{ix.\xi}f(\xi)$. Moreover, we have: $|G_\varepsilon| \leq G_0 \in L^1(\mathbb{R}^n)$. The Dominated Conver-

gence Theorem of Lebesgue (Theorem 1.13 and Remark 1.4) allows us to write:

$$\begin{aligned}\lim_{\varepsilon\to 0} I_\varepsilon(x) &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} G_\varepsilon(x)d\xi\\ &= \int_{\mathbb{R}^n} G_0(x)d\xi\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix.\xi}\widehat{f}(\xi)d\xi.\\ &= \frac{1}{(2\pi)^n}\overline{\mathcal{F}}(\widehat{f})\,.........(*)\end{aligned}$$

On the other hand, we have: $I_\varepsilon = F_\varepsilon * f$ où

$$F_\varepsilon(z) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{iz.\xi}e^{-\frac{\varepsilon^2\|\xi\|^2}{4}}d\xi.$$

Let's take the change of variable $\zeta = -\xi$, we get:

$$F_\varepsilon(z) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-iz.\zeta}e^{-\frac{\varepsilon^2\|\zeta\|^2}{4}}d\zeta = F_\varepsilon(-z).$$

Let's take the change of variable $\eta = \varepsilon\xi$, we obtain:

$$F_\varepsilon(z) = \frac{1}{(2\pi\varepsilon)^n}\int_{\mathbb{R}^n} e^{i\frac{z}{\varepsilon}.\eta}e^{-\frac{\|\eta\|^2}{4}}d\zeta = \frac{1}{\varepsilon^n}F_1\left(\frac{z}{\varepsilon}\right).$$

Then:

$$\begin{aligned}\int_{\mathbb{R}^n} F_1(z)dz &= \int_{\mathbb{R}^n} F_1(-z)dz\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-iz.\xi}e^{-\frac{\|\xi\|^2}{4}}d\xi dz.\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathcal{F}\left(e^{-\frac{\|\xi\|^2}{4}}\right)(z)dz.\end{aligned}$$

By following arguments similar to those in exercise 4.2, we can show that

$$\mathcal{F}\left(e^{-\frac{\|\xi\|^2}{4}}\right)(z) = (2\sqrt{\pi})^n e^{-z^2}.$$

Therefore:

$$\int_{\mathbb{R}^n} F_1(z)dz = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(2\sqrt{\pi})^n e^{-z^2}dz = \frac{1}{(2\pi)^n}.(2\sqrt{\pi})^n.(\sqrt{\pi})^n = 1,$$

which leads to:

$$\int_{\mathbb{R}^n}\frac{1}{\varepsilon^n}F_\varepsilon\left(\frac{z}{\varepsilon}\right)dz = \int_{\mathbb{R}^n} F_\varepsilon(t)dt = 1.$$

Applying the result follows:

We consider the sequence $(F_\varepsilon)\varepsilon > 0 \subset L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} F_\varepsilon(t)dt = 1$, and let $f \in L^1(\mathbb{R}^n)$. Then, $F_\varepsilon * f$ converges to f in $L^1(\mathbb{R}^n)$.

It follows that I_ε converges to f in $L^1(\mathbb{R}^\ltimes)\ldots\ldots\ldots(**)$.
From (*), (**), and considering that $f \in L^1(\mathbb{R}^n)$, we obtain the result. ■

Remark 4.4 *: There is another definition of the Fourier transform, which is:*

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi ix.\xi}dx. \tag{4.3}$$

In this case, $\mathcal{F}^{-1} = \overline{\mathcal{F}}$ *from* $L^1(\mathbb{R}^n)$ *to* $L^1(\mathbb{R}^n)$*, where:*

$$\overline{\mathcal{F}}(f)(x) = \int_{\mathbb{R}^n} f(\xi)e^{2\pi ix.\xi}d\xi. \tag{4.4}$$

4.2 Rapid Growth, Slow Decay

Definition 4.3 (***Schwartz space***) :

1. *A function* $\varphi : \mathbb{R}^n \to \mathbb{C}$ *is said to have rapid decay if, for every* $m \in \mathbb{N}$*, we have:*

$$\lim_{|x|\to+\infty} |x|^m\varphi(x) = 0.$$

2. *The Schwartz space* $\mathscr{S}(\mathbb{R}^n)$ *is the space of functions* $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ *such that, for every multi-index* $\alpha \in \mathbb{N}^n$*, the function* $D^\alpha\varphi$ *has rapid decay.*

It is evident that the space $\mathscr{S}(\mathbb{R}^n)$ is a vector space.

Remark 4.5 : *It is equivalent to say that* $\varphi \in \mathscr{S}(\mathbb{R}^n)$ *if the quantities*

$$\mathcal{N}_p(\varphi) = \sum_{|\alpha|\le p, |\beta|\le p} \|x^\alpha D^\beta\varphi(x)\|_{L^\infty(\mathbb{R}^n)},$$

are finite for all p.

Indeed, if $\varphi \in \mathscr{S}(\mathbb{R}^n)$, then we have $\lim_{|x|\to+\infty} |x^\alpha D^\beta\varphi(x)| = 0$, so $|x^\alpha D^\beta\varphi(x)|$ is bounded almost everywhere, which implies the boundedness of $\mathcal{N}_p(\varphi)$.

Conversely, if $\mathcal{N}_p(\varphi)$ is bounded, then $|x_i x^\alpha D^\beta\varphi(x)|$ is bounded for some i such that $|x_i|$ tends to infinity (i exists since $|x|$ tends to infinity).
Therefore: $\lim_{|x|\to+\infty} |x^\alpha D^\beta\varphi(x)| = \lim_{|x|\to+\infty} \dfrac{|x_i.x^\alpha D^\beta\varphi(x)|}{|x_i|} = 0.$

The space $\mathscr{S}(\mathbb{R}^n)$ is stable under differentiation and multiplication by polynomials. $\mathscr{S}(\mathbb{R}^n)$ is a topological vector space, and its seminorms are given by $(\mathcal{N}_p)_{p\in\mathbb{N}}$.

Definition 4.4 **(convergence in $\mathscr{S}(\mathbb{R}^n)$)** : *We say that a sequence of functions $\{\varphi_j\}_{j\in\mathbb{N}}$ in $\mathscr{S}(\mathbb{R}^n)$ converges to $\varphi \in \mathscr{S}(\mathbb{R}^n)$ if for every $p \in \mathbb{N}$, we have: $\lim\limits_{j\to+\infty} \mathcal{N}_p(\varphi_j - \varphi) = 0$.*

Proposition 4.2 : *For all $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we have: $x^\alpha\varphi \in L^1(\mathbb{R}^n)$, $\lim\limits_{|x|\to+\infty} |x^\alpha\varphi(x)| = 0$, and there exists constants C_p such that:*

$$\sum_{|\alpha|\leq p, |\beta|\leq p} \|x^\alpha D^\beta\varphi(x)\|_{L^1(\mathbb{R}^n)} \leq C_p\mathcal{N}_{p+n+1}(\varphi) \quad \forall\varphi \in \mathscr{S}(\mathbb{R}^n).$$

Proof: Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. It is clear $\lim\limits_{|x|\to+\infty} |x^\alpha\varphi(x)| = 0$.
Since $x^\alpha\varphi(x)$ is bounded, it is locally integrable.
It remains to prove that $\lim\limits_{A\to+\infty} \int_{|x|>A} |x^\alpha\varphi(x)|dx = 0$. We have:

$$\int_{|x|>A} |x^\alpha\varphi(x)|dx = \int_{|x|>A} \frac{|(x_1^2 + \cdots + x_n^2)x^\alpha\varphi(x)|}{|x|^2}dx$$

Since $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we have: $(x_1^2 + \cdots + x_n^2)x^\alpha\varphi(x) \in L^\infty(\mathbb{R}^n)$. Then: there exists $c > 0$ such that $|(x_1^2 + \cdots + x_n^2)x^\alpha\varphi(x)| < c$ a.e. which leads to:

$$\begin{aligned} \int_{|x|>A} |x^\alpha\varphi(x)|dx &= \int_{|x|>A} \frac{|(x_1^2 + \cdots + x_n^2)x^\alpha\varphi(x)|}{|x|^2}dx \\ &\leq \int_{|x|>A} \frac{c}{|x|^2}dx \overset{A\to+\infty}{\longrightarrow} 0 . \end{aligned}$$

Hence, $x^\alpha\varphi(x) \in L^1(\mathbb{R}^n)$.
Using the same arguments to prove that:

$$\sum_{|\alpha|\leq p, |\beta|\leq p} \|x^\alpha D^\beta\varphi(x)\|_{L^1(\mathbb{R}^n)} \leq C_p\mathcal{N}_{p+n+1}(\varphi) \quad \forall\varphi \in \mathscr{S}(\mathbb{R}^n).$$

■

Remark 4.6 *: Since $\mathscr{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, we can introduce the Fourier transform in $\mathscr{S}(\mathbb{R}^n)$. Moreover:*

> The properties of the Fourier transform (derivative, translation, dilation, convolution, and inversion) are always verified in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$.

Theorem 4.6 : *The Fourier transform maps the space $\mathscr{S}(\mathbb{R}^n)$ into itself, and for every $p \in \mathbb{N}$, there exists a constant C_p such that:*

$$\mathcal{N}_p(\widehat{\varphi}) \leq C_p\mathcal{N}_{p+n+1}(\varphi) \quad \forall\varphi \in \mathscr{S}(\mathbb{R}^n)$$

Proof: Let $\xi \in \mathbb{R}^n$ et $\alpha, \beta \in \mathbb{N}^n$. Then: $|\xi^\alpha D^\beta\widehat{\varphi}(\xi)| = |(i)^{|\beta|-|\alpha|}\mathcal{F}(D^\alpha(x^\beta\varphi))|$ and:

$$\begin{aligned}
\mathcal{N}_p(\widehat{\varphi}) &= \sum_{|\alpha|\le p,|\beta|\le p} \|\xi^\alpha D^\beta \widehat{\varphi}(x)\|_{L^\infty(\mathbb{R}^n)} \\
&= \sum_{|\alpha|\le p,|\beta|\le p} \|(i)^{|\beta|-|\alpha|}\mathcal{F}(D^\alpha(x^\beta\varphi))\|_{L^\infty(\mathbb{R}^n)} \\
&\le \sum_{|\alpha|\le p,|\beta|\le p} c_{\alpha,\beta}\|D^\alpha(x^\beta\varphi)\|_{L^1(\mathbb{R}^n)} \\
&\le \sum_{|\alpha|\le p,|\beta|\le p} c'_{\alpha,\beta}\|x^\beta D^\alpha\varphi\|_{L^1(\mathbb{R}^n)} \\
&\le C_p\mathcal{N}_{p+n+1}(\varphi).
\end{aligned}$$

■

Proposition 4.3 (***density of*** $\mathscr{D}(\mathbb{R}^n)$ ***in*** $\mathscr{S}(\mathbb{R}^n)$) : *Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then , there exists a sequence $(\varphi_j)_{j\in\mathbb{N}} \subset \mathscr{D}(\mathbb{R}^n)$ such that:*

$$\lim_{j\to+\infty} \mathcal{N}_p(\varphi_j - \varphi) = 0$$

Definition 4.5 (***slow decay***) :

1. *We say that a function $\varphi : \mathbb{R}^n \to \mathbb{C}$ has slow decay if there exist $m \in \mathbb{N}$ and $C > 0$ such that for all $x \in \mathbb{R}^n$, we have:*

$$|\varphi(x)| \le C(1+|x|)^m.$$

2. *$\mathscr{O}_M(\mathbb{R}^n)$ is the space of functions $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ such that for every $\alpha \in \mathbb{N}^n$, the function $D^\alpha\varphi$ has slow decay, i.e for all $\alpha \in \mathbb{N}^n$, there exists $C_\alpha > 0$ et $m_\alpha > 0$ such that for all $x \in \mathbb{R}^n$ we have:*

$$|D^\alpha f(x)| \le C_\alpha(1+|x|)^{m_\alpha}$$

t immediately follows from the above definition:

Theorem 4.7 : *Let $\psi \in \mathscr{O}_M(\mathbb{R}^n)$. Then: for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we have: $\psi.\varphi \in \mathscr{S}(\mathbb{R}^n)$.*

4.3 tempered distributions

Definition 4.6 : *Let $u \in \mathscr{D}'(\mathbb{R}^n)$. We say that u is a tempered distribution, denoted $u \in \mathscr{S}'(\mathbb{R}^n)$, if there exists $p \in \mathbb{N}$ and $C \ge 0$ such that:*

$$|\langle u, \varphi\rangle| \le C\mathcal{N}_p(\varphi) \qquad \forall\varphi \in \mathscr{D}(\mathbb{R}^n). \tag{4.5}$$

This concept refers to the continuity of the linear form u in the context of the trace topology from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{D}(\mathbb{R}^n)$. Based on the density of $\mathscr{D}(\mathbb{R}^n)$ in $\mathscr{S}(\mathbb{R}^n)$, and according to the Hahn-Banach Theorem (Corollary 1.1), we can extend the duality bracket $\langle ., .\rangle_{\mathscr{D}',\mathscr{D}}$ to the bracket $\langle ., .\rangle_{\mathscr{S}',\mathscr{S}}$ as follows:

Theorem 4.8 (***duality extension***) : *Let $u \in \mathscr{S}'(\mathbb{R}^n)$. The mapping $\varphi \to \langle u, \varphi\rangle$, defined on $\mathscr{D}(\mathbb{R}^n)$, uniquely extends to a linear form on $\mathscr{S}(\mathbb{R}^n)$ (denoted as $\varphi \to \langle u, \varphi\rangle_{\mathscr{S}',\mathscr{S}}$) which satisfies:*

$$|\langle u, \varphi\rangle_{\mathscr{S}',\mathscr{S}}| \leq C\mathcal{N}_p(\varphi) \qquad \forall \varphi \in \mathscr{S}(\mathbb{R}^n) \tag{4.6}$$

This extension of duality identifies $\mathscr{S}'(\mathbb{R}^n)$ with the space of linear forms on $\mathscr{S}(\mathbb{R}^n)$ that satisfy an estimate of the form (4.5).

