B- VECTORS

1) Definition of vector:

A vector is an oriented segment that has both magnitude (length) and direction. Graphically, vectors are often represented by arrows. The length of the arrow represents the magnitude of the vector, and the direction of the arrow indicates the direction of the vector. In handwritten equations or mathematical expressions, vectors are typically represented by placing a letter with an arrow above it such as $\vec{V}$ *. In physics, vectors offer a comprehensive representation of certain quantities that cannot be adequately described solely by stating their values and units. Vectors enable us to model and represent these quantities, allowing us to understand how they change in various spatial directions, essentially giving them a descriptive form associated with spatial orientations.*

2) The key characteristics of a vector are:

Tail (origin) ⟶ Tip (Head)

*1- **Magnitude (Length)**: Magnitude represents the length of the vector and expresses the scalar quantity of the vector. Its value is always positive. The symbol used to represent the magnitude or absolute value of a vector is indeed typically written as double vertical bars* (‖ ‖) *surrounding the vector* $\vec{V}$*, like this:* $\|\vec{V}\|$

*2- **Direction**: Direction indicates the orientation of the scalar quantity of the vector, geometrically defined by the angle between the vector and a specific axis or the arrow's direction.*

*3- **Support**: Support refers to the line connecting the starting point (tail or origin) of the vector to its endpoint (head or tip).*

3) Specifying a vector

The vector is typically defined using one of the following methods:

- ✓ *By specifying its length (magnitude) and the angle it makes with a specific axis (see figure 01).*
- ✓ *By determining its starting point (tail) and endpoint (head) using their coordinates in space (see figure 02).*
- ✓ *By describing its components in space, such as its three-dimensional coordinates (We will address this point later).*

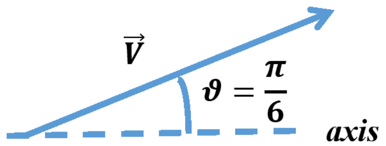


figure 01

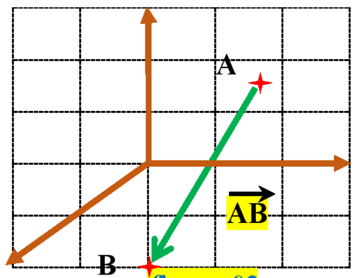
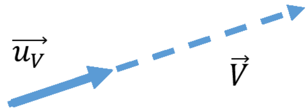


figure 02

4) The Unit Vector

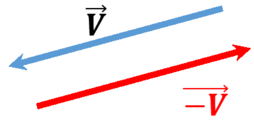


The unit vector notation for vector $\vec{V}$, which is denoted as $\overrightarrow{u_V}$, exhibits the following characteristics:

***Magnitude**: It has a magnitude of exactly one, denoted as $\|\overrightarrow{u_V}\| = 1$. This signifies that its length is normalized to unity, meaning it doesn't carry any specific units.*

***Direction**: $\overrightarrow{u_V}$ aligns precisely in the same direction as the original vector $\vec{V}$. In mathematical terms, this can be expressed as $\overrightarrow{u_V} = \vec{V}/\|\vec{V}\|$, where $\|\vec{V}\|$ represents the magnitude (length) of vector $\vec{V}$.*

5) The Negative of a Vector

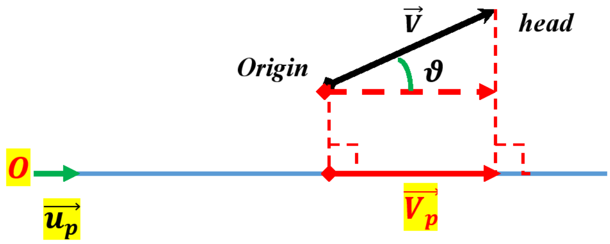


The negative vector is a vector with an equal magnitude to the original vector but directed in the opposite way.

