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Lessons and exercises on

Variational methods for nonlinear elliptic problems

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Chapter 1

Operators on Banach spaces

1.1 Some notions in functional analysis

1.1.1 Dual space

Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be two Banach spaces.

Definition 1.1 *Let $A : E \to F$ be a linear operator. One says that A is bounded (or continuous) if there is a constant $c \geq 0$ such that:*

$$\|Au\|_F \leq c\|u\|_E, \quad \forall u \in E.$$

Notation 1.1

- *We denote by $\mathcal{L}(E, F)$ the space of continuous (bounded) linear operators from E into F equipped with the norm*

$$\|T\|_{\mathcal{L}(E,F)} = \sup_{x \in E, \|x\|_E \leq 1} \|T(x)\|_F$$

- *If $E = F$, we write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$.*

- *Let E be a vector space over $\mathbb{R}$. We recall that **a functional** is a function defined on E, or on some subspace of E, with values in $\mathbb{R}$. We denote by E' the **dual space** of E, that is, the space of all continuous linear functionals on E; the dual norm on E (the norm in E') is defined by*

$$\|f\|_{E'} = \sup_{x \in E, \|x\|_E \leq 1} |f(x)| = \sup_{x \in E, \|x\|_E \leq 1} f(x).$$

- *When there is no confusion we shall also write $\|f\|$ instead of $\|f\|_{E'}$.*

- *Given $f \in E'$ and $x \in E$ we shall often write $\langle f, x\rangle$ instead of $f(x)$; we say that $\langle \cdot, \cdot \rangle$ is the scalar product for the duality E', E. So, we have the following inequality*

$$\forall f \in E', \forall x \in E : |\langle f, x\rangle| \leq \|f\|_{E'}\|x\|_E. \tag{1.1}$$

- *It is well known that E' is a Banach space, i.e., E' is complete (even if E is not); this follows from the fact that $\mathbb{R}$ is complete.*

- *If E is a Hilbert space, then, according to Riesz's theorem (see [1]), the spaces E and its dual E' can be identified, we write*

$$E' = E. \tag{1.2}$$

The equality (1.2) does not mean equal elements, but rather in the sense that there is a canonical isometry from E onto E'.

Example 1.1 *Let $1 < p < \infty$, we have $\Big(L^p(\Omega)\Big)' = L^q(\Omega)$ where $\frac{1}{p}+\frac{1}{q} = 1$. $\Big(L^1(\Omega)\Big)' = L^\infty(\Omega)$.*

1.1.2 Some notions of convergence

Definition 1.2 (Strong convergence) *Convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ in a normed vector space $(E, \|\cdot\|_E)$ to an element $x \in E$, defined in the following way:*

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \Rightarrow \|x_n - x\|_E \leq \varepsilon.$$

Or,

$$\lim_{n\to+\infty} \|x_n - x\|_E = 0.$$

We denote by $x_n \to +x$ as $n \to +\infty$.

Definition 1.3 *We say that the sequence $(x_n)_{n\in\mathbb{N}} \subset E$ converges weakly to $x \in E$ if and only if*

$$\langle f, x_n\rangle \to \langle f, x\rangle, \quad \forall f \in E'. \tag{1.3}$$

for $n \to +\infty$. This weak convergence is written as $x_n \rightharpoonup x$,

Remark 1.1

- *Note that the convergence in (1.3) is convergence for a sequence of real numbers.*
- *If $x_n \rightharpoonup x$, then $(x_n)_{n\in\mathbb{N}}$ is bounded in E; i.e.*

$$\exists M > 0 : \quad \|x_n\| \leq M.$$

Example 1.2

1. *Let $\{e_n\}_{n\in\mathbb{N}}$ be an Hilbertian base for a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Then, for each $x \in H$ can be decomposed as*

$$x = \sum_{n=1}^{\infty} \langle e_n, x\rangle e_n$$

Or

$$\|x - \sum_{n=1}^{m} \langle e_m, x\rangle e_m\| \to 0,$$

as $m \to +\infty$. From this it follows that $\langle e_n, x\rangle \to 0 = \langle 0, x\rangle$ for any $x \in H' = H$, this means that $e_n \rightharpoonup 0$ in H.

On the other hand, we have $\|e_n - 0\| = \|e_n\| = 1$, so $\{e_n\}$ does not converge. We deduce that $\{e_n\}_n$ is weakly convergent but it is not strongly convergent.