Definition 4.7 (***convergence in*** $\mathscr{S}'(\mathbb{R}^n)$) : *We say that the sequence (u_j) of elements in $\mathscr{S}'(\mathbb{R}^n)$ converges to u in $\mathscr{S}'(\mathbb{R}^n)$ if the following condition holds:*

$$\lim_{j\to+\infty} \langle u_j, \varphi\rangle = \langle u, \varphi\rangle \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^n)$$

Using the duality extension $\langle ., .\rangle_{\mathscr{S}',\mathscr{S}}$ to define the derivative of a tempered distribution u as follows:

$$\forall \alpha \in \mathbb{N}^n, \forall \varphi \in \mathscr{S}(\mathbb{R}^n) : \langle D^\alpha u, \varphi\rangle = (-1)^{|\alpha|}\langle u, D^\alpha \varphi\rangle.$$

The quantity above is well-defined, and furthermore, we have the following result:

Theorem 4.9 : *If $u \in \mathscr{S}'(\mathbb{R}^n)$, then all its partial derivatives belong to $\mathscr{S}'(\mathbb{R}^n)$. Moreover, if $u_j \to u$ in $\mathscr{S}'(\mathbb{R}^n)$, then $D^\alpha u_j \to D^\alpha u$ in $\mathscr{S}'(\mathbb{R}^n)$.*

Example 4.2 :

1. *$\delta \in \mathscr{S}'(\mathbb{R}^n)$ because for any $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we have:*

$$|\langle \delta, \varphi\rangle| = |\varphi(0)| \leq \mathcal{N}_0(\varphi).$$

2. *$L^1(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ because for any $f \in L^1(\mathbb{R}^n)$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we have:*

$$\begin{aligned} |\langle f, \varphi\rangle| &= \left|\int_{\mathbb{R}^n} f(x)\varphi(x)dx\right| \\ &\leq \|\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f(x)|dx \\ &= \|f\|_{L^1(\mathbb{R}^n)}\mathcal{N}_0(\varphi). \end{aligned}$$

3. *$L^\infty(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ because for any $f \in L^\infty(\mathbb{R}^n)$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we have:*

$$\begin{aligned} |\langle f, \varphi\rangle| &= \left|\int_{\mathbb{R}^n} f(x)\varphi(x)dx\right| \\ &\leq \|f\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\varphi(x)|dx \\ &= \|f\|_{L^\infty(\mathbb{R}^n)}.\|\varphi\|_{L^1(\mathbb{R}^n)} \\ &= C\|f\|_{L^\infty(\mathbb{R}^n)}\mathcal{N}_{n+1}(\varphi). \end{aligned}$$

4. *$L^2(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ because for any $f \in L^2(\mathbb{R}^n)$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we have:*

$$\begin{aligned}
|\langle f,\varphi\rangle| &= \left|\int_{\mathbb{R}^n} f(x)\varphi(x)dx\right| \\
&\leq \left(\int_{\mathbb{R}^n} f^2(x)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \varphi^2(x)dx\right)^{\frac{1}{2}} \\
&= \|f\|_{L^2(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} \frac{\varphi(x)}{(1+\|x\|)^{n+2}}.(1+\|x\|)^{n+2}.\varphi(x)dx\right)^{\frac{1}{2}} \\
&\leq C_1\|f\|_{L^2(\mathbb{R}^n)}(\|(1+\|x\|)^{n+2}.\varphi\|_{L^1(\mathbb{R}^n)})^{\frac{1}{2}} \\
&= C|f\|_{L^2}.\mathcal{N}_{1+[\frac{n}{2}]}(\varphi).
\end{aligned}$$

5. *$L^p(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ $(2 \leq p < +\infty)$ because for any $f \in L^p(\mathbb{R}^n)$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we have:*

$$|\langle f,\varphi\rangle| \leq C|f\|_{L^p}.\mathcal{N}_{1+[\frac{n}{p'}]}(\varphi) \quad \left(\frac{1}{p}+\frac{1}{p'}=1\right).$$

Proposition 4.4 :

1. *A distribution with compact support is tempered, i.e $\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$.*
2. *A tempered distribution is necessarily of finite order*

Proof: :

1. Let $u \in \mathscr{E}'(\mathbb{R}^n)$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Then, $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ and there exists a compact $K \subset \mathbb{R}^n$ and $m \in \mathbb{N}$ et $M > 0$ such that:

$$\begin{aligned}
|\langle u,\varphi\rangle| &\leq M.P_{K,m}(\varphi) \\
&= M.\sup_{x\in K,|\alpha|\leq m} |D^\alpha\varphi(x)| \\
&\leq M.\sup_{x\in \mathbb{R}^n,|\alpha|\leq m} |D^\alpha\varphi(x)| \\
&\leq M.\mathcal{N}_m(\varphi).
\end{aligned}$$

   Then: $u \in \mathscr{S}'(\mathbb{R}^n)$.

2. Let $u \in \mathscr{S}'(\mathbb{R}^n)$ and $K \subset$ be a compact. There exists $C \geq 0$ such that:

$$|\langle u,\varphi\rangle| \leq C\mathcal{N}_p(\varphi) \qquad \forall\varphi \in \mathscr{D}_K(\mathbb{R}^n).$$

   Hence:

$$|\langle u,\varphi\rangle| \leq C \sum_{|\alpha|\leq p,|\beta|\leq p} \|x^\alpha D^\beta\varphi(x)\|_{L^\infty(\mathbb{R}^n)} \qquad \forall\varphi \in \mathscr{D}_K(\mathbb{R}^n).$$

   Since K is a compact, then x^α is bounded, so, there exists $C_p > 0$ such that for all

$\varphi \in \mathscr{D}_K(\mathbb{R}^n)$ on a:

$$\begin{aligned} C \sum_{|\alpha|\leq p, |\beta|\leq p} \|x^\alpha D^\beta \varphi(x)\|_{L^\infty(\mathbb{R}^n)} &\leq C_p \sum_{|\beta|\leq p} \|D^\beta \varphi(x)\|_{L^\infty(\mathbb{R}^n)} \\ &\leq M \sup_{x\in K, \beta|\leq p} |D^\beta \varphi(x)| \\ &= M P_{K,p}(\varphi). \end{aligned}$$

Then: $|\langle u, \varphi\rangle| \leq M P_{K,p}(\varphi)$.
Therefore: u is with order less to p.

■

Theorem 4.10 : *Let $\psi \in \mathscr{O}_M(\mathbb{R}^n)$. Then:*

1. *For all $u \in \mathscr{S}'(\mathbb{R}^n)$ on a: $\psi.u \in \mathscr{S}'(\mathbb{R}^n)$.*
2. *Si $u_j \to u$ dans $\mathscr{S}'(\mathbb{R}^n)$ on a $\psi.u_j \to f.u$ dans $\mathscr{S}'(\mathbb{R}^n)$*

Proof: Let $\psi \in \mathscr{O}_M(\mathbb{R}^n)$. For any $\gamma \in \mathbb{N}^n$, there exists $C_\gamma > 0$ et $m_\gamma \in \mathbb{N}$ such that for all $x \in \mathbb{R}^n$ on a:

$$|D^\gamma \psi(x)| \leq C(1+|x|)^{m_\gamma}.$$

1. Let $u \in \mathscr{S}'(\mathbb{R}^n)$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Then: $\psi.\varphi \in \mathscr{D}(\mathbb{R}^n)$ and there exists $C_p > 0, p \in \mathbb{N}$ such that:

$$\begin{aligned} |\langle \psi u, \varphi\rangle| &= |\langle u, \psi\varphi\rangle| \\ &\leq C_p \mathcal{N}_p(\psi\varphi) \\ &= C_p \sum_{|\alpha|\leq p, |\beta|\leq p} \|x^\alpha D^\beta(\psi.\varphi)(x)\|_{L^\infty(\mathbb{R}^n)} \\ &= C_p \sum_{|\alpha|\leq p, |\gamma|\leq p, |\theta|\leq p} \|C_{\gamma,\theta} x^\alpha D^\gamma \psi(x).D^\theta \varphi(x)\|_{L^\infty(\mathbb{R}^n)} \\ &\leq C'_p \sum_{|\alpha|\leq p, |\gamma|\leq p, |\theta|\leq p} \|x^\alpha (1+|x|)^{m_\gamma}.D^\theta \varphi(x)\|_{L^\infty(\mathbb{R}^n)} \end{aligned}$$

There exists $q \in \mathbb{N}$ such that $\leq \max\{p, |\alpha| + m_\gamma\} \leq q$, which leads to:

$$|\langle \psi u, \varphi\rangle| \leq C_q \sum_{|\lambda|\leq q, |\theta|\leq q} \|x^\lambda.D^\theta \varphi(x)\|_{L^\infty(\mathbb{R}^n)} = C_q \mathcal{N}_q(\varphi)$$

. So, $\psi.u \in \mathscr{S}'(\mathbb{R}^n)$.

2. Based on the previous and Theorem 4.9.

■

4.4 Fourier transform for tempered distributions

Consider $u \in L^1(\mathbb{R}^n)$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then: $\widehat{u} \in L^1(\mathbb{R}^n)$ and we have:

$$\begin{aligned}
\langle \widehat{u}, \varphi \rangle &= \int_{\mathbb{R}^n} \widehat{u}(\xi)\varphi(\xi)d\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x)e^{-ix.\xi}\varphi(\xi)d\xi dx.
\end{aligned}$$

Let's take the change of variable $(y, \zeta) = (\xi, x)$, we get:

$$\begin{aligned}
\langle \widehat{u}, \varphi \rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(\zeta)e^{-iy.\zeta}\varphi(y)dyd\zeta. \\
&= \int_{\mathbb{R}^n} u(\zeta)\widehat{\varphi}(\zeta)d\zeta \\
&= \langle u, \widehat{\varphi} \rangle.
\end{aligned}$$

Taking into account that for $\varphi \in \mathscr{S}(\mathbb{R}^n)$ implies that $\widehat{\varphi} \in \mathscr{S}(\mathbb{R}^n)$, we can extend the above result as follows:

Definition 4.8 : *Let $u \in \mathscr{S}'(\mathbb{R}^n)$.*

1. *The Fourier transform of u is a tempered distribution denoted as $\widehat{u}$ or $\mathcal{F}u$, defined for any $\varphi \in \mathscr{S}(\mathbb{R}^n)$ as: $\langle \widehat{u}, \varphi \rangle = \langle u, \widehat{\varphi} \rangle$.*
2. *The conjugate $\overline{\mathcal{F}}$ of $\mathcal{F}$ is defined for any $\varphi \in \mathscr{S}(\mathbb{R}^n)$ as: $\langle \overline{\mathcal{F}}u, \varphi \rangle = \langle u, \overline{\mathcal{F}}\varphi \rangle$.*

It immediately follows from the definition and the properties of the Fourier transform in $\mathscr{S}(\mathbb{R}^n)$:

Theorem 4.11 ***(inverse)*** : *The Fourier transform is an isomorphism of $\mathscr{S}'(\mathbb{R}^n)$ onto itself, with the inverse $\mathcal{F}^{-1} = (2\pi)^{-n}\overline{\mathcal{F}}$.*

Theorem 4.12 (***continuity***) : *The Fourier transform on $\mathscr{S}'(\mathbb{R}^n)$ is continuous. If $u_j \to u$ in $\mathscr{S}'(\mathbb{R}^n)$, then: $\widehat{u_j} \to \widehat{u}$ in $\mathscr{S}'(\mathbb{R}^n)$*

Example 4.3 :

1. *We have:*

$$\begin{aligned}
\langle \widehat{\delta}, \varphi \rangle &= \langle \delta, \widehat{\varphi} \rangle = \widehat{\varphi}(0) \\
&= \int_{\mathbb{R}^n} e^{-i0.x}\varphi(x)dx. \\
&= \int_{\mathbb{R}^n} 1.\varphi(x)dx \\
&= \langle 1, \varphi \rangle.
\end{aligned}$$

Then: $\widehat{\delta} = 1$.

2. *We have:*

$$\begin{aligned}\langle \widehat{1}, \varphi \rangle &= \langle 1, \widehat{\varphi} \rangle \\ &= \int_{\mathbb{R}^n} \widehat{\varphi}(\xi) d\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix.\xi} \varphi(x) dx d\xi.\end{aligned}$$

Making the change of variable $\zeta = -\xi$, *we obtain:*

$$\begin{aligned}\langle \widehat{1}, \varphi \rangle &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix.\zeta} \varphi(x) dx d\zeta. \\ &= \int_{\mathbb{R}^n} \mathcal{F}(\varphi)(x) dx. \\ &= \langle 1, \mathcal{F}(\varphi) \rangle = \langle \mathcal{F}(1), \varphi \rangle.\end{aligned}$$

Hence: $\widehat{1} = \mathcal{F}(1)$, *which leads to:* $(2\pi)^{-n}\widehat{1} = (2\pi)^{-n}\mathcal{F}(1) = (2\pi)^{-n}\mathcal{F}(\widehat{\delta}) = \delta$.
Therefore: $\widehat{1} = (2\pi)^n \delta$.

Proposition 4.5 : *Let* $u \in \mathscr{S}'(\mathbb{R}^n), \alpha \in \mathbb{N}^n$ *and* $a \in \mathbb{R}^n$. *Then, we have:*

1. $\mathcal{F}(\tau_a u) = e^{-ia.\xi}\widehat{u}$, $\quad \mathcal{F}(e^{ia.\xi}u) = \tau_a \widehat{u}$.
2. $\mathcal{F}(D^\alpha u) = i^{|\alpha|}\xi^\alpha \widehat{u}$, $\quad D^\alpha \widehat{u} = \mathcal{F}((-i)^{|\alpha|}x^\alpha u)$.
3. $\widehat{\delta_a} = e^{-ia.\xi}$, $\quad \mathcal{F}(e^{ia.\xi}) = (2\pi)^n \delta_a$.
4. $\mathcal{F}(D^\alpha \delta) = i^{|\alpha|}\xi^\alpha$, $\quad \mathcal{F}(x^\alpha) = (2\pi)^n i^{|\alpha|} D^\alpha \delta$.

Proof: Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then:

1. *)

$$\begin{aligned}\langle \mathcal{F}(\tau_a u), \varphi \rangle &= \langle \tau_a u, \widehat{\varphi} \rangle \\ &= \langle u, \tau_{-a}\widehat{\varphi} \rangle \\ &= \langle u, \mathcal{F}(e^{ia.x}\varphi) \rangle\end{aligned}$$

Therefore: $\mathcal{F}(\tau_a u) = e^{-ia.\xi}\widehat{u}$.
**)

$$\begin{aligned}\langle \mathcal{F}(e^{ia.\xi}u), \varphi \rangle &= \langle e^{ia.\xi}u, \widehat{\varphi} \rangle \\ &= \langle u, e^{ia.\xi}\widehat{\varphi} \rangle \\ &= \langle u, \mathcal{F}(\tau_{-a}\varphi) \rangle \\ &= \langle \widehat{u}, \tau_{-a}\varphi \rangle \\ &= \langle \tau_a \widehat{u}, \varphi \rangle.\end{aligned}$$

Therefore: $\mathcal{F}(e^{ia.\xi}u) = \tau_a \widehat{u}$.

2. *)

$$\begin{aligned}\langle\mathcal{F}(D^\alpha u),\varphi\rangle &= \langle D^\alpha u,\widehat{\varphi}\rangle\\ &= (-1)^{|\alpha|}\langle u, D^\alpha\widehat{\varphi}\rangle\\ &= (-1)^{|\alpha|}\langle u,\mathcal{F}((-i)^{|\alpha|}x^\alpha\varphi)\rangle\\ &= \langle u,\mathcal{F}(i^{|\alpha|}x^\alpha\varphi)\rangle\\ &= \langle\widehat{u}, i^{|\alpha|}x^\alpha\varphi\rangle\\ &= \langle i^{|\alpha|}\xi^\alpha\widehat{u},\varphi\rangle.\end{aligned}$$

Therefore: $\mathcal{F}(D^\alpha u) = i^{|\alpha|}\xi^\alpha\widehat{u}$.

*)

$$\begin{aligned}\langle\mathcal{F}((-i)^{|\alpha|}x^\alpha u),\varphi\rangle &= \langle\xi^\alpha u,\widehat{\varphi}\rangle\\ &= \langle u,(-i)^{|\alpha|}\xi^\alpha\widehat{\varphi}\rangle\\ &= (-1)^{|\alpha|}\langle u,\mathcal{F}(D^\alpha\varphi)\rangle\\ &= (-1)^{|\alpha|}\langle\widehat{u}, D^\alpha\varphi)\rangle\\ &= \langle D^\alpha\widehat{u},\varphi\rangle.\end{aligned}$$

Therefore: $D^\alpha\widehat{u} = \mathcal{F}((-i)^{|\alpha|}x^\alpha u)$.

3. *) $\widehat{\delta_a} = \mathcal{F}(\tau_a\delta) = e^{-ia.\xi}\widehat{\delta} = e^{-ia.\xi}$.
   *) $\mathcal{F}(e^{ia.\xi}) = \tau_a\widehat{1} = (2\pi)^n\tau_a\delta = (2\pi)^n\delta_a$.

4. *) $\mathcal{F}(D^\alpha\delta) = i^{|\alpha|}\xi^\alpha\widehat{\delta} = i^{|\alpha|}\xi^\alpha$.
   *) $\mathcal{F}(x^\alpha) = \dfrac{1}{(-i)^{|\alpha|}}D^\alpha\widehat{1} = (2\pi)^n i^{|\alpha|}D^\alpha\delta$.