6) The vertical projection of a vector onto an axis

The vertical projection of a vector $\vec{V}$ onto an axis (OP) is a vector $\vec{V_p}$ aligned with that axis. Its starting point (origin or tail) is the projection of the vector's origin point onto the axis, and its endpoint is the projection of the vector's head(tip) onto the axis. Its length is the result of multiplying the vector's length by the cosecant of the angle (ϑ) enclosed between them.

$$\|\vec{V_p}\| = \|\vec{V}\| \times \cos(\vartheta)$$



If the axis onto which a vector is projected is oriented with a specific direction, such as being represented by a unit vector, we can express the projection by taking the scalar product (dot product) between the original vector and the unit vector of the axis. We'll delve into the concept of the dot product *between two vectors in more detail later on.,and therefore, we can write the following:*

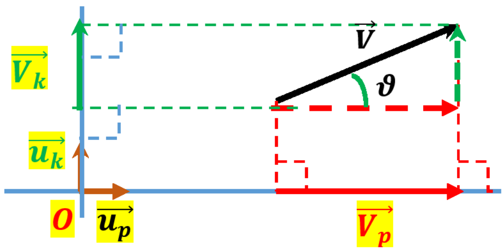
$$\overrightarrow{V_p} = \|\overrightarrow{V_p}\|\,\overrightarrow{u_p} = \|\vec{V}\| \times \cos(\vartheta)\,\overrightarrow{u_p}$$

*The length of the vector obtained through the projection can be called the '**component** of the original vector on the axis. This component represents a part of the original vector and allows you to analyze the original vector into its **subcomponents** along multiple axes.*

7) The analytical representation of a vector

The analytical representation of a vector means representing the vector using its individual components relative to specific axes. These components are typically the outcomes of projecting the vector orthogonally onto these axes. In other words, the analytical representation of a vector is the process of rewriting it using its components and unit vectors extending in specific directions along particular axes.

Example01:



The vector $\vec{V}$ in the corresponding figure has a magnitude of 2 and makes an angle of 30 degrees with the (OP) axis. We can find its components along the two axes (OP) and (OK), which carry the unit vectors $\overrightarrow{u_p}$ and $\overrightarrow{u_k}$, respectively. We can write its analytical expression along these two axes using the following method:

To find the component along the (op) axis. Firstly, we need to perform orthogonal projections of the original vector onto both axes, as detailed in the figure.

On (OP) axis $\Rightarrow \overrightarrow{V_p} = \|\overrightarrow{V_p}\|\,\overrightarrow{u_p} = \|\vec{V}\| \times \cos(\vartheta)\overrightarrow{u_p} = \frac{2\sqrt{3}}{2}\,\overrightarrow{u_p} = \sqrt{3}\,\overrightarrow{u_p}$

On (OK) axis $\Rightarrow \overrightarrow{V_k} = \|\overrightarrow{V_k}\|\,\overrightarrow{u_k} = \|\vec{V}\| \times \sin(\vartheta)\overrightarrow{u_k} = 2 \times \frac{1}{2}\overrightarrow{u_k} = \overrightarrow{u_k}$

$$\vec{V} = \overrightarrow{V_p} + \overrightarrow{V_k} = \sqrt{3}\,\overrightarrow{u_p} + \overrightarrow{u_k}$$

So $\sqrt{3}$ and 1 *values resulting from projecting the vector $\vec{V}$ on the two principal axes (OP, and OK) are called the components of the vector $\vec{V}$.*

- *When vector $\vec{A}$ analyzed into its components in a Cartesian coordinate system* $(O, \vec{\imath}, \vec{\jmath}, \vec{k})$ *, the analytical expression for this vector is written as follows:*

$$\vec{A} = a_x\,\vec{\imath} + a_y\vec{\jmath} + a_z\,\vec{k} \quad or \quad \vec{A} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix}$$

Where the relationship between the components of the vector, its magnitude, and the angle it forms with a given axis is expressed as follows:

$$\begin{cases} a_x = |\vec{A}| \cos\theta \\ a_y = |\vec{A}| \sin\theta \\ |\vec{A}| = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2} \\ \tan(\theta) = a_y / a_x \end{cases}$$