2. *Let* $\Omega \subset \mathbb{R}^N$ *an open set, let* $1 < p < \infty$*. A sequence* $u_n \rightharpoonup u$ *in* $L^p(\Omega)$ *means that*

$$\int_\Omega u_n v \, dx \to \int_\Omega uv \, dx, \quad \forall \in v \in L^q(\Omega),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

3. *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be the sequence of functions defined by*

$$f_n(x) = \begin{cases} \frac{1}{\sqrt{n}}, & \text{if } x \in [n, 2n]; \\ 0, & \text{if not.} \end{cases}$$

We can verify that $\{f_n\}_{n\in\mathbb{N}}$ *converges weakly to* 0 *in* $L^2([0;+\infty[)$ *but does not converge strongly in* $L^2([0;+\infty[)$.

1.1.3 Reflexive space

Definition 1.4 *Let* $(E, \|\cdot\|)$ *be a Banach space and let* E' *be the dual space. The bidual* E'' *is the dual of* E' *with norm*

$$\|f\|_{E''} = \sup_{g\in E', \|g\|_{E'}\leq 1} |\langle f, g\rangle|, \quad \forall f \in E''.$$

There is a canonical injection $i : E \to E''$ defined as follows:

$$\begin{aligned} i : E &\to E'' \\ x &\mapsto ix, \end{aligned}$$

where ix is defined as:

$$\begin{aligned} ix : E' &\to \mathbb{R} \\ f &\mapsto \langle ix, f\rangle_{E'',E'} = \langle f, x\rangle_{E',E}, \end{aligned}$$

It is clear that i is linear and that i is an isometry, that is,

$$\|ix\|_{E''} = \|x\|_E. \qquad \text{(exercice)}$$

Definition 1.5 (Reflexive space) *The space* E *is said to be reflexive if the canonical injection* $i : E \to E''$ *is surjective, i.e.* $i(E) = E''$.

When E is reflexive, E'' is usually identified with E; $(E'' = E)$.

Example 1.3 1. *finite dimensional spaces are reflexive (since* $\dim E = \dim E' = \dim E''$*).*

2. L^p *spaces are reflexive for* $1 < p < \infty$*.* L^1 *and* L^∞ *are not reflexive.*

3. *Hilbert spaces are reflexive.*

The following result shows an important compactness property in reflexive spaces (see [1])

Theorem 1.2 *Assume that* E *is a reflexive Banach space and let*(x_n) *be a bounded sequence in* E*. Then there exists a subsequence* (x_{n_k}) *that converges weakly to some* x *in* E*.*

1.1.4 Continuity, coercivity and convexity of operators

In the following, we state some definitions concerning continuity properties.

Definition 1.6 *The operator $T : E \to F$ is called continuous at the point $x_0 \in E$ if for any sequence $(x_n) \subset E$*

$$\forall (x_n) \subset E : \quad x_n \to x_0 \ \text{ in } \ E \ \Rightarrow \ T(x_n) \to T(x_0) \ \text{ in } F$$
$$\Big(\|x_n - x_0\|_E \to 0 \ \Rightarrow \ \|T(x_n) - T(x_0)\|_F \to 0 \Big)$$

This definition is equivalent to:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in E : \|x - x_0\|_E \leq \delta \Rightarrow \|T(x) - T(x_0)\|_F \leq \varepsilon.$$

Example 1.4 *Let $(E, \|\cdot\|)$ be a Banach space. We consider $N : E \to \mathbb{R}$; $N(x) = \|x\|$. N is a continuous functional. Indeed, for any $(x_n)_n \subset E$ such that $x_n \to x$ in E, i.e. ($\|x_n - x\| \to 0$). Since*

$$\Big| N(x_n) - N(x) \Big| = \Big| \|x_n\| - \|x\| \Big| \leq \|x_n - x\|,$$

then we have $N(x_n) \to N(x)$ in $\mathbb{R}$.

There are some other definitions that are related to the concept of weak convergence.

Definition 1.7 *Let $T : E \to F$ be an operator. T is called weakly continuous at $x_0 \in E$ if for every sequence $(x_n) \subset E$, we have*

$$x_n \rightharpoonup x_0 \quad \text{in } \ E \Rightarrow T(x_n) \rightharpoonup T(x_0).$$

Remark 1.2 *For functionals defined on E, the concepts of convergence and weak convergence coincide. So the functional $T :\to \mathbb{R}$ is called weakly continuous if for every sequence $(x_n) \subset E$, we have*

$$x_n \rightharpoonup x_0 \quad \text{in } \ E \Rightarrow T(x_n) \to T(x_0) \quad \text{in} \mathbb{R}.$$

Note that the concept of weak continuity is stronger than the concept of continuity because the convergence implies the weak convergence.