■

Theorem 4.13 (***convolution***) : *Let $T\in\mathscr{S}'(\mathbb{R}^n)$ and $S\in\mathscr{E}'(\mathbb{R}^n)$, then:*

$$T*S\in\mathscr{S}'(\mathbb{R}^n)\ et\ \widehat{T*S} = \widehat{T}.\widehat{S}.$$

Example 4.4 : *We provide two examples used in partial differential equations.*

1. *Consider in $\mathscr{S}'(\mathbb{R}^n)$ the Laplace equation:*

$$\Delta u = 0.$$

Using the Fourier transform, we obtain: $\widehat{\Delta u} = 0$. But:

$$\begin{aligned}\widehat{\Delta u} &= \mathcal{F}\left(\sum_{i=1}^n\frac{\partial^2 u}{\partial x_i^2}\right) = \sum_{i=1}^n\mathcal{F}\left(\frac{\partial^2 u}{\partial x_i^2}\right)\\ &= \sum_{i=1}^n(-ix_j)^2\widehat{u}\\ &= -|x|^2\widehat{u}\end{aligned}$$

Then: $\widehat{u}_{|\mathbb{R}^n_+} = 0$ and $\operatorname{supp}\widehat{u} = \{0\}$. Therefore: $\widehat{u} = \sum_{|\alpha|\le m\in\mathbb{N}} a_\alpha D^\alpha\delta$.

which leads to: $u = \sum_{|\alpha|\leq m} (2\pi)^{-n} a_\alpha \mathcal{F}(D^\alpha \delta) = \sum_{|\alpha|\leq m} (2\pi)^{-n} a_\alpha i^{|\alpha|} \xi^\alpha = \sum_{|\alpha|\leq m} b_\alpha \xi^\alpha.$
Hence: u *is a polynomial.*

2. *Consider in* $\mathscr{S}'(\mathbb{R}^n)$ *the equation:*

$$-\Delta u + \lambda u = f, \;\; où\ \lambda > 0, f \in \mathscr{S}'(\mathbb{R}^n)$$

. *Using the Fourier transform, we obtain:* $\mathcal{F}(-\Delta u + \lambda u) = \widehat{f}$.
Then: $(|x|^2 + \lambda)\widehat{u} = \widehat{f}$.
Therefore: $\widehat{u} = \dfrac{\widehat{f}}{|x|^2 + \lambda}$.
Finely: $u = (2\pi)^{-n} \overline{\mathcal{F}} \left[\dfrac{\widehat{f}}{|x|^2 + \lambda} \right]$.
For $f = \delta$ *we obtain the elementary solution* $u_0 = (2\pi)^{-n} \overline{\mathcal{F}} \left[\dfrac{1}{|x|^2 + \lambda} \right]$.

Exercices

Exercise 4.1 : *Calculate* $\widehat{f}$ *in the follows cases:*

1. $f(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}$.
2. $f(x) = e^{-\alpha|x|}$ $(\alpha > 0)$.
3. $f(x) = H(x)e^{-\alpha x}$ $(\alpha > 0)$.

Exercise 4.2 :

1. *Show that the function* $\xi \mapsto \widehat{f}(\xi) = \mathcal{F}(e^{-x^2})(\xi)$ *satisfy the differential equation :*

$$y'(\xi) + \frac{\xi}{2}\widehat{f}(\xi) = 0.$$

2. *Calculate* $\widehat{f}(0)$, *and then determine the solution to the differential equation.*
3. *Use the dilation property to establish the result:*

$$\mathcal{F}(e^{-ax^2})(\xi) = \sqrt{\frac{\pi}{a}}.e^{-\frac{\xi^2}{4a}} \;\; (a > 0).$$

Exercise 4.3 : *Let* T *be the linear operator on* $\mathscr{S}(\mathbb{R}^2)$ *defined as follows:*

$$\forall \varphi \in \mathscr{S}(\mathbb{R}^2) : \langle T, \varphi \rangle = \int_{\mathbb{R}} \varphi(x, -x)dx$$

1. *Verify that* $T \in \mathscr{S}'(\mathbb{R}^2)$.

2. *Calculate in the distribution sense:* $\dfrac{\partial T}{\partial x} - \dfrac{\partial T}{\partial y}$.

Exercise 4.4 : *Consider the sequence of functions* $\{f_j\}_{j\in\mathbb{N}}$ *such that:* $f_j = \chi_{[-j,j]}$.

1. *Calculate* $\widehat{f_j}$.

2. *Determinate* $\lim\limits_{j\to+\infty} \dfrac{\sin(j\xi)}{\xi}$ *in* $\mathscr{S}'(\mathbb{R}^n)$.

Exercise 4.5 : *By using the equality* $\delta' * H = \delta$ *calculate* $\widehat{H}$.

Exercise 4.6 : *Let* $\psi \in \mathscr{D}(\mathbb{R})$ *such that* $\psi = 1$ *in the neighbourhood of* 0. *Set* $u = \psi.H$

1. *Calculate* u' *in function of* ψ.

2. *Calculate* $\widehat{u}$ *in function of* $\widehat{\psi'.H}$.

Exercise 4.7 : *Consider the function* f *defined as:* $f(x) = 1 - 10x^2 + 20x^{20}$.
Show that $f \in \mathscr{S}'(\mathbb{R})$, *then calculate* $\widehat{f}$.

Exercise 4.8 :

1. *Show that* $v_p\frac{1}{x} \in \mathscr{S}'(\mathbb{R})$.

2. *Find all tempered distributions* u *such that* $\dfrac{d\widehat{u}}{d\xi} = 0$.

3. *Show that all tempered distributions* u *such that* $xu = 0$ *are of the form* $u = \lambda\delta$ $(\lambda \in \mathbb{R})$.

4. *What are the tempered distributions* u *such that* $xu' + u = 0$.

Solutions of exercises

Solution 4.1 : *Let* $\xi \in \mathbb{R}$

1. $f(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}$.

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{-\infty}^{+\infty} e^{-ix.\xi}\chi_{[-\frac{1}{2},\frac{1}{2}]}(x)dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-ix.\xi}dx. \\
&= \begin{cases} 1 & : \ \xi = 0 \\ \dfrac{e^{\frac{i\xi}{2}} - e^{-\frac{i\xi}{2}}}{i\xi} & : \ \xi \neq 1. \end{cases}
\end{aligned}$$

Then: $\widehat{f}(\xi) = \begin{cases} 1 & : \ \xi = 0 \\ \dfrac{\sin\frac{\xi}{2}}{\frac{\xi}{2}} & : \ \xi \neq 1. \end{cases}$

2. $f(x) = e^{-\alpha|x|}$ $(\alpha > 0)$.

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{-\infty}^{+\infty} e^{-ix.\xi} e^{-\alpha|x|} dx \\
&= \int_{-\infty}^{0} e^{(\alpha - i\xi).x} dx + \int_{0}^{+\infty} e^{-(\alpha + i\xi).x} dx \\
&= \left[\frac{e^{(\alpha - i\xi).x}}{\alpha - i\xi} \right]_{-\infty}^{0} - \left[\frac{e^{-(\alpha + i\xi).x}}{\alpha + i\xi} \right]_{0}^{+\infty} \\
&= \frac{1}{\alpha - i\xi} + \frac{1}{\alpha + i\xi} \\
&= \frac{2\alpha}{\alpha^2 + \xi^2}.
\end{aligned}$$

3. $f(x) = H(x)e^{-\alpha x}$ $(\alpha > 0)$.

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{-\infty}^{+\infty} e^{-ix.\xi} H(x) e^{-\alpha x} dx \\
&= \int_{0}^{+\infty} e^{-(\alpha + i\xi).x} dx \\
&= -\left[\frac{e^{-(\alpha + i\xi).x}}{\alpha + i\xi} \right]_{0}^{+\infty} \\
&= \frac{1}{\alpha + i\xi} \\
&= \frac{\alpha - i\xi}{\alpha^2 + \xi^2}.
\end{aligned}$$

Solution 4.2 :

1. $f(x) = e^{-x^2}$ $\quad f'(x) = -2xe^{-x^2} = -2xf(x)$ $\quad \widehat{f}(\xi) = \mathcal{F}(e^{-x^2})(\xi)$.

$$\frac{d\widehat{f}}{d\xi} = \mathcal{F}(-ixf) = \frac{i}{2}\mathcal{F}(-2xf) = \frac{i}{2}\mathcal{F}(f') = -\frac{\xi}{2}\widehat{f}(\xi)$$

. Then: $\widehat{f}$ *verify the differential equation :*

$$y'(\xi) + \frac{\xi}{2}\widehat{f}(\xi) = 0.$$

2. $\widehat{f}(0) = \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$. *Then:*

$$\widehat{f}(\xi) = \sqrt{\pi} e^{-\frac{\xi^2}{4}}.$$

3. *We have:*

$$\mathcal{F}(e^{-ax^2})(\xi) = \mathcal{F}(f(\sqrt{a}x))(\xi) = \sqrt{\frac{1}{a}}.\widehat{f}\left(\frac{\xi}{\sqrt{a}}\right) = \sqrt{\frac{\pi}{a}}.e^{-\frac{\xi^2}{4a}}.$$

Solution 4.3 : $\forall\varphi \in \mathscr{S}(\mathbb{R}^2) : \langle T, \varphi\rangle = \displaystyle\int_{\mathbb{R}} \varphi(t,-t)dt.$

1. *Let* $\varphi \in \mathscr{D}(\mathbb{R}^2)$. *Then, we have:*

$$\begin{aligned} |\langle T,\varphi\rangle| &\leq \int_{\mathbb{R}} |\varphi(t,-t)|dt \\ &= \int_{\mathbb{R}} \frac{1}{1+t^2}.|(1+t^2)\varphi(t,-t)|dt \\ &\leq \|(1+x^2)\varphi(x,y)\|_{L^\infty(\mathbb{R}^2)}.\textstyle\int_{\mathbb{R}} \frac{1}{1+t^2}dt \\ &\leq \pi.\mathcal{N}_2(\varphi). \end{aligned}$$

Therefore: $T \in \mathscr{S}'(\mathbb{R}^2)$.

2. *Let* $\varphi \in \mathscr{S}(\mathbb{R}^2)$.

$$\begin{aligned} \left\langle \frac{\partial T}{\partial x} - \frac{\partial T}{\partial y}, \varphi \right\rangle &= -\left\langle T, \frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial y} \right\rangle \\ &= -\int_{-\infty}^{+\infty} \left[\frac{\partial \varphi}{\partial x}(t,-t) - \frac{\partial \varphi}{\partial y}(t,-t)\right] dt \end{aligned}$$

Set: $\Phi(t) = \varphi(t,-t)$. *Then:* $\Phi'(t) = \dfrac{\partial \varphi}{\partial x}(t,-t) - \dfrac{\partial \varphi}{\partial y}(t,-t)$.
Therefore:

$$\begin{aligned} \left\langle \frac{\partial T}{\partial x} - \frac{\partial T}{\partial y}, \varphi \right\rangle &= -\int_{-\infty}^{+\infty} \Phi'(t)dt \\ &= [-\Phi(t)]_{-\infty}^{+\infty} \\ &= [-\varphi(t,-t)]_{-\infty}^{+\infty} = 0. \end{aligned}$$

So, $\dfrac{\partial T}{\partial x} - \dfrac{\partial T}{\partial y} = 0$.

Solution 4.4 : $f_j = \chi_{[-j,j]}$ $(j \in \mathbb{N}^*)$.

1. *We have:*

$$\begin{aligned} \widehat{f_j}(\xi) &= \int_{-\infty}^{+\infty} e^{-ix.\xi}\chi_{[-j,j]}(x)dx \\ &= \int_{-j}^{j} e^{-ix.\xi}dx. \\ &= \begin{cases} 2j & : \ \xi = 0 \\ \dfrac{e^{ij} - e^{-ij}}{i\xi} & : \ \xi \neq 1. \end{cases} \end{aligned}$$

Then: $\widehat{f_j}(\xi) = \begin{cases} 2j & : \ \xi = 0 \\ \dfrac{2\sin(j\xi)}{\xi} & : \ \xi \neq 1. \end{cases}$

2. *Let* $\varphi \in \mathscr{S}'(\mathbb{R}^n)$. *Then:*

$$\begin{aligned}\lim_{j\to+\infty}\langle f_j,\varphi\rangle &= \lim_{j\to+\infty}\int_{-\infty}^{+\infty} f_j(x)\varphi(x)dx\\ &= \lim_{j\to+\infty}\int_{-j}^{j}\varphi(x)dx\\ &= \int_{-\infty}^{+\infty}\varphi(x)dx\\ &= \langle 1,\varphi\rangle.\end{aligned}$$

So, $\lim\limits_{j\to+\infty} f_j = 1$, *which leads to:* $\lim\limits_{j\to+\infty}\widehat{f_j} = 2\pi\delta$.
Then: $\lim\limits_{j\to+\infty}\dfrac{2\sin(j\xi)}{\xi} = 2\pi\delta$ *,i.e* $\lim\limits_{j\to+\infty}\dfrac{\sin(j\xi)}{\xi} = \pi\delta$.

Solution 4.5 : *We have* $\delta' * H = \delta$, *then:* $\widehat{\delta' * H} = 1$.
Therefore: $\mathcal{F}(\delta').\widehat{H} = 1$, *which leads to:* $i\xi.\widehat{H} = 1$.
finely: $\widehat{H} = \dfrac{1}{i\xi} = -\dfrac{i}{\xi}$.

Solution 4.6 : $\psi \in \mathscr{D}(\mathbb{R})$ *such that* $\psi = 1$ *in the neighbourhood of* 0. $u = \psi.H$

1. *On a:*

$$\begin{aligned}\langle u',\varphi\rangle &= -\langle u,\varphi'\rangle\\ &= -\langle \psi.H,\varphi'\rangle\\ &= -\langle H,\psi.\varphi'\rangle\\ &= -\int_0^{+\infty}\psi(x)\varphi'(x)dx\\ &= -[\psi(x)\varphi(x)]_0^{+\infty} + \int_0^{+\infty}\psi'(x)\varphi(x)dx\\ &= \varphi(0) + \int_{-\infty}^{+\infty} H(x).\psi'(x)\varphi(x)dx\\ &= \langle \delta + \psi'.H,\varphi\rangle.\end{aligned}$$

Then: $u' = \delta + \psi'.H$.

2. $\widehat{u} = \mathcal{F}(\delta + \psi'.H) = 1 + \widehat{\psi'.H}$.

Solution 4.7 : $f(x) = 1 - 10x^2 + 20x^{20}$.
$*$) *let* $\varphi \in \mathscr{D}(\mathbb{R})$. *on a:*

$$\begin{aligned}|\langle f,\varphi\rangle| &= \int_{-\infty}^{+\infty}(1-10x^2+20x^{20}).\varphi(x)dx\\ &= \|(1-10x^2+20x^{20}).\varphi\|_{L^1(\mathbb{R}^n)}\\ &\le C_{20}\mathcal{N}_{22}(\varphi).\end{aligned}$$

Then: $f \in \mathscr{S}'(\mathbb{R})$.

$*$) *We have:*

$$\begin{aligned}\widehat{f} &= \widehat{1} - 10\mathcal{F}(x^2) + 20\mathcal{F}(x^{20}) \\ &= 2\pi\delta - 10(2\pi).i^2\delta'' + 20(2\pi).i^{20}\delta^{(20)} \\ &= 2\pi(\delta + 10\delta'' + 20\delta^{(20)}).\end{aligned}$$

Solution 4.8 :

1. *Let* $\varphi \in \mathscr{D}(\mathbb{R})$. *We know from example 2.9 that:*

$$|\langle v_p\frac{1}{x}, \varphi\rangle| \leq \int_{-\infty}^{+\infty}\int_0^1 |\varphi'(tx)|dtdx.$$

Then:

$$\begin{aligned}|\langle v_p\tfrac{1}{x}, \varphi\rangle| &\leq \int_{-\infty}^{+\infty}\frac{1}{1+x^2}\int_0^1 |(1+x^2)\varphi'(tx)|dtdx \\ &\leq \|(1+x^2)\varphi'\|_{L^\infty(\mathbb{R})}\int_{-\infty}^{+\infty}\frac{1}{1+x^2}dx \\ &= \pi\mathcal{N}_2(\alpha).\end{aligned}$$

Hence: $v_p\frac{1}{x} \in \mathscr{S}'(\mathbb{R})$.