8) Belonging of vector to a specific plane defined by two vectors

We classify the vector $\vec{V}$ as part of the plane (P) defined by vectors $\vec{V_1}$ and $\vec{V_2}$ when we can represent it using these two vectors. In other words, if we can find values "a" and "b" such that $\vec{V} = a\,\vec{V_1} + b\,\vec{V_2}$, we can conclude that vector $\vec{V}$ can be expressed in relation to vectors $\vec{V_1}$ and $\vec{V_2}$ and is situated within the specified plane.

Example02: *We consider the following vectors:*

$\vec{V_1} = 2\vec{\imath} + \vec{\jmath}$ $\quad \vec{V_2} = \vec{\imath} + 3\vec{\jmath} + \vec{k}$ $\quad \vec{V_3} = 4\vec{\imath} + 7\vec{\jmath} + 2\vec{k}$

To prove that the vector $\vec{V_3}$ belongs to the plane (P) formed by vectors $\vec{V_1}$ and $\vec{V_2}$

$$\vec{V_3} = a\,\vec{V_1} + b\,\vec{V_2} = 2a\,\vec{\imath} + a\,\vec{\jmath} + b\,\vec{\imath} + 3b\,\vec{\jmath} + b\,\vec{k} = (2a+b)\,\vec{\imath} + (a+3b)\,\vec{\jmath} + b\,\vec{k}$$

$$(2a+b)\vec{\imath} + (a+3b)\vec{\jmath} + b\,\vec{k} = 4\vec{\imath} + 7\vec{\jmath} + 2\vec{k} \Rightarrow \begin{cases} 2a + b = 4 \\ a + 3b = 7 \\ b = 2 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 2 \end{cases}$$

$\vec{V_3} = \vec{V_1} + 2\,\vec{V_2}$ *we can write $\vec{V_3}$ as function of both $\vec{V_1}$ and $\vec{V_2}$, thus, the vector $\vec{V_4}$ belongs to the plane (P)*

9) The linear dependence of two vectors

We say that two vectors are linearly dependent if one of them can be expressed as the scalar product of the other by a real number. In other term, the vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ are linearly dependent if there exists a real number $\boldsymbol{\alpha}$ that satisfies the following relationship: $\vec{\boldsymbol{A}} = \boldsymbol{\alpha}.\ \vec{\boldsymbol{B}}$

Example03:

We consider the following vectors: $\quad \vec{A} = \vec{\imath} + 2\vec{\jmath} + \vec{k} \quad \vec{B} = 4\vec{\imath} + 8\vec{\jmath} + 2\vec{k}$

- *Verify whether the two vectors are linearly dependent*

$$\vec{A} = \alpha\,\vec{B} \Rightarrow \alpha\left(\vec{\imath} + 2\vec{\jmath} + \vec{k}\right) = 4\vec{\imath} + 8\vec{\jmath} + 2\vec{k} \Rightarrow \begin{cases} \alpha = 4 \\ 2\,\alpha = 8 \\ \alpha = 2 \end{cases} \Rightarrow = \begin{matrix} \alpha = 4 \\ \alpha = 4 \\ \alpha = 2 \end{matrix}$$

Therefore, It can be said that these two vectors are not linearly dependent, as there is no single real number that satisfies the relation $\vec{A} = \alpha\,\vec{B}$

10) Vector Addition:

The sum of two vectors $\vec{A}$ and $\vec{B}$ *is a vector with its tail at the beginning of the first vector* $\vec{A}$ *and its head at the end of the second vector* $\vec{B}$*, while its length is obtained as follows:*

$$\vec{R} = \vec{A} + \vec{B} \Rightarrow (\vec{R})^2 = (\vec{A} + \vec{B})^2 = (\vec{A})^2 + (\vec{B})^2 + 2\,(\vec{A}.\vec{B})$$