Definition 1.8 (Compact operator) *An operator $T : E \to F$ is called* compact *if it is continuous and $T(B)$ is relatively compact set in F (i.e. $\overline{T(B)}$ is compact set in F); here $B = \{x \in E : \|x\|_E < 1\}$.*

There is an equivalent definition using sequences.

Definition 1.9 *An operator $T : E \to F$ is called* compact *if it is continuous and if every bounded sequence $(x_n)_{n \in \mathbb{N}} \subset E$ has a subsequence (x_{n_k}) for which $\Big(T(x_{n_k})\Big)$ converges.*

Remark 1.3 *If T is a linear operator then the requirement that T is continuous on E, which then equivalent that T is bounded operator, is superfluous*

The compact operators defined on reflexive spaces have an important property that appears in the following theorem.

Theorem 1.3 *Let E be a reflexive Banach space and F a Banach space. If $T : E \to F$ is linear and compact. Then we have*

$$x_n \rightharpoonup x_0 \ \text{ in } \ E \Rightarrow T(x_n) \to T(x_0) \ \text{ in } F$$

Example 1.5 *The application norm $E \ni x \mapsto \|x\|$ is a compact operator.*

In the search for the global minimum, which we will discuss in Chapter 2, a related concept is convexity and coercivity.

Definition 1.10 1. *We say that a part K of E is convex if On dit qu'une partie K de X est convexe si:*

$$\forall x, y \in K, \quad \forall \theta \in [0,1], \quad \theta x + (1-\theta)y \in K.$$

2. *When K is convex and $J : K \longrightarrow \mathbb{R}$ is a functional. We say that J is convex if:*

$$\forall x, y \in K, \quad \forall \theta \in [0,1], \quad J(\theta x + (1-\theta)y) \leq \theta J(x) + (1-\theta)J(y).$$

We say that J is strictly convex if:

$$\forall x, y \in K, \quad avec \quad x \neq y \quad \forall \theta \in]0,1[, \quad J(\theta x + (1-\theta)y) < \theta J(x) + (1-\theta)J(y).$$

3. *We say that J is (strictly) concave if $-J$ is (strictly) convex.*

Definition 1.11 *A functional $J : E \to \mathbb{R}$ on a Banach space E is called coercive if, for every sequence $(u_n)_{n\in\mathbb{N}} \subset E$,*

$$\lim_{\|u_n\| \to +\infty} J(u_n) = +\infty.$$

1.1.5 Lower and Upper Semi-contiwuity

The following concepts are often found in minimization theory of functionals.

Definition 1.12 1. *A functional $J : E\mathbb{R}$ is called* lower semi-continuous *at a point $x_0 \in E$ if for every sequence $(x_n) \subset E$, we have:*

$$x_n \to x_0 \Rightarrow f(x_0) \leq \liminf_{n\to+\infty} f(x_n).$$

2. *$J : E\mathbb{R}$ is called* weakly lower semi-continuous ***(w.l.s.c)*** *at a point $x_0 \in E$ if for every sequence $(x_n) \subset E$, we have:*

$$x_n \rightharpoonup x_0 \Rightarrow f(x_0) \leq \liminf_{n\to+\infty} f(x_n).$$

3. *$J : E\mathbb{R}$ is called* upper semi-continuous *at a point $x_0 \in E$ if for every sequence $(x_n) \subset E$, we have:*

$$x_n \to x_0 \Rightarrow f(x_0) \geq \limsup_{n\to+\infty} f(x_n).$$

4. $J : E\mathbb{R}$ *is called* weakly upper semi-continuous ***(w.u.s.c)*** *at a point* $x_0 \in E$ *if for every sequence* $(x_n) \subset E$*, we have:*

$$x_n \rightharpoonup x_0 \Rightarrow f(x_0) \geq \limsup_{n\to+\infty} f(x_n).$$