2. $\dfrac{d\widehat{u}}{d\xi} = 0$ *implies that* $\widehat{u} = \lambda$ $(\lambda \in \mathbb{R})$. *Then:* $u = \lambda\delta$.

3. *We have:* $xu = 0$, *then:* $\widehat{-ixu} = 0$, *i.e* $\dfrac{d\widehat{u}}{d\xi} = 0$. *Therefore:* $u = \lambda\delta$ $(\lambda \in \mathbb{R})$.

4. *We have:* $xu' + u = 0$, *so:* $(xu)' = 0$ (*see solution of exercise 2.8*).
Then: $xu = \lambda\delta$ $(\lambda \in \mathbb{R})$.
Let $\chi \in \mathscr{D}'(\mathbb{R})$ *such that* $\chi = 1$ *in the neighbourhood of* 0. *Set:* $\widetilde{\varphi} = \varphi - \varphi(0).\chi$.
Then: $\widetilde{\varphi}(0) = 0$. *The Taylor formula can be written as:*

$$\widetilde{\varphi}(x) = x\int_0^1 \psi'(tx)dt = x\theta_\varphi(x) \ (\theta_\varphi \in \mathscr{D}(\mathbb{R})).$$

Set: $\langle u, \varphi\rangle = \langle\lambda\delta, \theta_\varphi\rangle = \lambda\theta_\varphi(0)$.
taking into account: $\widetilde{x\varphi} = x\varphi$, *then:* $\theta_{x\varphi} = \varphi$. *Therefore:*
$\langle xu, \varphi\rangle = \langle u, x\varphi\rangle = \langle\lambda\delta, \varphi\rangle$, *i.e* $xu = \lambda\delta$.
Using arguments similar to the ones in the first question to prove that u is a tempered distribution.

CHAPTER 5

SOBOLEV SPACES

Let $\Omega \subset \mathbb{R}^n$ be an open set, $\Gamma = \partial\Omega$, $p \in \mathbb{R}$ with $1 \leq p \leq +\infty$, and p' the conjugate de p, i.e $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Note that a function $u \in L^p(\Omega)$ identifies a distribution over Ω, also denoted u. We can define $\dfrac{\partial u}{\partial x_i}$ $(i \in \{1, cdots, n\})$ as a distribution over Ω and ∇u as a vectorial distribution over Ω.

The purpose of introducing Sobolev spaces is to provide a functional framework for certain partial differential equations and boundary value problems that can have solutions referred to as «weak solutions».

I) Consider the following boundary value problem:

$$(P_1) \qquad \begin{cases} -u''(x) + u(x) = f(x) & : \quad x \in [a,b], \\ u(a) = u(b) = 0. \end{cases}$$

where $f \in \mathscr{C}([a,b])$.

A classical (strong) solution of problem (P_1) is a function in $\mathscr{C}^2([a,b])$. We will seek other solutions of problem (P_1), which are regular distributions. By multiplying both terms of the first equation by a function $\varphi \in \mathscr{D}(a,b)$ and integrating over (a,b), we obtain:

$$\int_a^b -u''(x)\varphi(x)dx + \int_a^b u(x)\varphi(x)dx = \int_a^b f(x)\varphi(x)dx.$$

Using integration by parts and considering $\varphi(a) = \varphi(b) = 0$, we get:

$$\int_a^b u'(x)\varphi'(x)dx + \int_a^b u(x)\varphi(x)dx = \int_a^b f(x)\varphi(x)dx. \tag{5.1}$$

Note that $\mathscr{D}(a,b) \subset L^2(a,b)$, then: φ et φ' can be considered in $L^2(a,b)$ and the equation (5.1) makes sense for $u, u' \in L^2(a,b)$ where u' is the derivative of u in the distributional sense, i.e

$$\langle u', \varphi \rangle = -\int_a^b u(x)\varphi'(x)dx, \qquad \forall \varphi \in \mathscr{D}(a,b).$$

This involves the existence of a function $g \in L^2(a,b)$, satisfying:

$$\int_a^b u(x)\varphi'(x)dx = -\int_a^b g(x)\varphi(x)dx, \qquad \forall \varphi \in \mathscr{D}(a,b).$$

II) Now, consider the following boundary value problem:

$$(P_n) \quad \begin{cases} -\Delta u(x) + u(x) = f(x) & : \quad x \in \overline{\Omega}, \\ u(x) = 0 & : \quad x \in \Gamma, \end{cases}$$

A classical (strong) solution of problem (P_n) for $f \in \mathscr{C}(\overline{\Omega})$ is a function in $\mathscr{C}^2(\overline{\Omega})$. We will seek other solutions to problem (P_n), which are regular distributions. By multiplying both terms of the first equation by a function $\varphi \in \mathscr{D}(\Omega)$ and integrating over Ω, we obtain:

$$\int_\Omega -\Delta u(x)\varphi(x)dx + \int_\Omega u(x)\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx.$$

Applying Green's formula and considering $\varphi(x) = 0$ on Γ, we obtain:

$$\sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x_i}(x).\frac{\partial \varphi}{\partial x_i}(x)dx + \int_\Omega u(x)\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx. \tag{5.2}$$

Note that $\varphi, \dfrac{\partial \varphi}{\partial x_i}$ $(i \in \{1, \cdots n\})$ can be considered in $L^2(\Omega)$ and the equation (5.2) makes sense for $u, \dfrac{\partial u}{\partial x_i} \in L^2(\Omega)$ $(i \in \{1, \cdots n\})$ where $\dfrac{\partial u}{\partial x_i}$ is the partial derivative of u in the distributional sense in the direction i, i.e.,

$$\left\langle \frac{\partial u}{\partial x_i}, \varphi \right\rangle = -\int_\Omega u(x)\frac{\partial \varphi}{\partial x_i}(x)dx, \qquad \forall \varphi \in \mathscr{D}(\Omega).$$

This involves the existence of functions $g_i \in L^2(\Omega)$, satisfying:

$$\int_\Omega u(x)\frac{\partial \varphi}{\partial x_i}dx = \int_\Omega g_i(x)\varphi(x)dx, \qquad \forall \varphi \in \mathscr{D}(\Omega),\ \forall i \in \{1, \cdots n\}.$$

Sometimes, it is necessary to consider that φ and its partial derivatives belong to $L^{p'}(\Omega)$, from which u and its partial derivatives belong to $L^p(\Omega)$. Such a space satisfying the above properties is called a Sobolev space based on $L^p(\Omega)$. In general, we have:

5.1 Espace $W^{m,p}(\Omega)$

Definition 5.1 : *The Sobolev space of order* 1*, denoted as* $W^{1,p}(\Omega)$*, is defined as:*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega), \exists g_1, g_2, \cdots, g_n \in L^p(\Omega) : \int_\Omega u.\frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i\varphi; \forall \varphi \in \mathscr{D}(\Omega) \right\}.$$

In particular, we set $H^1(\Omega) = W^{1,2}(\Omega)$.

Remark 5.1 *:*

i) *The function g_i, if it exists, is unique. Indeed, suppose there exist two functions, $g_{1,i}$ and $g_{2,i}$, in $L^p(\Omega)$ that satisfy:*

$$\int_\Omega u.\frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_{1,i}\varphi = -\int_\Omega g_{2,i}\varphi; \quad \forall \varphi \in \mathscr{D}(\Omega).$$

Then:

$$\int_\Omega (g_{1,i} - g_{2,i})\varphi = 0; \quad \forall \varphi \in \mathscr{D}(\Omega).$$

According to Dubois-Reymond's Lemma (Theorem 2.1), $g_{1,i} = g_{2,i}$ a.e in Ω.

ii) *The function g_i is called the weak derivative of u in the direction i, and we have $\dfrac{\partial u}{\partial x_i} = g_i$.*

iii) *If $\dfrac{\partial u}{\partial x_i}$ exists in the usual sense and $\dfrac{\partial u}{\partial x_i} \in L^p(\Omega)$ then: $u \in W^{1,p}(\Omega)$.*

Remark 5.2 *: One can use a test function in $\mathscr{D}^1(\Omega)$ instead of a test function in $\mathscr{D}(\Omega)$ due to the density of $\mathscr{D}(\Omega)$ in $\mathscr{D}^1(\Omega)$.*

Example 5.1 *Let u be the function from $]-1,1[$ to $\mathbb{R}$, defined as: $u(x) = |x|$. We have:*

$$\int_{-1}^1 |u(x)|^p dx = \int_{-1}^1 |x|^p dx = 2\int_0^1 x^p dx = \frac{2}{p+1}.$$

Then: $u \in L^p(]-1,1[)$.
Let Now $\varphi \in \mathscr{D}(]-1,1[)$. Then:

$$\begin{aligned}
\int_{-1}^1 u(x)\varphi'(x)dx &= \int_{-1}^1 |x|\varphi'(x)dx \\
&= -\int_{-1}^0 x\varphi'(x)dx + \int_0^1 x\varphi'(x)dx \\
&= [-x\varphi'(x)]_{-1}^0 + \int_{-1}^0 \varphi(x)dx + [x\varphi'(x)]_0^1 - \int_0^1 x\varphi(x)dx \\
&= \int_{-1}^0 \varphi(x)dx - \int_0^1 \varphi(x)dx.
\end{aligned}$$

So, $u'(x) = \begin{cases} -1 & : \ x \in]-1,0[, \\ 1 & : \ x \in]0,1[. \end{cases}$

$$\int_{-1}^1 |u'(x)|^p dx = \int_{-1}^0 dx + \int_0^1 = 2.$$

Therefore: $u' \in L^p(]-1,1[)$. Hence: $u \in W^{1,p}(]-1,1[)$.

It is clear that $W^{1,p}(\Omega)$ is a sub-space of $L^p(\Omega)$.
We equip $W^{1,p}(\Omega)$ with the norm:

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}$$

or the equivalent norm:

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} = (\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{(L^p(\Omega))^n})^{\frac{1}{p}}.$$

We equip $H^1(\Omega)$ with the scalar product:

$$\begin{aligned}(u,v)_{H^1(\Omega)} &= (u,v)_{L^2(\Omega)} + \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i},\frac{\partial v}{\partial x_i}\right)_{L^2(\Omega)},\\ &= \int_\Omega u(x).v(x)dx + \sum_{i=1}^{n}\int_\Omega \frac{\partial u}{\partial x_i}(x).\frac{\partial v}{\partial x_i}(x)dx,\\ &= \int_\Omega u(x).v(x)dx + \int_\Omega \nabla u(x).\nabla v(x)dx.\end{aligned}$$

Theorem 5.1 : *$W^{1,p}(\Omega)$ is a Banach space and $H^1(\Omega)$ is a Hilbert space.*

Proof: Let $(u_j)_{j\in\mathbb{N}}$be a Cauchy sequence in dans $W^{1,p}(\Omega)$. Then, $(u_j)_{j\in\mathbb{N}}$, $\left(\frac{\partial u_j}{\partial x_i}\right)$ $(1 \le i \le n)$ are Cauchy sequence in $L^p(\Omega)$. Since $L^p(\Omega)$ is a Banach space, it follows that $(u_j)_{j\in\mathbb{N}}$ converges to $u \in L^p(\Omega)$ and $\left(\frac{\partial u_j}{\partial x_i}\right)$ converges to $g_i \in L^p(\Omega)$ for $1 \le i \le n$.
Now, let $\varphi \in \mathscr{D}(\Omega)$. Then, we have:

$$\int_\Omega u_j(x)\frac{\partial \varphi}{\partial x_i}(x)dx = -\int_\Omega \frac{\partial u_j}{\partial x_i}(x)\varphi(x)dx.$$

Taking the limit, we obtain:

$$\int_\Omega u(x)\frac{\partial \varphi}{\partial x_i}(x)dx = -\int_\Omega g_i(x)\varphi(x)dx.$$

Then: $u \in W^{1,p}(\Omega)$.
The case of $H^1(\Omega)$ is a particular case of this result. ■

Theorem 5.2 : *The space $W^{1,p}(\Omega)$ is separable for $1 \le p < +\infty$, reflexive for $1 < p < +\infty$.*

Proof: Consider the operator A de $W^{1,p}(\Omega)$ in $(L^p(\Omega))^{n+1}$, defined as:

$$\forall u \in W^{1,p}(\Omega) : Au = \left(u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right).$$

We equip $(L^p(\Omega))^{n+1}$ with the norm:

$$\|(u_0,\cdots,u_n)\|_{(L^p(\Omega))^{n+1}} = \sum_{i=0}^{n}\|u_i\|_{L^p(\Omega)}.$$

Then: for any $u \in W^{1,p}(\Omega)$ we have:

$$\|Au\|_{(L^p(\Omega))^{n+1}} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)} = \|u\|_{W^{1,p}(\Omega)}.$$

Then: the operator A is an isometric, so it is homeomorphism from $W^{1,p}(\Omega)$ to a closed part B of $(L^p(\Omega))^{n+1}$.
Since $L^p(\Omega)$ is separable for $1 \leq p < +\infty$, reflexive fo $1 < p < +\infty$, then $(L^p(\Omega))^{n+1}$ and its closed subsets share the same properties.
As a result, $W^{1,p}(\Omega)$ is separable for $1 \leq p < +\infty$ and reflexive for $1 < p < +\infty$. ■

Proposition 5.1 : *Let $u \in L^p(\Omega)$. The following properties are equivalents:*

1. $u \in W^{1,p}(\Omega)$.
2. $\exists c > 0 : \left|\int_\Omega u(x)\frac{\partial \varphi}{\partial x_i}(x)dx\right| \leq c\|\varphi\|_{L^{p'}(\Omega)}, \quad \forall \varphi \in \mathscr{D}(\Omega), \quad \forall i = 1, \dots, n.$

Proof:

⇒ Let $u \in W^{1,p}(\Omega)$ et $\varphi \in \mathscr{D}(\Omega)$. Then, $u \in L^p(\Omega), u \in L'^p(\Omega)$ and we have:

$$\begin{aligned}
\left|\int_\Omega u(x)\frac{\partial \varphi}{\partial x_i}(x)dx\right| &= \left|\int_\Omega \frac{\partial u}{\partial x_i}(x)\varphi(x)dx\right|, \\
&\leq \int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|.|\varphi(x)|dx, \\
&\leq \left(\int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|^p dx\right)^{\frac{1}{p}}.\left(\int_\Omega |\varphi(x)|^{p'}dx\right)^{\frac{1}{p'}}, \\
&= \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}.\|\varphi\|_{L^{p'}(\Omega)}, \\
&\leq c.\|\varphi\|_{L^{p'}(\Omega)},
\end{aligned}$$

where $c = \max_{1\leq i\leq n} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}$.

⇐ Let $u \in W^{1,p}(\Omega)$ be such that

$$\exists c > 0 : \left|\int_\Omega u(x)\frac{\partial \varphi}{\partial x_i}(x)dx\right| \leq c\|\varphi\|_{L^{p'}(\Omega)}, \quad \forall \varphi \in \mathscr{D}(\Omega), \quad \forall i = 1, \dots, n.$$

Then, the operator $A_i : \mathscr{D}(\Omega) \to L^{p'}(\Omega)$ defined as: $A_i\varphi = \int_\Omega u(x)\frac{\partial \varphi}{\partial x_i}(x)dx$ est is continuous, and since $\mathscr{D}(\Omega)$ is dense in $L^{p'}(\Omega)$ we can extend the operator A to $L^{p'}(\Omega)$. From the Riez's theorem of representation (Theorem 1.12) there exists $g_i \in L^p(\Omega)$ such that

$$A_i\varphi = -\int_\Omega g_i(x)\varphi(x)dx,$$

i.e $\int_\Omega u(x)\frac{\partial \varphi}{\partial x_i}(x)dx = -\int_\Omega g_i(x)\varphi(x)dx$. Then: $u \in W^{1,p}(\Omega)$.