$\|\vec{\mathbf{R}}\|^2 = \|\vec{\mathbf{A}}\|^2 + \|\vec{\mathbf{B}}\|^2 + \mathbf{2}\,\|\vec{\mathbf{A}}\|.\|\vec{\mathbf{B}}\|\,\mathbf{cos}(\boldsymbol{\vartheta})$ *where* $\boldsymbol{\vartheta}$ *denotes the angle enclosed between* $\vec{A}$ and $\vec{B}$ *vectors*

$$\|\vec{\mathbf{R}}\| = \sqrt{\|\vec{\mathbf{A}}\|^2 + \|\vec{\mathbf{B}}\|^2 + \mathbf{2}\,\|\vec{\mathbf{A}}\|.\|\vec{\mathbf{B}}\|\,\mathbf{cos}(\boldsymbol{\vartheta})}$$

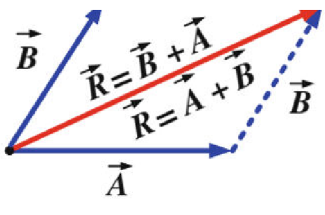
Geometrically*, vectors can be added by parallel displacement of the second vector to the first vector, such that the end of the first vector (the head) matches the starting point of the second vector (the tail). The resulting vector from the addition has its starting point at the origin of the first vector and its endpoint at the head of the second vector, as shown in the following figure where* $\vec{R} = \vec{A} + \vec{B}$

Analytically:

Let us consider $\vec{A}$ *and* $\vec{B}$ *are two vectors have their components in the Cartesian coordinates system (O, XYZ), where "O" is the origin, and "XYZ" represents the three axes with three basis orthogonal unit vectors* ($\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$) *that correspond to the three main coordinate axes. We can sum* $\vec{A}$ *and* $\vec{B}$ *analytically by adding all the components multiplied by the same unit vector as follows:*

$$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\,\vec{k} \qquad \vec{B} = B_x\vec{\imath} + B_y\vec{\jmath} + B_z\,\vec{k}$$

$$\vec{A} + \vec{B} = (A_x + B_x)\,\vec{\imath} + (A_y + B_y)\,\vec{\jmath} + (A_z + B_z)\,\vec{k}$$



We can use another writing shape for vectors sum

$$\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix} \quad and \quad \vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix} \Rightarrow \vec{A} + \vec{B} \begin{pmatrix} A_x + B_x \\ A_y + B_y \\ A_z + B_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix}$$

The magnitude of $(\vec{A} + \vec{B})$ *vector is given as follow:*

$$\|\vec{A} + \vec{B}\| = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2 + (A_z + B_z)^2}$$

To find the coordinates of the starting point of the vector sum of vectors $\vec{A}$ and $\vec{B}$, where vector $\vec{A}$ has its tail at point (x_1, y_1) and its head at point (x_2, y_2), and vector $\vec{B}$ has its tail at point (x_3, y_3) and its head at point (x_4, y_4), the resultant of their addition, vector $\vec{C} = \vec{A} + \vec{B}$, will have its starting coordinates at (x_1+x_3, y_1+y_3) and its head at ($x_2 + x_4$, $y_2 + y_4$).

Important Note:

1- *Subtracting two vectors is a special case of vector addition. Geometrically, we add the first vector to the negative of the second vector. Analytically, it is expressed as follows:*

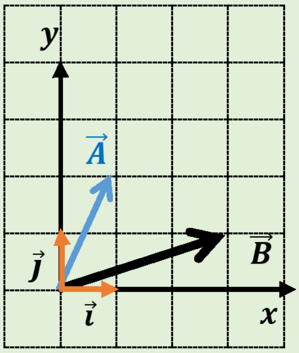
$$\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix} \quad \text{and} \quad \vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix} \Rightarrow \vec{A} - \vec{B} \begin{pmatrix} A_x - B_x \\ A_y - B_y \\ A_z - B_z \end{pmatrix}\begin{pmatrix} \vec{\imath} \\ \vec{\jmath} \\ \vec{k} \end{pmatrix}$$