Example 1.6 *Let* $(H, \langle\cdot,\cdot\rangle$ *be a Hilbert space. The functional* $N : E \ni x \mapsto N(x) = \|x\|$*, where* $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$*. We have* N *is w.l.s.c. Indeed; let* $(x_n) \subset H$ *and* $x_0 \in H$ *such that* $x_n \rightharpoonup x_0$*, then*

$$\langle x_n, y\rangle \to \langle x, y\rangle, \forall y \in H' = H,$$

in particular, taking $y = x_0 \in H$*, we get*

$$\langle x_n, x_0\rangle \to \langle x_0, x_0\rangle = \|x_0\|^2. \tag{1.4}$$

By a direct calculation, we get

$$\begin{aligned} \|x_n - x_0\|^2 &= \langle x_n - x_0, x_n - x_0\rangle \\ &= \|x_n\|^2 - 2\langle x_n, x_0\rangle + \|x_0\|^2. \end{aligned}$$

Since $\|x_n - x_0\|^2 \geq 0$*, we deduce that*

$$\|x_n\|^2 \geq 2\langle x_n, x_0\rangle - \|x_0\|^2. \tag{1.5}$$

Passing to the limit in (1.5), and using (1.4), we get

$$\liminf_{n\to+\infty} \|x_n\|^2 \geq 2\|x_0\|^2 - \|x_0\|^2 = \|x_0\|^2.$$

which implies that N *is w.l.s.c.*

1.2 Functional spaces

In this chapter, we briefly mention some spaces and their most important properties and theories, which we will need in the following lessons. For more details see e.g [2].

1.2.1 L^p Spaces

Definition 1.13 *Let* $p \in \mathbb{R}$ *with* $1 \leq p < \infty$*, we set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}^N ; f \text{ is measurable and } \int_\Omega |f(x)|^p \, dx < \infty \right\}.$$

with

$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p \, dx\right]^{\frac{1}{p}}.$$

Definition 1.14 *We set*

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R}^N ; f \text{is measurable and } |f(x)| \leq c \ \ a.e.on \ \ \Omega \right\},$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = inf\,\{c : |f(x)| \leq c \ \ a.e.on \ \ \Omega\},$$

with c *is a constant.*

If $f \in L^\infty(\Omega)$ *then we have* $|f(x)| \leq \|f\|_\infty$ *a.e. in* Ω.

Theorem 1.4 *$L^p(\Omega)$ is a Banach space for any $1 \leq p \leq \infty$, and reflexive for $1 \leq p < \infty$, and separable for $1 \leq p < \infty$.*

Notation 1.5 *Let $1 \leq p \leq \infty$, we denote by p' the conjugate exponent*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 1.6 (Hölder's inequality) *Assume that $f \in L^p$ and $g \in L^p$ with $1 \leq p \leq \infty$, then $fg \in L^1$ and*

$$\int |fg| \leq \|f\|_{L^p} \|g\|_{L^{p'}}.$$

1.2.2 Some Results about Integration

Theorem 1.7 *(Lebesgue's Dominated Convergence Theorem) Let (f_n) be a sequence of functions of L^1. We suppose that*

1. *$f_n(x) \longrightarrow f(x)$ a.e.on Ω,*
2. *there exists a function $g \in L^1$ such that for every n, $|f_n(x)| \leq g(x)$ a.e.in Ω.*

We have $f \in L^1(\Omega)$ and $\|f_n - f\|_{L^1} \longrightarrow 0$.

Theorem 1.8 *(Lebesgue's Dominated Convergence Inverse Theorem) Let (f_n) be a sequence in L^p and let $f \in L^p$ be such that $\|f_n - f\|_{L^p} \longrightarrow 0$.*
Then, there exist a subsequence (f_{n_k}) and a function $h \in L^p$ such that

1. *$f_{n_k}(x) \longrightarrow f(x)$ a.e.on Ω,*
2. *$|f_{n_k}(x)| \leq h(x) \quad \forall k$, a.e.in Ω.*

Lemma 1.1 (Fatou Lemma) *Let (f_n) be a sequence of functions in L^1 that satisfy*

1. *for all $n \in \mathbb{N}$, $f_n \geq 0$ a.e.*
2. *$\sup_n \int f_n < \infty$.*

For almost $x \in \Omega$ we set $f(x) = \liminf_{n \to +\infty} f_n(x)$. Then $f \in L^1$ and

$$\int f \leq \liminf_{n \to +\infty} \int f_n.$$

1.2.3 $W^{1.p}(\Omega)$ Spaces

Let $\Omega \in \mathbb{R}^N$ be an open set, and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.15 *We denote by $\mathcal{D}(\Omega)$ the set of function of class $\mathcal{C}^\infty(\Omega)$ with support compact include in Ω. The Sobolev space $W^{1.p}(\Omega)$ is defined by*

$$W^{1.p}(\Omega) = \left\{ u \in L^p(\Omega), \exists g_i \in L^p(\Omega) \text{such that:} \int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega g_i \varphi dx, \forall \varphi \in \mathcal{D}(\Omega), \forall i = 1, 2, ..., N \right\}.$$