■

Theorem 5.3 : *Suppose that Ω is bounded, Lipschitz (or $\Omega = \mathbb{R}^n_+$). Then, for any $u \in W^{1,p}(\Omega)$, there exists $U \in W^{1,p}(\mathbb{R}^n)$, and a constant $c = c(\Omega) > 0$ such that:*

i) $U_{|\Omega} = u$,

ii) $\|U\|_{L^p(\mathbb{R}^n)} \leq c\|u\|_{L^p(\Omega)}$,

iii) $\|U\|_{W^{1,p}(\mathbb{R}^n)} \leq c\|u\|_{W^{1,p}(\Omega)}$.

Definition 5.2 : *Let $m \in \mathbb{N}$ (with $m \geq 2$). The Sobolev space $W^{m,p}(\Omega)$ of order m is defined as follows:*

$$W^{m,p}(\Omega) = \left\{ u \in W^{m-1,p}(\Omega), \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\Omega), \forall i = 1, 2, \ldots, n \right\}.$$

In other words:

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega), \forall \alpha \in \mathbb{N}^n (|\alpha| \leq m), \exists g_\alpha \in L^p(\Omega) : \int_\Omega u.D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g_\alpha \varphi; \forall \varphi \in \mathscr{D}(\Omega) \right\}.$$

In particular, we set $H^m(\Omega) = W^{m,2}(\Omega)$.

We equip $W^{m,p}(\Omega)$ with the norm:

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(\Omega)}$$

$H^m(\Omega)$ is a Hilbert space, equipped with the scalar product:

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^p(\Omega)}.$$

Using similar arguments as in Theorem 5.1 and Theorem 5.2, we can obtain the following two theorems:

Theorem 5.4 : *$W^{m,p}(\Omega)$ is a Banach spac and $H^m(\Omega)$ is a Hilbert space.*

Theorem 5.5 : *$W^{m,p}(\Omega)$ is a separable space for $1 \leq p < +\infty$ and reflexive space for $1 < p < +\infty$.*

The following lemma is important to proof the density of test functions space in certain Sobolev spaces:

Lemma 5.1 *Let $f \in L^1(\mathbb{R}^n)$ and $u \in W^{1,p}(\mathbb{R}^n)$.*
Then: $f * u \in W^{1,p}(\mathbb{R}^n)$ and for any $i \in \{1, \cdots, n\}$ we have: $\frac{\partial}{\partial x_i}(f * u) = f * \frac{\partial u}{\partial x_i}$.

Proof: Let's first assume that f has compact support. In this case, $(fu) \in L^p(\mathbb{R}^n)$, and for all $\varphi \in \mathscr{D}(\mathbb{R}^n)$, we have:

$$\begin{aligned}
\int_{\mathbb{R}^n} (f*u)(x)\frac{\partial \varphi}{\partial x_i}(x)dx &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)u(y)\frac{\partial \varphi}{\partial x_i}(x)dxdy \\
&= \int_{\mathbb{R}^n} u(y)dy \int_{\mathbb{R}^n} f(x-y)\frac{\partial \varphi}{\partial x_i}(x)dx \\
&= \int_{\mathbb{R}^n} u(y)dy \int_{\mathbb{R}^n} \check{f}(y-x)\frac{\partial \varphi}{\partial x_i}(x)dx \\
&= \int_{\mathbb{R}^n} u(x)\left(\check{f}*\frac{\partial \varphi}{\partial x_i}\right)(x)dx \\
&= \int_{\mathbb{R}^n} u(x)\frac{\partial}{\partial x_i}(\check{f}*\varphi)(x)dx \\
&= -\int_{\mathbb{R}^n} \frac{\partial u}{\partial x_i}(x)(\check{f}*\varphi)(x)dx \\
&= -\int_{\mathbb{R}^n} \left(f*\frac{\partial u}{\partial x_i}\right)(x)\varphi(x)dx.
\end{aligned}$$

Now, let's consider the case where f does not have compact support. There exists a sequence $\rho_j j = 1^{+\infty}$ in $\mathscr{D}(\mathbb{R}^n)$ converging to f in $L^1(\mathbb{R}^n)$. We then have:

$$\int_{\mathbb{R}^n} (\rho_j*u)(x)\frac{\partial \varphi}{\partial x_i}(x)dx = -\int_{\mathbb{R}^n}\left(\rho_j*\frac{\partial u}{\partial x_i}\right)(x)\varphi(x)dx. \tag{5.3}$$

Also, we have:

$$\rho_j*u \longrightarrow f*u \text{ in } L^p(\mathbb{R}^n) \qquad \rho_j*\frac{\partial u}{\partial x_i} \longrightarrow f*\frac{\partial u}{\partial x_i} \text{ in } L^p(\mathbb{R}^n).$$

Using the Lebesgue Dominated Convergence Theorem (Theorem 1.13), we obtain:

$$\int_{\mathbb{R}^n} (f*u)(x)\frac{\partial \varphi}{\partial x_i}(x)dx = -\int_{\mathbb{R}^n}\left(f*\frac{\partial u}{\partial x_i}\right)(x)\varphi(x)dx. \tag{5.4}$$

Hence, we get the result. ■

Theorem 5.6 : *$\mathscr{D}(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$.*

Proof: Let $u \in W^{1,p}(\mathbb{R}^n)$ and the function $\chi \in \mathscr{D}(\mathbb{R}^n)$ such that $0 \leq \chi \leq 1$, $\operatorname{supp}\chi \subset B(0,2)$ and $\chi = 1$ on $B(0,1)$. Consider the sequence $\{\chi_j\}_{j=1}^{+\infty}$ defined as: $\chi_j(x) = \chi\left(\frac{x}{j}\right)$. Then: $\chi_j.u$ converges to u a.e. and $|\chi_j.u| \leq |u|$ for all j. From the Lebesgue dominated convergence theorem (Theorem 1.13), the sequence $\{\chi_j.u\}_{j=1}^{+\infty}$ converges to u in $L^p(\mathbb{R}^n)$. Let $\{\rho_j.u\}_{j=1}^{+\infty}$ be a regularization sequence as in Definition 1.24. Set $\varphi_j = \chi_j.(\rho_i*u)$. Then: $\varphi_j \in \mathscr{D}(\mathbb{R}^n)$ and we have:

$$\varphi_j - u = \chi_j.[(\rho_j*u)-u] + [\chi_j.u-u].$$

Knowing that:

$$\begin{aligned}
\|\chi_j.[(\rho_j * u) - u]\|_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\chi_j.[(\rho_j * u) - u]|^p dx \\
&\leq \int_{\mathbb{R}^n} |(\rho_j * u) - u|^p dx \\
&= \|(\rho_j * u) - u\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

we deduce that:

$$\|\varphi_j - u\|_{L^p(\mathbb{R}^n)} \leq \|(\rho_j * u) - u\|_{L^p(\mathbb{R}^n)} + \|\chi_j.u - u\|_{L^p(\mathbb{R}^n)} \longrightarrow 0.$$

Using Lemma 5.1 we get:

$$\frac{\partial \varphi_j}{\partial x_i} = \frac{\partial \chi_j}{\partial x_i}.(\rho_j * u) + \chi_j \left(\rho_j * \frac{\partial u}{\partial x_i}\right).$$

Then:

$$\frac{\partial \varphi_j}{\partial x_i} - \frac{\partial u}{\partial x_i} = \frac{\partial \chi_j}{\partial x_i}.(\rho_j * u) + \chi_j \left[\left(\rho_j * \frac{\partial u}{\partial x_i}\right) - \frac{\partial u}{\partial x_i}\right] + \left[\chi_j.\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial x_i}\right].$$

Noting that:

$$\begin{aligned}
\left\|\frac{\partial \chi_j}{\partial x_i}.(\rho_j * u)\right\|_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left|\frac{\partial \chi_j}{\partial x_i}.(\rho_j * u)\right|^p dx \\
&\leq \left\|\frac{\partial \chi_j}{\partial x_i}\right\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\rho_j * u|^p dx \\
&\leq \left\|\frac{\partial \chi_j}{\partial x_i}\right\|_{L^\infty(\mathbb{R}^n)} \|\rho_j\|_{L^1(\mathbb{R}^n)} \|u\|_{L^p(\mathbb{R}^n)}, \\
&= \frac{1}{j} \left\|\frac{\partial \chi}{\partial x_i}\right\|_{L^\infty(\mathbb{R}^n)} \|u\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

Then:

$$\begin{aligned}
\left\|\frac{\partial \varphi_j}{\partial x_i} - \frac{\partial u}{\partial x_i}\right\|_{L^p(\mathbb{R}^n)} &\leq \left\|\frac{\partial \chi_j}{\partial x_i}.(\rho_j * u)\right\|_{L^p(\mathbb{R}^n)} + \left\|\left(\rho_j * \frac{\partial u}{\partial x_i}\right) - \frac{\partial u}{\partial x_i}\right\|_{L^p(\mathbb{R}^n)} + \left\|\chi_j.\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial x_i}\right\|_{L^p(\mathbb{R}^n)} \\
&\leq \frac{1}{j} \left\|\frac{\partial \chi}{\partial x_i}\right\|_{L^\infty(\mathbb{R}^n)} \|u\|_{L^p(\mathbb{R}^n)} + \left\|\left(\rho_j * \frac{\partial u}{\partial x_i}\right) - \frac{\partial u}{\partial x_i}\right\|_{L^p(\mathbb{R}^n)} + \left\|\chi_j.\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial x_i}\right\|_{L^p(\mathbb{R}^n)} \\
&\overset{j\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

■

Using similar arguments as above, we can present other density results:

Theorem 5.7 : *$\mathscr{D}(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$.*

Theorem 5.8 : *$\mathscr{C}^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.*

Remark 5.3 : *If Ω is bounded and of class $\mathscr{C}^m$, then:*

i) $\mathscr{D}(\overline{\Omega})$ *is dense in* $W^{m,p}(\Omega)$.

ii) *For all* $k \geq m$, *the space* $\mathscr{D}^k(\overline{\Omega})$ *is dense in* $W^{m,p}(\Omega)$. *In particular,* $\mathscr{D}^k(\mathbb{R}^n)$ *is dense in* $W^{m,p}(\mathbb{R}^n)$ *for all* $k \geq m$.

5.2 Inequalities and Sobolev embeddings

First, we have the following lemma:

Lemma 5.2 *: Suppose that* $n \geq 2$ *and let* $f_1, \cdots, f_n \in L^{n-1}(\mathbb{R}^{n-1})$. *For all* $x \in \mathbb{R}^n$ *and* $i \in \{1, \cdots, n\}$ *we set:* $\hat{x}_i = (x_1, \cdots, x_i, x_{i+1}, \cdots, x_n)$ *and let* f *be the function defined as:* $f(x) = f_1(\hat{x}_1).f_2(\hat{x}_2)\cdots f_n(\hat{x}_n)$. *Then:* $f \in L^1(\mathbb{R}^n)$ *and we have:*

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \prod_{i=1}^{n} \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$

Theorem 5.9 (**Gagliardo – Nirenberg – Sobolev**) : *Assume that* $n \geq 2$ *and* $1 \leq p \leq n$. *Given* p^* *such that* $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}$ *(Sobolev exponent), then:* $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$, *and there exists a constant* $c = c(p,n) > 0$ *such that:*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq c\|\nabla u\|_{L^p(\mathbb{R}^n)}, \forall u \in W^{1,p}(\mathbb{R}^n)$$

Proof: Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Then, we have:

$$|\varphi(x)| = \left| \int_{-\infty}^{x_i} \frac{\partial \varphi}{\partial x_i}(x_1, \cdots, x_i, t, x_{i+1}, \cdots, x_n) dt \right| \leq \int_{-\infty}^{+\infty} \left| \frac{\partial \varphi}{\partial x_i}(x_1, \cdots, x_i, t, x_{i+1}, \cdots, x_n) \right| dt$$

Set: $f_i(\hat{x}_i) = \displaystyle\int_{-\infty}^{+\infty} \left| \frac{\partial \varphi}{\partial x_i}(x_1, \cdots, x_i, t, x_{i+1}, \cdots, x_n) \right| dt.$

Then: $|\varphi(x)|^n \leq \displaystyle\prod_{i=1}^{n} f_i(\hat{x}_i)$, which leads to: $|\varphi(x)|^{\frac{n}{n-1}} \leq \displaystyle\prod_{i=1}^{n} f_i^{\frac{1}{n-1}}(\hat{x}_i)$.

From Lemma 5.2, on obtain:

$$\begin{aligned}
\int_{\mathbb{R}^n} |\varphi(x)|^{\frac{n}{n-1}} dx &\leq \prod_{i=1}^{n} \|f_i^{\frac{1}{n-1}}\|_{L^{n-1}(\mathbb{R}^{n-1})}, \\
&= \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} f_i dx \right)^{\frac{1}{n-1}}, \\
&= \prod_{i=1}^{n} \|f_i\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}}, \\
&= \prod_{i=1}^{n} \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n-1}}.
\end{aligned}$$

Then:

$$\|\varphi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \prod_{i=1}^{n} \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n}}. \tag{5.5}$$

For $t \geq 1$ we replace φ by $|\varphi|^{t-1}.\varphi$, so we obtain: $\left|\dfrac{\partial(|\varphi|^{t-1}.\varphi)}{\partial x_i}\right| = t|\varphi|^{t-1}\dfrac{\partial \varphi}{\partial x_i}$. Then:

$$\||\varphi|^{t-1}.\varphi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq t \prod_{i=1}^{n} \left\| |\varphi|^{t-1}. \frac{\partial \varphi}{\partial x_i} \right\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n}}.$$

Note that:

$$\||\varphi|^{t-1}.\varphi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\varphi|^{\frac{tn}{n-1}} dx\right)^{\frac{n-1}{n}} = \|\varphi\|^t_{L^{\frac{tn}{n-1}}(\mathbb{R}^n)},$$

$$\left\| |\varphi|^{t-1}. \frac{\partial \varphi}{\partial x_i} \right\|_{L^1(\mathbb{R}^n)} \leq \||\varphi|^{t-1}\|_{L^{p'}(\mathbb{R}^n)}. \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^p(\mathbb{R}^n)} = \|\varphi\|^{t-1}_{L^{p'(t-1)}(\mathbb{R}^n)}. \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^p(\mathbb{R}^n)}.$$

Therefore:

$$\|\varphi\|^t_{L^{\frac{tn}{n-1}}(\mathbb{R}^n)} \leq t\|\varphi\|^{t-1}_{L^{p'(t-1)}(\mathbb{R}^n)}. \prod_{i=1}^{n} \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^p(\mathbb{R}^n)}^{\frac{1}{n}}.$$

Taking $t = \dfrac{n-1}{n}p^* = \dfrac{(n-1)p}{n-p}$. Then: $\dfrac{tn}{n-1} = p'(t-1) = p^*$, which leads to

$$\|\varphi\|^t_{L^{p^*}(\mathbb{R}^n)} \leq \frac{(n-1)p}{np} \|\varphi\|^{t-1}_{L^{p^*}(\mathbb{R}^n)}. \prod_{i=1}^{n} \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^p(\mathbb{R}^n)}^{\frac{1}{n}}.$$

Then:

$$\|\varphi\|_{L^{p^*}(\mathbb{R}^n)} \leq \frac{(n-1)p}{np}. \prod_{i=1}^{n} \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^p(\mathbb{R}^n)}^{\frac{1}{n}} \leq c \prod_{i=1}^{n} \left\| \frac{\partial \varphi}{\partial x_i} \right\|_{L^p(\mathbb{R}^n)}.$$

Hence:

$$\|\varphi\|_{L^{p^*}(\mathbb{R}^n)} \leq c\|\nabla u\|_{L^p(\mathbb{R}^n)}, \forall \varphi \in \mathscr{D}(\mathbb{R}^n).$$

From the density of $\mathscr{D}(\mathbb{R}^n)$ in $W^{1,p}(\mathbb{R}^n)$ we obtain the result. ■

Corollary 5.1 :

i) *For $n \geq 2$ et $1 \leq p < n$ we have:*

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \qquad \forall q \in [p, p^*].$$

ii) *For $n \geq 2$ we have:*

$$W^{1,n}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \qquad \forall q \in [n, +\infty[.$$

Theorem 5.10 (**Morry**) : *Let $p > n$. Then:*

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n).$$

Moreover, for any $u \in W^{1,p}(\mathbb{R}^n)$ we have:

$$|u(x) - u(y)| \leq c(n,p)|x-y|^{\frac{p-n}{p}}, \qquad a.e\ x, y \in \mathbb{R}^n.$$

The previous results remain valid for sufficiently regular open sets (see [1, 5]):

Theorem 5.11 : *Assume that Ω is of class $\mathscr{C}^1$ with $\Gamma = \partial\Omega$ is bounded (or $\Omega = \mathbb{R}^n_+$), and $1 \leq p \leq +\infty$. Then:*

1. *If $1 \leq p < n$, we have: $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.*
2. *If $p = n$, we have: $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \forall q \in [p, +\infty[$.*
3. *If $p > n$ we have: $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. Moreover, for any $u \in W^{1,p}(\Omega)$ we have:*

$$|u(x) - u(y)| \leq c(n,p)|x-y|^{\frac{p-n}{p}}, \qquad a.e\ x, y \in \Omega.$$

In particular: $W^{1,p}(\Omega) \hookrightarrow \mathscr{C}(\overline{\Omega})$.