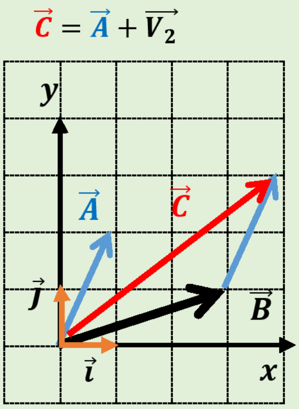
- ✓ *Vector addition has the commutative property* $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
- ✓ *Vector addition has the associative property* $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$
- ✓ *Vector addition has the distributive property* $\lambda \times (\vec{A} + \vec{B}) = \lambda \times \vec{A} + \lambda \times \vec{B}$

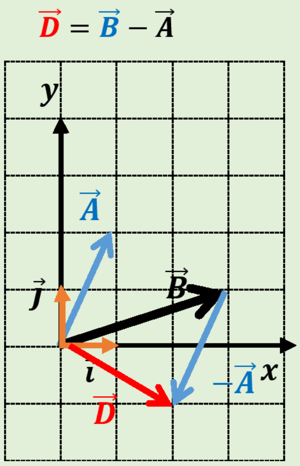
Example03:

We consider the following vectors: $\vec{A} = 3\vec{\imath} + 1\vec{\jmath}$ $\quad \vec{B} = 1\vec{\imath} + 2\vec{\jmath}$

Using the graphical and analytical methods, find the sum and subtraction of these vectors







Addition ***Subtraction***

Analytically:

$$\vec{C} = \vec{A} + \vec{B} = 3\vec{\imath} + 1\vec{\jmath} + 1\vec{\imath} + 2\vec{\jmath} = 4\vec{\imath} + 3\vec{\jmath}$$

$$\vec{D} = \vec{B} - \vec{A} = 1\vec{\imath} + 2\vec{\jmath} - (3\vec{\imath} + 1\vec{\jmath}) = -2\vec{\imath} + 1\vec{\jmath}$$

11) Scalar (Dot) Product, Vector (Cross) Product, And Mixed Product

11-1) Scalar (Dot) Product

- ✓ *The scalar (dot) product of two vectors* $\vec{A}$ and $\vec{B}$ *is denoted* $\vec{A} \cdot \vec{B}$
- ✓ *The scalar (dot) product of two vectors* $\vec{A}$ and $\vec{B}$ *produces a* **scalar**.
- ✓ *The scalar product of vectors* $\vec{A}$ and $\vec{B}$ *is given in terms of their magnitudes and the angle (*$\boldsymbol{\vartheta}$*) enclosed between them as follows:*

$$\vec{\boldsymbol{A}}.\vec{\boldsymbol{B}} = \|\vec{\boldsymbol{A}}\|.\|\vec{\boldsymbol{B}}\| \ \boldsymbol{cos}\,(\boldsymbol{\vartheta})$$

We can distinguish three special cases based on the value of the angle

$$\vec{A}.\vec{B} = \|\vec{A}\|.\|\vec{B}\| \ cos\,(\vartheta) = \begin{cases} \|\vec{A}\|.\|\vec{B}\| \ if\ \vartheta = 0\ \vec{A} \text{ and } \vec{B} \text{ are parallel vectors} \\ -\|\vec{A}\|.\|\vec{B}\| \ if\ \vartheta = \pi\ \vec{A} \text{ and } \vec{B} \text{ are anti parallel vectors} \\ 0\ \ if\ \vartheta = \frac{\pi}{2}\ \ so\ \vec{A} \perp \vec{B} \end{cases}$$