We set

$$H^1(\Omega) = W^{1.2}(\Omega).$$

For $u \in W^{1.p}(\Omega)$ *we set* $\frac{\partial u}{\partial x_i} = g_i$, *and we write*

$$\nabla u = grad\ u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N}).$$

The space $W^{1.p}$ *is equipped with the norm* $\|u\|_{W^{1.P}} = \|u\|_{L^P} + \|\nabla u\|_{L^P}$, *or sometimes with the equivalent norm* $\|u\|_{W^{1.P}} = (\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p)^{\frac{1}{p}}$ *if* $(1 \leq p < \infty)$.
The space $H^1(\Omega)$ *is equipped with the scalar product*

$$\int_\Omega uvdx + \int_\Omega \nabla u \nabla v dx.$$

The associated norm $\|u\|_{H^1} = (\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)^{\frac{1}{2}}$ *is equivalent to the* $W^{1.2}$ *norm.*

Proposition 1.1 *[?].*$W^{1.p}(\Omega)$ *is a Banach space for* $1 \leq p \leq \infty$, *reflexive for* $1 < p < \infty$, *and separable for* $1 \leq p < \infty$.
In particular $H^1(\Omega)$ *is reflexive, separable and Hilbert space.*

1.2.4 Sobolev Injections and Inequalities

Let $(E, \|.\|_E), (F, \|.\|_F)$ Banach spaces

Notation 1.9

1. E is injected continuously into F, means that the canonical injection $j : E \to F$ is continuous i.e, $\exists c > 0, \forall x \in E : \|x\|_F \leq c\|x\|_E$, and we denote by $E \hookrightarrow F$.

2. E is injected in compact into F means that the canonical injection $j : E \to F$ is compact i.e for all sequence bounded u_n in E we can extract subsequence u_{nk} convergent in F, and we denote by $E \hookrightarrow_c F$.
   If $1 \leq p < \infty$, the Sobolev exponent of p defined by $p^* = \frac{Np}{N-p}$ or $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

Theorem 1.10 *Let* $1 \leq p \leq \infty$, *we suppose that* Ω *is on open set of class* $\mathcal{C}^1$ *a bounded frontier, and we take* $\Omega = \mathbb{R}_+^N$

1. $W^{1.p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [1, p^*[\quad$ *if* $p < N$.
2. $W^{1.p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [p, \infty[\quad$ *if* $p = N$.
3. $W^{1.p}(\Omega) \hookrightarrow L^\infty(\Omega)$ *if* $p > N$.

Theorem 1.11 (Rellich-Kondrachon) *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain of class* c^1

1. $W^{1.p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [1.p^*[\quad$ *if* $p < N$.
2. $W^{1.p}(\Omega) \hookrightarrow_c L^q(\Omega) \quad \forall q \in [p.\infty[\quad$ *if* $p = N$.
3. $W^{1.p}(\Omega) \subset \mathcal{C}\left(\overline{\Omega}\right)$ *if* $p > N$.

1.2.5 $W_0^{1.p}(\Omega)$ Space

Definition 1.16 *.Let $1 \leq p \leq \infty$, $W_0^{1.p}$ means the closing of $\mathcal{D}(\Omega)$ in $W^{1.p}$, we notice*

$$\begin{aligned} W_0^{1.p}(\Omega) &= \overline{\mathcal{D}(\Omega)}^{W^{1.p}} \\ &= \left\{ u \in W^{1.P}(\Omega) : u = 0 \ sur \ \partial\Omega \right\}, \end{aligned}$$

and

$$H_0^1(\Omega) = W_0^{1.2}(\Omega).$$

The space $W_0^{1.p}(\Omega)$ provided with norm induced by $W^{1.p}$, H_0^1 is a Hilbert space for the scalar product of H^1 we put $\|u\|_{H_0^1} = \|u\|$.