Theorem 5.12 : *Let $m \in \mathbb{N}^*$ and $1 \leq p \leq +\infty$.*

1. *If $1 - \frac{m}{n} > 0$, then: $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, where $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$.*
2. *If $1 - \frac{m}{n} = 0$, then: $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, for any $q \in [p, +\infty[$.*
3. *If $1 - \frac{m}{n} < 0$, then: $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$.*

The following theorem provides a more precise result:

Theorem 5.13 (**Rellich – Kondrachov**) : *Suppose that Ω is bounded and of class $\mathscr{C}^1$. Then:*

1. *If $p < n$, then: $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, for any $q \in [1, p^*[$.*
2. *If $p = n$, then: $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, for any $q \in [1, +\infty[$.*
3. *If $p > n$ then: $W^{1,p}(\Omega) \hookrightarrow \mathscr{C}(\overline{\Omega})$.*

These embeddings are compact.

Remark 5.4 *: For $n = 1$ and $\Omega = I$ be an interval we have the following properties:*

i) *For any $u \in W^{1,p}(I)$, there exists $\tilde{u} \in \mathscr{C}(\bar{I})$ such that $u = \tilde{u}$ a.e in I and*

$$\tilde{u}(x) - \tilde{u}(y) = \int_x^y u'(t)dt, \quad \forall x, y \in \bar{I}.$$

ii) *For a function* $u \in L^\infty(I)$ *to be in* $W^{1,\infty}(I)$*, it is necessary and sufficient that there exists* $c > 0$ *such that:*

$$u(x) - u(y) \leq c|x - y|, \quad p.p.\ x, y \in I.$$

iii) *If* I *is bounded then:*
**)* $W^{1,p}(I) \hookrightarrow L^\infty(\Omega),\ \forall\ 1 \leq p \leq +\infty.$
***)* $W^{1,p}(I) \hookrightarrow C(\overline{I}),\ \forall\ 1 < p \leq +\infty$ *with compactness.*
****) The embedding* $W^{1,1}(I) \hookrightarrow \mathscr{C}(\overline{I})$ *is continuous but n0t pas compact.*
*****) The embedding* $W^{1,1}(I) \hookrightarrow L^q(I)$ *is compact for any* $1 \leq q \leq +\infty$.

5.3 The space $W_0^{1,p}(\Omega)$

Suppose that $1 \leq p < +\infty$.

Definition 5.3 : *The space* $W_0^{1,p}(\Omega)$ *is the closure of* $\mathscr{D}(\Omega)$ *dans* $W^{1,p}(\Omega)$.
In particular $H_0^1(\Omega) = W_0^{1,p}(\Omega)$.

Remark 5.5 : *From the density of* $\mathscr{D}(\mathbb{R}^n)$ *in* $W^{1,p}(\mathbb{R}^n)$*, we deduce that:* $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$.

Theorem 5.14 : *Suppose that* Ω *is of class* $\mathscr{C}^1$*, and let* $u \in L^p(\Omega)$ $(1 < p < +\infty)$*. Then, the following properties are equivalents:*

i) $u = 0$ *on* Γ,

ii) $u \in W_0^{1,p}(\Omega)$.

Here is another characterization of the space $W_0^{1,p}(\Omega)$:

Theorem 5.15 : *Suppose that* Ω *is of class* $\mathscr{C}^1$*, and let* $u \in W^{1,p}(\Omega) \cap \mathscr{C}(\overline{\Omega})$*. Then, the following properties are equivalents:*

i) $u \in W_0^{1,p}(\Omega)$,

ii) *there exists* $c > 0$ *such that for any* $i \in \{1, \cdots n\}$ *and for any* $\varphi \in \mathscr{D}(\mathbb{R}^n)$ *we have:*

$$\left| \int_\Omega u \frac{\partial \varphi}{\partial x_i} dx \right| \leq c.\|\varphi\|_{L^{p'}(\Omega)}.$$

iii) *The function* $\widetilde{u}$*, defined as:* $\widetilde{u}(x) = \begin{cases} u(x) & : \ x \in \Omega, \\ 0 & : \ x \notin \Omega, \end{cases}$ *belongs to* $W^{1,p}(\mathbb{R}^n)$.

In this case we have: $\dfrac{\partial \widetilde{u}}{\partial x_i} = \widetilde{\left(\dfrac{\partial u}{\partial x_i}\right)}$.

Theorem 5.16 (***Poincaré's inequality***) : *Suppose that Ω is bounded in a direction, i.e there exists $i \in \{1, \cdots, n\}, a_i, b_i \in \mathbb{R}$ such that: $a \leq x_i \leq b, \ \forall x \in \Omega$. Then, there exists a constant $c = c(\Omega, p)$ such that:*

$$\|u\|_{L^p(\Omega)} \leq c\|\nabla u\|_{L^p(\Omega)}, \forall u \in W_0^{1,p}(\Omega)$$

Proof: From the density of $\mathscr{D}(\Omega)$ in $W_0^{1,p}(\Omega)$, it suffices to prove this theorem for functions in $\mathscr{D}(\Omega)$. So let $\varphi \in \mathscr{D}(\Omega)$. Then, we have:

$$\begin{aligned}
|\varphi(x)| &= \left|\int_a^{x_i} \frac{\partial \varphi}{\partial x_i}(x_1, \cdots, x_{i-1}, t, x_{i-1}, \cdots, x_n)dt\right|, \\
&\leq \int_a^b \left|\frac{\partial \varphi}{\partial x_i}(x)dx_i\right|, \\
&\leq \left(\int_a^b 1.dx_i\right)^{\frac{1}{p'}} \left(\int_a^b \left|\frac{\partial \varphi}{\partial x_i}(x)dx_i\right|^p\right)^{\frac{1}{p}}, \\
&= (b-a)^{\frac{p-1}{p}} \left(\int_a^b \left|\frac{\partial \varphi}{\partial x_i}(x)dx_i\right|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\|\varphi\|^p_{L^p(\Omega)} &= \int_{\mathbb{R}^n} |\varphi(x)|^p dx \\
&\leq (b-a)^{p-1} \int_\Omega \int_a^b \left|\frac{\partial \varphi}{\partial x_i}(x)\right|^p dx_i dx, \\
&= (b-a)^{p-1} \int_a^b \int_\Omega \left|\frac{\partial \varphi}{\partial x_i}(x)\right|^p dx dx_i, \\
&\leq (b-a)^p \left\|\frac{\partial \varphi}{\partial x_i}\right\|^p_{L^p(\Omega)}, \\
&\leq (b-a)^p \|\nabla \varphi\|^p_{L^p(\Omega)}.
\end{aligned}$$

Hence, the result follows by density. ■

Corollary 5.2 : *For all $u \in W_0^{1,p}(\Omega)$, we have: $\|\nabla u\|_{L^p(\Omega)} \leq \|u\|_{W^{1,p}(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$. Hence, we can consider $\|\nabla u\|_{L^p(\Omega)}$ as a norm on $W_0^{1,p}(\Omega)$ that is equivalent to $\|u\|_{W^{1,p}(\Omega)}$.*

here is another version of the Poincaré inequality:

Theorem 5.17 (***Poincaré-Wirtinger inequality***) : *Suppose that Ω is connected, of class $\mathscr{C}^1$ and with bounded measure $|\Omega|$. set: $u_\Omega = \frac{1}{|\Omega|}\int_\Omega u(x)dx$. Then, there exists $c > 0$ such that:*

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq c\|\nabla u\|_{L^p(\Omega)}, \forall u \in W^{1,p}(\Omega).$$

5.4 The space $W^{-1,p'}(\Omega)$

Definition 5.4 : *We denote by $W^{-1,p'}(\Omega)$ the dual space of $W_0^{1,p}(\Omega)$ and by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$.*

Proposition 5.2 : *We have*

1. $H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ *with density.*
2. *If* Ω *is bounded and* $\dfrac{2n}{n+2} \leq p < +\infty$ *then:* $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow W^{-1,p}(\Omega)$*, with density.*
3. *Si* Ω *is not bounded and* $\dfrac{2n}{n+2} \leq p \leq 2$ *then:* $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow W^{-1,p}(\Omega)$.

We have the following characterization of elements in $W^{-1,p'}(\Omega)$:

Theorem 5.18 : *Let* $f \in W^{-1,p'}(\Omega)$*. Then, there exists* $G_0 \in L^{p'}(\Omega), G = (G_1, G_2, \dots, G_n) \in (L^{p'}(\Omega))^n$ *such that:*

$$\langle f, v\rangle = \int_\Omega G_0.u + \int_\Omega G.\nabla u, \qquad \forall u \in W_0^{1,p}(\Omega)$$

et $\max\limits_{0\leq i\leq n} \|G_i\|_{L^{p'}(\Omega)} = \|f\|$.
If Ω *is bounded, we can take* $G_0 = 0$.

Proof: Consider the space $E = (L^p(\Omega))^{n+1}$, equipped with the norm:

$$\|V\|_{(L^p(\Omega))^{n+1}} = \|(v_0, v_1, \cdots v_n)\|_{(L^p(\Omega))^{n+1}} = \sum_{i=0}^{n} \|v_i\|_{L^p(\Omega)}.$$

The operator A from $W_0^{1,p}(\Omega)$ to $(L^p(\Omega))^{n+1}$, defied as:

$$\forall u \in W_0^{1,p}(\Omega) : Au = \left(u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right).$$

is an isometric (see proof de Theorem 5.2). set $F = A^{-1}(W_0^{1,p}(\Omega))$, we equip F with the induced norm from E.
Let the linear continuous mapping ℓ defied on F by: $\ell(v) = \langle f, A^{-1}v\rangle$. From Hahn-Banach theorem de (Corollary 1.1), we can extend $\ell$to a linear continuous mapping L defined on E with: $\|L\|_{E'} = \|f\|$.
The Riesz representation Theorem (Theorem 1.12) allows us to write:

$$\langle L, h\rangle = \sum_{i=0}^{n} \int_\Omega G_i.v_i, \qquad \forall v_i \in E.$$

aking into account $v_0 = u_0$ and $v_i = \dfrac{\partial u_i}{\partial x_i}$ (for $1 \leq i \leq n$), we obtain the result.
For a bounded Ω, using the norm $\|\nabla u\|_{L^p(\Omega)}$ on $W_0^{1,p}(\Omega)$, we can take $G_0 = 0$. ■

5.5 Sobolev spaces with fractional order, trace theorem and Green's formula

In this section, we provide a brief overview of Sobolev spaces of fractional order and the trace theorem. First, we have the following theory, which demonstrates the relationship between the spaces $H^m(\mathbb{R}^n)$ and the space of tempered distributions.

Theorem 5.19 *: $H^m(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$. Moreover, we have:*

$$H^m(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : (1 + |\xi|^2)^{\frac{m}{2}} \widehat{u} \in L^2(\mathbb{R}^n), \xi \in \mathbb{R}^n\}.$$

This fundamental property has been leveraged to extend the concept of Sobolev spaces of integer order to more general spaces known as Sobolev spaces with fractional order, which are introduced in the following definition:

Definition 5.5 *: Let $s \in \mathbb{R}$. We define the space $H^s(\mathbb{R}^n)$ as follows:*

$$H^s(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : (1 + |\xi|^2)^{\frac{s}{2}} \widehat{u} \in L^2(\mathbb{R}^n), \xi \in \mathbb{R}^n\}.$$

Generally, we have the following definition:

Definition 5.6 *: Let $0 < s < 1$ et $p[1, +\infty[$. We define l'espace $W^{s,p}(\Omega)$ as follows:*

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{s + \frac{n}{p}}} \in L^p(\Omega \times \Omega).$$

Si $s > 1$, by writing $s = m + r$ où $m \in \mathbb{N}$ et $0 < r < 1$ we define $W^{s,p}(\Omega)$ as follows:

$$W^{s,p}(\Omega) = \{u \in W^{m,p}(\Omega) : D^\alpha u \in W^{s,p}(\Omega), \ \forall \alpha \in \mathbb{N}^n, |\alpha| = 1\}.$$

Theorem 5.20 (**trace**) : *Suppose that Ω is of class $\mathscr{C}^1$. Then:*

The map $\gamma_0 : \mathscr{D}(\overline{\Omega}) \to C(\Gamma)$, defined as $\gamma_0 v = v_{|\Gamma}$, can be continuously extended to a continuous linear map from $H^1(\Omega)$ into $L^2(\Gamma)$, also denoted as γ_0.

γ_0 is called the trace map, and $\gamma_0 v$ is called the trace of v on Γ.

The map γ is surjective from $H^1(\Omega)$ into $H^{\frac{1}{2}}(\Gamma)$.

In general, we can define the trace map γ_0 from $W^{1,p}(\Omega)$ into $L^p(\Gamma)$. This map is surjective from $W^{1,p}(\Omega)$ into $W^{1-\frac{1}{p},p}(\Gamma)$.

Note that $W^{1-\frac{1}{p},p}(\Gamma)$ is a Sobolev space defined on the submanifold Γ of dimension $n-1$, using a specified coordinate system.

Remark 5.6 *: $W_0^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega) : \gamma_0 v = v_{|\Gamma} = 0\}$.*

An important result of the trace theorem is the following:

Theorem 5.21 (***Green's formua***) : *Suppose that Ω is bounded, of class $\mathscr{C}^1$ by pieces and let ν be the outward normal vector of Γ. Then:*

i) *For all $u, v \in H^1(\Omega)$ we have:*

$$\int_\Omega \frac{\partial u}{\partial x_i}.vdx = -\int_\Omega u.\frac{\partial v}{\partial x_i} + \int_\Gamma u.v.\nu_i d\sigma(x).$$

ii) *For all $u, v \in W^{2,p}(\Omega)$ we have:*

$$\int_\Omega -\Delta u.vdx = \int_\Omega \nabla u.\nabla vdx - \int_\Gamma \frac{\partial u}{\partial \nu}.v\sigma(x).$$

Exercises

Exercise 5.1 : *Let $p \in [1, +\infty[$, H be the Heaviside function, and let $\psi \in \mathscr{D}(\mathbb{R})$.*
Determine the conditions on ψ for $H \cdot \psi \in W^{1,p}(\mathbb{R})$

Exercise 5.2 : *Let u be the function defined on $]-1, 1[$ as: $u(x) = \dfrac{x + |x|}{2}$*

1. *Show that $u \in H^1(]-1, 1[)$.*
2. *Is $u \in H^2(]-1, 1[)$?*

Exercise 5.3 : *Let $p \in [1, +\infty[$ and let f be the function defined as:*

$$f(x) = \begin{cases} x & : \ x \in [0,1], \\ -x+2 & : \ x \in [1,2], \\ 0 & : \ x \notin [1,2]. \end{cases}$$

1. *show that $f \in W^{1,p}(\mathbb{R})$.*
2. *Is $f \in W^{2,p}(\mathbb{R})$?*

Exercise 5.4 : *Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, and u be the function defined on $\Omega \setminus \{(0, 0)\}$ as:*

$$u(x,y) = \left| \ln \frac{\sqrt{x^2+y^2}}{2} \right|^\alpha$$

where $0 < \alpha < \dfrac{1}{2}$.
Show that $u \in H^1(\Omega)$ but does not have a continuous representation on Ω.