- ✓ *The scalar product of vectors* $\vec{A}$ and $\vec{B}$ *is given in terms of the components of* $\vec{A}$ and $\vec{B}$ *as follows:*

$$\vec{A}.\vec{B} = (A_x \times B_x) + (A_y \times B_y) + (A_z \times B_z)$$

- ✓ *The scalar product has the commutative property* $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- ✓ *The scalar product has the distributive property* $\vec{C} \cdot (\vec{A} + \vec{B}) = (\vec{C} \cdot \vec{A}) + (\vec{C} \cdot \vec{B})$
- ✓ *The scalar product of the unit vector is given as follow:*

$$\vec{\imath}.\vec{\jmath} = \vec{\imath}.\vec{\jmath} = \vec{\imath}.\vec{k} = \vec{k}.\vec{\imath} = \vec{k}.\vec{\jmath} = \vec{\jmath}.\vec{k} = 0$$

$$\vec{\imath}.\vec{\imath} = \vec{\jmath}.\vec{\jmath} = \vec{k}.\vec{k} = 1$$

- ✓ *If the components of vectors* $\vec{A}$ and $\vec{B}$ *are time-dependent, the derivative of their scalar product is expressed as follows:*

$$\frac{d(\vec{A} \cdot \vec{B})}{dt} = \frac{d(\vec{A})}{dt} \cdot \vec{B} + \vec{A} \cdot \frac{d(\vec{B})}{dt}$$

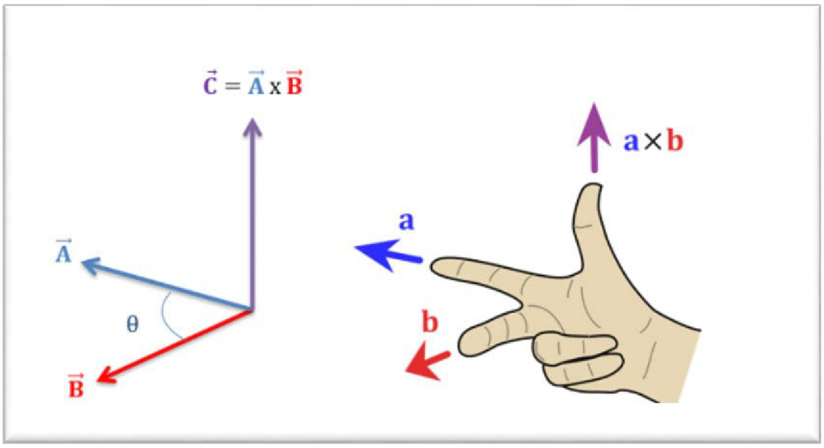
11-2) Vector (Cross) Product

- ✓ *The Vector (Cross) product of two vectors* $\vec{A}$ and $\vec{B}$ *is denoted* $\vec{A} \wedge \vec{B}$
- ✓ *The Vector (Cross) product of two vectors* $\vec{A}$ and $\vec{B}$ *produces a* ***vector*** *that is perpendicular on the plane formed by* $\vec{A}$ and $\vec{B}$*, and its direction is determined using* ***the right-hand rule****. Its magnitude is given in terms of is given in term of* $\vec{A}$ and $\vec{B}$ *magnitudes and the angle (*$\boldsymbol{\vartheta}$*) enclosed between them as follows:*

$$\vec{C} = \|\vec{A} \wedge \vec{B}\| = \|\vec{A}\| . \|\vec{B}\| \, \sin(\vartheta)$$

We can distinguish two special cases based on the value of the angle

$$\vec{A} \wedge \vec{B} = \|\vec{A}\| . \|\vec{B}\| \, sin\,(\vartheta) = \begin{cases} 0 & if \;\; \vec{A} \text{ and } \vec{B} \text{ are parallel vectors } (\vec{A} \,|\,|\vec{B}) \\ \pm \|\vec{A}\| . \|\vec{B}\| & if \;\; \vec{A} \text{ and } \vec{B} \text{ are orthogonal vectors } (\vec{A} \perp \vec{B}) \end{cases}$$