Exercise 5.5 : *Let B be the unit ball in $\mathbb{R}^n$, and let u be the function defined on $B \setminus 0$ as:*

$$u(x) = |x|^\alpha.$$

Investigate the membership of u in $H^1(B)$.

Exercise 5.6 : *Let* $p \in [1, +\infty[$ *and* f *be the function defined from* $]0, 1[$ *to* $\mathbb{R}$ *as follow:*

$$\forall x \in]0, 1[: f(x) = x^{-\frac{1}{p+1}}.$$

1. *Show that* $f \in L^p(]0, 1[)$.
2. *Find the function* g *such that:* $\forall \varphi \in \mathscr{D}(]0, 1[) : \int_0^1 f(x)\varphi'(x)dx = -\int_0^1 g(x)\varphi(x)dx.$
3. *Is* $f \in W^{1,p}(]0, 1[)$?
4. *Let* $u \in W_0^{1,p}(]0, 1[)$ *and* $\{\varphi_n\}_{n=1}^{\infty}$ *be a sequence belongs to* $\mathscr{D}(]0, 1[)$, *converges to* v *in* $W_0^{1,p}(]0, 1[)$ (*i.e* $\lim_{n \to +\infty} \|\varphi'_n - u'\|_{L^p(]0,1[)} = 0$).
   Show that: $\forall n \in \mathbb{N} : \left| \int_0^1 f(x)\varphi_n(x)dx \right| \leq \frac{p+1}{p} \|\varphi'_n\|_{L^p(]0,1[)}.$
5. *Conclude.*

Exercise 5.7 : *let* $\delta : \mathscr{D}(] - 1, 1[) \to \mathbb{R}$, *defined as:* $< \delta, \varphi >= \varphi(0)$.

1. *show that* $\delta \in H^{-1}(] - 1, 1[)$.
2. *Find* $u_0 \in H_0^1(] - 1, 1[)$ *solution of the equation:*

$$-T'' = \delta \text{ in } \mathscr{D}'(] - 1, 1[).$$

3. *Show that this solution is unique.*

Solutions of exercises

Solution 5.1 : $p \in [1, +\infty[, H$ *the Heaviside function,* $\psi \in \mathscr{D}(\mathbb{R})$. *We have:*

$$\int_{-\infty}^{+\infty} |H(x).\psi(x)|^p dx = \int_0^{+\infty} |\psi(x)|^p dx = |\psi|^p_{L^p(]0,+\infty[)} < +\infty.$$

Then: $H.\psi \in L^p(\mathbb{R})$.
Now, let $\phi \in \mathscr{D}(\mathbb{R})$. *Then:*

$$\begin{aligned} \int_{-\infty}^{+\infty} H(x).\psi(x)\varphi'(x)dx &= \int_0^{+\infty} \psi(x)\varphi'(x)dx \\ &= [\psi(x)\varphi(x)]_0^{+\infty} - \int_0^{+\infty} \psi'(x)\varphi(x)dx \\ &= -\psi(0)\varphi(0) - \int_0^{+\infty} \psi'(x)\varphi(x)dx \\ &= -\langle \psi(0)\delta + \psi', \varphi \rangle. \end{aligned}$$

For $H \cdot \psi \in W^{1,p}(\mathbb{R})$, *it is necessary that* $\psi(0) = 0$.

Solution 5.2 : $x \in]-1,1[, u(x) = \dfrac{x+|x|}{2} = \begin{cases} 0 & : & x \in]-1,0], \\ x & : & x \in]0,1[\end{cases}$

1. *We have:* $\displaystyle\int_{-1}^{1} u^2(x)dx = \int_0^1 x^2 dx = \frac{1}{3}$. *Then:* $u \in L^2(]-1,1[)$.
   Let $\varphi \in \mathscr{D}(]-1,1[)$. *Then:*

$$\begin{aligned} \int_{-1}^{1} u(x)\varphi'(x)dx &= \int_0^1 x\varphi'(x)dx \\ &= [x\varphi(x)]_0^1 - \int_0^1 \varphi(x)dx, \\ &= -\int_0^1 \varphi(x)dx. \end{aligned}$$

   Therefore: $u'(x) = \begin{cases} 0 & : & x \in]-1,0], \\ 1 & : & x \in]0,1[\end{cases}$
   $\displaystyle\int_{-1}^{1} u'^2(x)dx = \int_0^1 dx = \frac{1}{2}$. *So,* $u' \in L^2(]-1,1[)$.
   Hence: $u \in H^1(]-1,1[)$.

2. *Let* $\varphi \in \mathscr{D}(]-1,1[)$. *Then:*

$$\begin{aligned} \int_{-1}^{1} u'(x)\varphi'(x)dx &= \int_0^1 \varphi'(x)dx \\ &= [\varphi(x)]_0^1, \\ &= -\varphi(0). \end{aligned}$$

   Hence: $u'' = \delta \notin L^2(]-1,1[)$. *So,* $u \notin H^2(]-1,1[)$.

Solution 5.3 : $p \in [1,+\infty[,\ f(x) = \begin{cases} x & : & x \in [0,1], \\ -x+2 & : & x \in [1,2], \\ 0 & : & x \notin [1,2]. \end{cases}$

1. *We have:* $\forall x \in \mathbb{R} : |f(x)| \leq 1$, *then:* $|f(x)|^p \leq 1$. *Hence:* $f \in L^p(\mathbb{R})$.
   Let $\varphi \in \mathscr{D}(]-1,1[)$. *Then:*

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x)\varphi'(x)dx &= \int_0^1 x\varphi'(x)dx + \int_1^2 (-x+2)\varphi'(x)dx \\ &= [x\varphi(x)]_0^1 - \int_0^1 \varphi(x)dx + [(-x+2)\varphi(x)]_1^2 + \int_1^2 \varphi(x)dx, \\ &= -\int_0^1 \varphi(x)dx + \int_1^2 \varphi(x)dx. \end{aligned}$$

   Therefore: $f'(x) = \begin{cases} 1 & : & x \in]0,1[, \\ -1 & : & x \in]1,2[, \\ 0 & : & x \notin]0,2[. \end{cases}$
   $\displaystyle\int_{-\infty}^{+\infty} |f'(x)|^p dx = \int_0^1 dx + \int_1^2 dx = 2$. *So,* $f' \in L^p(]-1,1[)$.

hence: $f \in W^{1,p}(\mathbb{R})$.

2. *Let* $\varphi \in \mathscr{D}(]-1,1[)$. *Then:*

$$\begin{aligned}\int_{-\infty}^{+\infty} f'(x)\varphi'(x)dx &= \int_0^1 \varphi'(x)dx - \int_1^2 \varphi'(x)dx \\ &= [\varphi(x)]_0^1 - [\varphi(x)]_1^2, \\ &= 2\varphi(1) - \varphi(2) - \varphi(0).\end{aligned}$$

Hence: $f'' = 2\delta_1 - \delta_2 - \delta \notin L^p(\mathbb{R}[)$. *So,* $f \notin W^{2,p}(\mathbb{R})$.

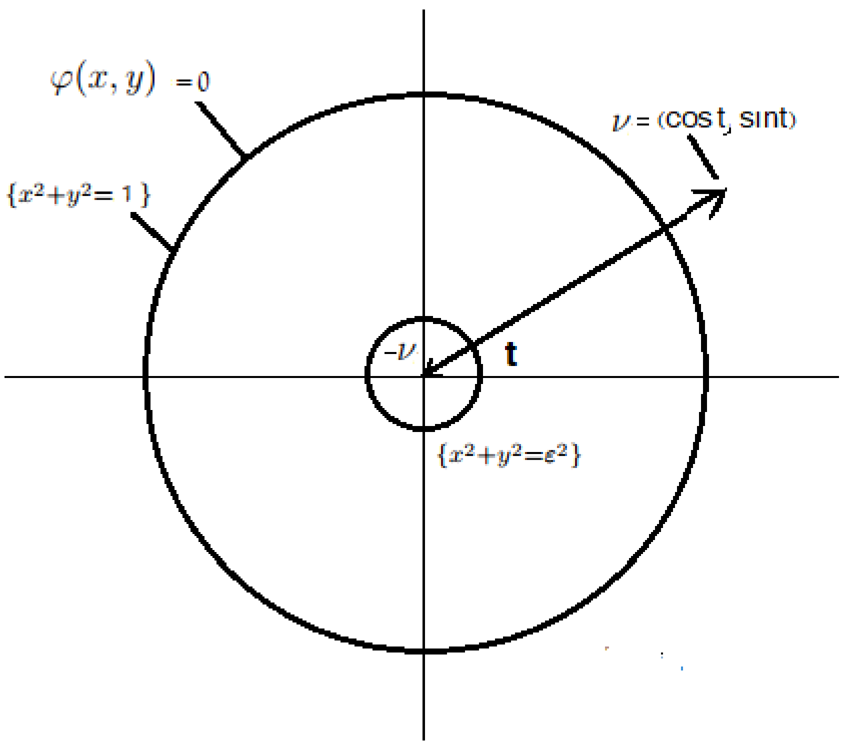
Solution 5.4 : $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}, u(x,y) = \left|\ln \frac{\sqrt{x^2+y^2}}{2}\right|^\alpha, 0 < \alpha < \frac{1}{2}$.

We have:

$$\begin{aligned}\int_\Omega u^2(x,y)dxdy &= \int_\Omega \left|\ln \frac{\sqrt{x^2+y^2}}{2}\right|^{2\alpha} dxdy \\ &= \int_{-\pi}^{\pi}\int_0^1 r\left|\ln \frac{r}{2}\right|^{2\alpha} drd\theta \\ &= 2\pi \int_0^1 r\left|\ln \frac{r}{2}\right|^{2\alpha} drd\theta < +\infty.\end{aligned}$$

Then: $u \in L^2(\Omega)$.
Let $\varphi \in \mathscr{D}(\Omega)$. *Then:*

$$\begin{aligned}\int_\Omega u(x,y)\frac{\partial\varphi}{\partial x}dxdy &= \int_\Omega \left(-\ln \frac{\sqrt{x^2+y^2}}{2}\right)^\alpha \frac{\partial\varphi}{\partial x}dxdy \\ &= \lim_{\varepsilon\to 0}\int_{\Omega\cap\{x^2+y^2>\varepsilon^2\}} \left(-\ln \frac{\sqrt{x^2+y^2}}{2}\right)^\alpha \frac{\partial\varphi}{\partial x}dxdy\end{aligned}$$



Since $\partial\Omega = \{x^2 + y^2 = \varepsilon^2\} \cup \{x^2 + y^2 = 1\}$ *and* $\varphi = 0$ *on* $\{x^2 + y^2 = 1\}$ *we obtain:*

$$\begin{aligned}
\int_{\Omega\cap\{x^2+y^2>\varepsilon^2\}} \left(-\ln\frac{\sqrt{x^2+y^2}}{2}\right)^{\alpha} \frac{\partial\varphi}{\partial x} dxdy &= -\int_{\{x^2+y^2=\varepsilon^2\}} \left(-\ln\frac{\varepsilon}{2}\right)^{\alpha} \varphi(x,y)\nu_x d\sigma(x,y) \\
&\quad + \int_{\Omega\cap\{x^2+y^2>\varepsilon^2\}} \frac{\alpha x}{2\sqrt{x^2+y^2}} \left|\ln\frac{\sqrt{x^2+y^2}}{2}\right|^{\alpha-1} dxdy \\
&= -\left(-\ln\frac{\varepsilon}{2}\right)^{\alpha} \int_{-\pi}^{\pi} \varphi(\varepsilon\cos t, \varepsilon\sin t)\cos t dt \\
&\quad + \int_{\Omega\cap\{x^2+y^2>\varepsilon^2\}} \frac{\alpha x}{2\sqrt{x^2+y^2}} \left|\ln\frac{\sqrt{x^2+y^2}}{2}\right|^{\alpha-1} dxdy
\end{aligned}$$

Noting that:

$$\begin{aligned}
\varphi(\varepsilon\cos t, \varepsilon\sin t)\cos t &\simeq \varphi(0,0).\cos t + \varepsilon\left(\cos t\frac{\partial\varphi}{\partial x}(0,0) + \sin t\frac{\partial\varphi}{\partial y}(0,0)\right) \\
&= \varphi(0,0).\cos t + \frac{\varepsilon}{2}\left((1+\cos 2t)\frac{\partial\varphi}{\partial x}(0,0) + \sin 2t\frac{\partial\varphi}{\partial y}(0,0)\right)
\end{aligned}$$

Then:

$$-\left(-\ln\frac{\varepsilon}{2}\right)^{\alpha} \int_{-\pi}^{\pi} \varphi(\varepsilon\cos t, \varepsilon\sin t)\cos t dt \simeq -\pi\varepsilon\left(-\ln\frac{\varepsilon}{2}\right)^{\alpha},$$

i.e $\lim_{\varepsilon\to 0} -\left(-\ln\frac{\varepsilon}{2}\right)^{\alpha} \int_{-\pi}^{\pi} \varphi(\varepsilon\cos t, \varepsilon\sin t)\cos t dt = 0.$

Therefore:

$$\int_{\Omega} u(x,y)\frac{\partial\varphi}{\partial x} dxdy = \int_{\Omega} \frac{\alpha x}{2\sqrt{x^2+y^2}} \left|\ln\frac{\sqrt{x^2+y^2}}{2}\right|^{\alpha-1} dxdy.$$

Similarly:

$$\int_{\Omega} u(x,y)\frac{\partial\varphi}{\partial y} dxdy = \int_{\Omega} \frac{\alpha y}{2\sqrt{x^2+y^2}} \left|\ln\frac{\sqrt{x^2+y^2}}{2}\right|^{\alpha-1} dxdy.$$

Then:

$$\frac{\partial u}{\partial x} = -\frac{\alpha x}{2\sqrt{x^2+y^2}} \left|\ln\frac{\sqrt{x^2+y^2}}{2}\right|^{\alpha-1}, \qquad \frac{\partial u}{\partial y} = -\frac{\alpha y}{2\sqrt{x^2+y^2}} \left|\ln\frac{\sqrt{x^2+y^2}}{2}\right|^{\alpha-1}.$$

$$\begin{aligned}
\int_{\Omega} \left(\frac{\partial u}{\partial x}\right)^2 (x,y) dxdy &= \int_{\Omega} \frac{\alpha^2 x^2}{4(x^2+y^2)} \left|\ln\frac{\sqrt{x^2+y^2}}{2}\right|^{2\alpha-2} dxdy \\
&= \frac{\alpha^2}{4} \int_{-\pi}^{\pi}\int_0^1 r\cos^2\theta \left|\ln\frac{r}{2}\right|^{2\alpha-2} drd\theta \\
&\leq \frac{\pi\alpha^2}{2} \int_0^1 r\left|\ln\frac{r}{2}\right|^{2\alpha-2} drd\theta < +\infty.
\end{aligned}$$

Hence: $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^2(\Omega)$.

Therefore: $u \in H^1(\Omega)$.

On a: $\lim\limits_{(x,y)\to(0,0)} u(x,y) = +\infty.$
So,u does not have a continuous representation on Ω.

Solution 5.5 : *Let* $B = \{x \in \mathbb{R}^n : |x| \leq 1\}, u(x) = |x|^\alpha$.