- ✓ *The vector (Cross) product has not the commutative property* $\vec{A} \wedge \vec{B} = -\vec{B} \wedge \vec{A}$
- ✓ *The vector (Cross) product has the distributive property* $\vec{C} \wedge (\vec{A} + \vec{B}) = (\vec{C} \wedge \vec{A}) + (\vec{C} \wedge \vec{B})$
- ✓ *The vector (Cross) product of vectors* $\vec{A}$ and $\vec{B}$ *is given in terms of the determinant as follows:*

$$\vec{A} \wedge \vec{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \vec{\imath} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \vec{\jmath} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \vec{k}$$

$$\vec{A} \wedge \vec{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y\, B_z - B_y A_z)\vec{\imath} - (A_x\, B_z - B_x A_z)\vec{\jmath} + (A_x\, B_y - B_x A_y)\,\vec{k}$$

- ✓ *The vector (Cross) product of the unit vector is given as follow:*

$$\vec{\imath} \wedge \vec{\jmath} = -\vec{\jmath} \wedge \vec{\imath} = \vec{k}$$
$$\vec{\jmath} \wedge \vec{k} = -\vec{k} \wedge \vec{\jmath} = \vec{\imath}$$
$$\vec{k} \wedge \vec{\imath} = -\vec{\imath} \wedge \vec{k} = \vec{\jmath}$$
$$\vec{\imath} \wedge \vec{\imath} = \vec{\jmath} \wedge \vec{\jmath} = \vec{k} \wedge \vec{k} = \vec{0}$$

- ✓ *If the components of vectors* $\vec{A}$ and $\vec{B}$ *are time-dependent, the derivative of the vector (cross) product is expressed as follows:*

$$\frac{d(\vec{A} \wedge \vec{B})}{dt} = \frac{d(\vec{A})}{dt} \wedge \vec{B} + \vec{A} \wedge \frac{d(\vec{B})}{dt}$$

11-3) Mixed Product

The result of the mixed product of vectors is a scalar value, calculated by the determinant as follows:

$$\vec{C}\cdot(\vec{A}\wedge\vec{B}) = \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} C_x - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} C_y + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} C_z$$

$$\vec{C}\cdot(\vec{A}\wedge\vec{B}) = \left(A_y B_z - B_y A_z\right)C_x - \left(A_x B_z - B_x A_z\right)C_y + \left(A_x B_y - B_x A_y\right) C_z$$

Exercise 01:

Consider the following vectors:

$\vec{A} = 4\vec{\imath} + 2\vec{\jmath}$ $\qquad$ $\vec{B} = 2\vec{\imath} + \vec{\jmath} - \vec{k}$

1- Calculate the magnitude of $\vec{A}$ and $\vec{B}$, the dot product $\vec{A}.\vec{B}$ and the cross product $\vec{A}\Lambda\vec{B}$

2- Calculate the angle θ formed by the two vectors

Solution

1- Calculate the magnitude of $\vec{A}$ and $\vec{B}$, the dot product $\vec{A}.\vec{B}$ and the cross product $\vec{A}\Lambda\vec{B}$

$\|\vec{A}\| = \sqrt{(4)^2 + (2)^2} = \sqrt{20}$

$\|\vec{B}\| = \sqrt{(2)^2 + (1)^2 + (-1)^2} = \sqrt{6}$

$\vec{A}.\vec{B} = 4 * 2 + 2 * 1 + 0 * (-1) = 10$

$$\vec{A}\wedge\vec{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 4 & 2 & 0 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix}\vec{\imath} - \begin{vmatrix} 4 & 0 \\ 2 & -1 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix}\vec{k} = -2\vec{\imath} - 4\vec{\jmath} + 0\,\vec{k}$$

2- Calculate the angle θ formed by the two vectors

$$\vec{A}.\vec{B} = \|\vec{A}\| \times \|\vec{B}\| \cos(\vartheta) = 10$$

$$\|\vec{A}\| \times \|\vec{B}\| \cos(\vartheta) = \frac{\vec{A}.\vec{B}}{\|\vec{A}\| \times \|\vec{B}\|} = \frac{10}{\sqrt{20 \times 6}} = 0.912$$

$$\vartheta = 24.21^0$$