1. **) Suppose that* $n = 1$. *Then:*

$$\begin{aligned}\int_\Omega u^2(x)dx &= \int_{-1}^{1} |x|^{2\alpha}dx \\ &= 2\int_0^1 x^{2\alpha}dx < +\infty \; si \; \alpha > -\frac{1}{2}.\end{aligned}$$

hence: $u \in L^2(B)$ *si* $\alpha > -\frac{1}{2}$.
Let $\varphi \in \mathscr{D}(B)$. *Then:*

$$\begin{aligned}\int_B u(x)\varphi'(x)dx &= \int_{-1}^{1} |x|^\alpha \varphi'(x)dx \\ &= \int_{-1}^{0} (-x)^\alpha \varphi'(x)dx + \int_0^1 x^\alpha \varphi'(x)dx \\ &= \lim_{\varepsilon\to 0}\left(\int_{-1}^{-\varepsilon} (-x)^\alpha \varphi'(x)dx + \int_\varepsilon^1 x^\alpha \varphi'(x)dx\right) \\ &= \lim_{\varepsilon\to 0}\left(\varepsilon^\alpha(\varphi(\varepsilon) - \varphi(-\varepsilon)) + \alpha\int_{-1}^{-\varepsilon} (-x)^{\alpha-1}\varphi(x)dx - \alpha\int_\varepsilon^1 x^{\alpha-1}\varphi(x)dx\right)\end{aligned}$$

We have: $\lim\limits_{\varepsilon\to 0} \varepsilon^\alpha(\varphi(\varepsilon) - \varphi(-\varepsilon)) = \lim\limits_{\varepsilon\to 0} 2\varepsilon^{1+\alpha}\frac{\varphi(\varepsilon) - \varphi(-\varepsilon)}{2\varepsilon} = 0$ *si* $\alpha > -1$.
Then: for $\alpha > -1$ *we have:*

$$\int_B u(x)\varphi'(x)dx = \int_{-1}^{0} (-x)^{\alpha-1}\varphi(x)dx + \alpha\int_0^1 x^{\alpha-1}\varphi(x)dx.$$

In this case, $u'(x) = \alpha sign(x)|x|^{\alpha-1}$, *where* $sign(x)$ *represents the sign of* x.

$$\begin{aligned}\int_B u'^2(x)dx &= \alpha^2\int_{-1}^{1} |x|^{2\alpha-2}dx \\ &= 2\alpha^2\int_0^1 x^{2\alpha-2}dx < +\infty \; si \; \alpha > \frac{1}{2}.\end{aligned}$$

Therefore: $u \in H^1(B)$ *pour* $\alpha > \frac{1}{2}$.
***) Suppose that* $n = 2$. *Then:*

$$\begin{aligned}\int_B u^2(x,y)dxdy &= \int_B (x^2 + y^2)^\alpha dxdy \\ &= \int_{-\pi}^{\pi}\int_0^1 r^{2\alpha+1}drd\theta \\ &= 2\pi\int_0^1 r^{2\alpha+1}dr < +\infty \; si \; \alpha > -1.\end{aligned}$$

Hence: $u \in L^2(B)$ *si* $\alpha > -1$.

Let $\varphi \in \mathscr{D}(B)$*. Using the same arguments as in Exercise 5.4, we obtain for* $\alpha > -1$*:*

$$\begin{aligned}
\int_B u(x,y)\frac{\partial \varphi}{\partial x}dxdy &= \int_B (x^2+y^2)^{\frac{\alpha}{2}}\frac{\partial \varphi}{\partial x}dxdy \\
&= \lim_{\varepsilon\to 0}\int_{B\cap\{x^2+y^2>\varepsilon^2\}}(x^2+y^2)^{\frac{\alpha}{2}}\frac{\partial \varphi}{\partial x}dxdy \\
&= \lim_{\varepsilon\to 0}\left[-\pi\varepsilon^{1+\alpha}-\int_{B\cap\{x^2+y^2>\varepsilon^2\}}\alpha x(x^2+y^2)^{\frac{\alpha-2}{2}}\varphi(x,y)dxdy\right] \\
&= \int_B -\alpha x(x^2+y^2)^{\frac{\alpha-2}{2}}\varphi(x,y)dxdy.
\end{aligned}$$

Hence: $\dfrac{\partial u}{\partial x} = \alpha x(x^2+y^2)^{\frac{\alpha-2}{2}} \qquad \dfrac{\partial u}{\partial y} = \alpha y(x^2+y^2)^{\frac{\alpha-2}{2}}$.

$$\begin{aligned}
\int_B\left(\frac{\partial u}{\partial x}\right)^2 dxdy &= \alpha^2\int_B x^2(x^2+y^2)^{\alpha-2}dxdy \\
&= \int_{-\pi}^{\pi}\int_0^1 \cos^2\theta . r^{2\alpha-1}drd\theta \\
&= \pi\int_0^1 r^{2\alpha-1}dr < +\infty \text{ si } \alpha > 0.
\end{aligned}$$

$$\begin{aligned}
\int_B\left(\frac{\partial u}{\partial x}\right)^2 dxdy &= \alpha^2\int_B y^2(x^2+y^2)^{\alpha-2}dxdy \\
&= \int_{-\pi}^{\pi}\int_0^1 \sin^2\theta . r^{2\alpha-1}drd\theta \\
&= \pi\int_0^1 r^{2\alpha-1}dr < +\infty \text{ si } \alpha > 0.
\end{aligned}$$

So, $u \in H^1(B)$ *si* $\alpha > 0$.

****) Suppose that* $n \geq 2$ *and set:* $x = (x_1, \cdots, x_n)$*, where*

$$\begin{cases}
x_1 = r\cos\theta_1\cos\theta_2\cdots\cos\theta_{n-1} \\
x_2 = r\cos\theta_1\cos\theta_2\cdots\sin\theta_{n-1} \\
x_2 = r\cos\theta_1\cos\theta_2\cdots\sin\theta_{n-2} \\
\vdots \\
x_{n-2} = r\cos\theta_1\sin\theta_2 \\
x_{n-1} = r\sin\theta_1.
\end{cases} , r\in]0,1[, \theta_1,\theta_2,\theta_{n-2}\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[, \theta_{n-1}\in]-\pi,\pi[.$$

Then:

$$\begin{aligned}
\int_B u^2(x)dx &= \int_B |x|^{2\alpha}dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\pi}^{\pi}\int_0^1 r^{n-1}\cos^{n-2}\theta_1\cdots\cos\theta_{n-1}r^{2\alpha}drd\theta_1 d\theta_2\cdots d\theta_{n-1} \\
&= M\int_0^1 r^{2\alpha+n-1}dr < +\infty \; si \; \alpha > -\frac{n}{2}.
\end{aligned}$$

Hence: $u \in L^2(B)$ *si* $\alpha > -\frac{n}{2}$.

Let $\varphi \in \mathscr{D}(B)$.*Using the same arguments as in Exercise 5.4, we obtain for* $\alpha > -\frac{n}{2}$*:*

$$\begin{aligned}
\int_B u(x)\frac{\partial \varphi}{\partial x_i}dxdy &= \int_B |x|^\alpha \frac{\partial \varphi}{\partial x_i}dx \\
&= \lim_{\varepsilon\to 0}\int_{B\cap\{|x|>\varepsilon\}} |x|^\alpha \frac{\partial \varphi}{\partial x}dxdy \\
&= \lim_{\varepsilon\to 0}\left[\int_{\{|x|=\varepsilon\}} \varepsilon^\alpha \varphi(x)\nu_i(x)d\sigma(x) - \int_{\{|x|>\varepsilon\}} \alpha x_i|x|^{\alpha-2}\varphi(x)dx\right] \\
&= -\int_B \alpha x_i|x|^{\alpha-2}\varphi(x)dx.
\end{aligned}$$

So, $\frac{\partial u}{\partial x_i} = \alpha x_i|x|^{\alpha-2}$.

$$\begin{aligned}
\int_B \left(\frac{\partial u}{\partial x_i}\right)^2 dx &= \alpha^2\int_B x_i^2|x|^{2\alpha-4}dx \\
&= M'\int_0^1 r^{2\alpha+n-3}dr < +\infty \; si \; \alpha > \frac{2-n}{2}.
\end{aligned}$$

Solution 5.6 : $p \in [1,+\infty[, \forall x \in]0,1[$: $f(x) = x^{-\frac{1}{p+1}}$.

1. *We have:*

$$\begin{aligned}
\int_0^1 |f(x)|^p dx &= \int_0^1 x^{-\frac{p}{p+1}}dx \\
&= p+1 < +\infty.
\end{aligned}$$

   Then: $f \in L^p(]0,1[)$.

2. *Let* $\varphi \in \mathscr{D}(]0,1)$. *Then:*

$$\begin{aligned}
\int_0^1 f(x)\varphi'(x)dx &= \int_0^1 x^{-\frac{1}{p+1}}\varphi'(x)dx \\
&= [x^{-\frac{1}{p+1}}\varphi(x)]_0^1 + \frac{1}{p+1}\int_0^1 x^{-\frac{p+2}{p+1}}\varphi(x)dx, \\
&= \frac{1}{p+1}\int_0^1 x^{-\frac{p+2}{p+1}}\varphi(x)dx.
\end{aligned}$$

   Then: $g(x) = -\frac{1}{p+1}x^{-\frac{p+2}{p+1}}$.

3. *We have:*

$$\int_0^1 |g(x)|^p dx = \frac{1}{(p+1)^p}\int_0^1 x^{-\frac{p(p+2)}{p+1}}dx = \infty \; car \; -\frac{p(p+2)}{p+1} \le -1.$$

   So, $f \notin W^{1,p}(]0,1[)$.

4. $u \in W_0^{1,p}(]0,1[), \{\varphi_n\}_{n=1}^\infty \subset \mathscr{D}(]0,1[), \lim_{n\to+\infty}\|\varphi_n' - u'\|_{L^p(]0,1[)} = 0$.

We have:

$$\begin{aligned}\int_0^1 f(x)\varphi_n(x)dx &= \int_0^1 x^{-\frac{1}{p+1}}\varphi_n(x)dx\\ &= \frac{p+1}{p}[x^{\frac{p}{p+1}}\varphi(x)]_0^1 - \frac{p+1}{p}\int_0^1 x^{\frac{p}{p+1}}\varphi_n'(x)dx,\\ &= -\frac{p+1}{p}\int_0^1 x^{\frac{p}{p+1}}\varphi_n'(x)dx.\end{aligned}$$

Then:

$$\begin{aligned}\left|\int_0^1 f(x)\varphi_n(x)dx\right| &= \left|\frac{p+1}{p}\int_0^1 x^{\frac{p}{p+1}}\varphi_n'(x)dx\right|\\ &\leq \frac{p+1}{p}\int_0^1 x^{\frac{p}{p+1}}|\varphi_n'(x)|dx\\ &\leq \frac{p+1}{p}\int_0^1 |\varphi_n'(x)|dx\\ &\leq \frac{p+1}{p}\left(\int_0^1 1^{p'}dx\right)^{\frac{1}{p'}}\left(\int_0^1 |\varphi_n'(x)|^p dx\right)^{\frac{1}{p}}\\ &= \frac{p+1}{p}\|\varphi_n'\|_{L^p(]0,1[)}.\end{aligned}$$

5. *From the previous question we have:*

$$\left|\int_0^1 f(x)\varphi_n(x)dx\right| \leq \frac{p+1}{p}\|\varphi_n\|_{W_0^{1,p}(]0,1[)}.$$

By density:

$$\forall u \in W_0^{1,p}(]0,1[) : \left|\int_0^1 f(x)\varphi_n(x)dx\right| \leq \frac{p+1}{p}\|u\|_{W_0^{1,p}(]0,1[)}.$$

Then: $f \in W_0^{-1,p'}(]0,1[)$.

Solution 5.7 : $\delta : \mathscr{D}(]-1,1[) \to \mathbb{R}, <\delta,\varphi>= \varphi(0)$

1. *Let* $\varphi \in \mathscr{D}(]-1,1[)$. *Then:*

$$\begin{aligned}|\langle\delta,\varphi\rangle| &= |\varphi(0)|\\ &= \left|\int_{-1}^0 \varphi'(x)dx\right|\\ &\leq \int_{-1}^1 |\varphi'(x)|dx\\ &\leq \left(\int_{-1}^1 1^2dx\right)^{\frac{1}{2}}\left(\int_0^1 |\varphi'(x)|^2dx\right)^{\frac{1}{2}}\\ &= \sqrt{2}\|\varphi\|_{H_0^1(]-1,1[)}.\end{aligned}$$

By density, we deduce that $\delta \in H^{-1}(]-1,1[)$.

2. *From the example 3.9 and Corollary 3.1, solutions of the equation* $-T'' = \delta$ *in* $\mathscr{D}'(]-1,1[)$ *are restrictions of the function* $f(x) = \alpha|x| + \beta$.
   Since $u_0 \in H_0^1(]-1,1[)$ *we have:* $u_0(-1) = u_0(1) = 0$, *So,* u_0 *can be expressed in the form:*
$$u_0(x) = \begin{cases} a(x+1) & : \ x \in [-1,0] \\ b(x-1) & : \ x \in]0,1]. \end{cases}$$
   Hence: pour $\varphi \in \mathscr{D}(]-1,1[)$ *on a:*
$$\begin{aligned} \int_{-1}^{1} u_0(x)\varphi'(x)dx &= a\int_{-1}^{0}(x+1)\varphi'(x)dx + b\int_{0}^{1}(x-1)\varphi'(x)dx \\ &= a[(x+1)\varphi(x)]_{-1}^{0} - a\int_{-1}^{0}\varphi(x)dx + b[(x-1)\varphi(x)]_0^1 - b\int_0^1 \varphi(x)dx \\ &= (a+b)\varphi(0) - a\int_{-1}^{0}\varphi(x)dx - b\int_0^1\varphi(x)dx. \end{aligned}$$
   Therefore: $u_0' = -(a+b)\delta + f$, *où:*
$$f(x) = \begin{cases} a & : \ x \in [-1,0] \\ b & : \ x \in]0,1]. \end{cases}$$
   $u \in H_0^1(]-1,1[)$ *implicate to* $u' \in L^2(]-1,1[)$. *Then:* $a + b = 0$, *i.e* $b = -a$. *So,*
$$u_0'(x) = \begin{cases} a & : \ x \in [-1,0] \\ -a & : \ x \in]0,1]. \end{cases}$$
   Hence: for $\varphi \in \mathscr{D}(]-1,1[)$ *we have:*
$$\begin{aligned} \int_{-1}^{1} u_0'(x)\varphi'(x)dx &= a\int_{-1}^{0}\varphi'(x)dx - a\int_0^1\varphi'(x)dx \\ &= a[\varphi(x)]_{-1}^{0} - a[\varphi(x)]_0^1 \\ &= 2a\varphi(0). \end{aligned}$$
   Then: $u'' = 2a$, *which leads to:* $a = \dfrac{1}{2}$. *Hence:*
$$u_0(x) = \frac{1-|x|}{2} = \begin{cases} \dfrac{1+x}{2} & : \ x \in [-1,0] \\ \dfrac{1-x}{2} & : \ x \in]0,1]. \end{cases}$$

3. *Suppose there exists another function* $u_1 \in H_0^1(]-1,1[)$ *satisfying* $-u_1'' = \delta$.
   Then, $(u_1 - u_0)'' = 0$. *Hence:*
$$\int_{-1}^{1}(u_1-u_0)'(x)v'(x)dx = 0, \qquad \forall v \in H_0^1(]-1,1[).$$

Set: $v = u_1 - u_0$, *we obtain:* $\displaystyle\int_{-1}^{1}(u_1' - u_0')^2(x)dx = 0.$
Then:

$$\|u_1 - u_0\|_{H_0^1(]-1,1[)} = \|u_1' - u_0'\|_{L^2(]-1,1[)} = 0.$$

So, $u_1 = u_0$.