Remark 1.4 *When $\Omega = \mathbb{R}^N$, we know that $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{1.p}(\mathbb{R}^N)$, and there for*

$$W_0^{1.p}(\mathbb{R}^N) = W^{1.p}(\mathbb{R}^N).$$

Proposition 1.2 (Poincaré's inequality) *Let $\Omega \subset \mathbb{R}^N$ on open set, Then there exists a constant $c > 0$ such that*

$$\|u\|_{W^{1.p}(\Omega)} \leq c \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1.p}(\Omega).$$

In other words, on $W_0^{1.p}$, the quantity $\|\nabla u\|_{L^p(\Omega)}$ is a norm equivalent to the $W^{1.p}$ norm.

Theorem 1.12 (Young's inequality) *. For $a, b \geq 0$ and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

1.3 Differentiability of Functionals and Critical Points

In what follows, we present the two principal definitions of differentiability and their main properties, we start with the directional derivative.

1.3.1 Gâteaux differentiability

Definition 1.17 *Let E be a Banach space, $\Omega \subset E$ an open set, and let $I : \Omega \longrightarrow \mathbb{R}$ be a functional, we say that I is Gâteau differentiable (G-differentiable) at $u \in \Omega$, if there exists $A \in E'$ (A linear and continuous), denoted by $I'_G(u)$ such that, for all $v \in E$,*

$$\lim_{t \to 0} \frac{I(u + tv) - I(u)}{t} = \langle A, v \rangle, \tag{1.6}$$

If I is Gâteaux differentiable at u, there exists only one linear functional $A \in E'$ satisfying (1.6). It is called the Gâteaux differential of I at u and is denoted by $I'_G(u)$.

Definition 1.18 *If the functional I is Gâteaux differentiable at every u of an open set $U \subset E$, we say that I is Gâteaux differentiable on U. The map*

$$\begin{array}{rcl} I'_G : E & \to & E' \\ u & \mapsto & I'_G(u) \end{array}$$

is called the Gâteaux derivative of I.

Example 1.7 *Let $1 < p < \infty$, the functional*

$$\begin{array}{rcl} I : L^p(\Omega) & \to & \mathbb{R} \\ u & \mapsto & I(u) = \int_\Omega |u|^p \, dx \end{array}$$

is Gâteaux differentiable and we have

$$\langle I'_G(u), v\rangle = p \int_\Omega |u|^{p-2} uv dx.$$

Indeed; let $v \in L^p(\Omega)$, we have

$$\lim_{t\to 0} \frac{I(u+tv) - I(u)}{t} = \lim_{t\to 0} \frac{\int_\Omega |u+tv|^p \, dx - \int_\Omega |u|^p \, dx}{t}.$$

We define

$$\begin{array}{rcl} g_{u,v} : [0,t] & \to & \mathbb{R} \\ s & = & g_{u.v}(s) = |u+sv|^p \, . \end{array}$$

since $g_{u,v}$ is continuous on $[0,t]$ and differentiable on $]0,t[$, then according to the mean value theorem, there exists a real $c_t \in]0,t[$ such that

$$g_{u,v}(t) - g_{u,v}(0) = g'_{u,v}(c_t)t$$

or,

$$|u+tv|^p - |u|^p = p \, |u+c_t v|^{p-2} \, (u+c_t v)vt,$$

when $t \to 0$, we have $c_t \to 0$, so

$$\begin{aligned} \lim_{t\to 0} \frac{|u+tv|^p - |u|^p}{t} &= \lim_{c_t \longrightarrow 0} p|u+c_t v|^{p-2}(u+c_t v)v \\ &= p|u|^{p-2}uv. \end{aligned} \tag{1.7}$$

Let us recall by the following preliminary inequality,

$$(a+b)^p \leq 2^{p-1}(a^p + b^p), \forall p > 1, \ ; \forall a, b > 0. \tag{1.8}$$

Using (1.8), we get

$$\left| |u+c_t v|^{p-2}(u+c_t v)v \right| \leq |u+c_t v|^{p-1}|v| \leq 2^{p-1} \leq C(|u|^{p-1}|v| + |v|^p).$$

We verify that $|u|^{p-1}|v|+|v|^p)\in L^1(\Omega)$*; that is, by Hölder inequality we get*

$$\int_\Omega |u|^{p-1}|v|\ dx \le \|u\|_{L^p}^{p-1}\|v\|_{L^p}<\infty$$

then $|u|^{p-1}|v|+|v|^p)\in L^1(\Omega)$*. Using this and (1.7), applying Dominated Convergence Theorem, we get*

$$\lim_{t\to 0}\frac{\int_\Omega |u+tv|^p\ dx-\int_\Omega |u|^p\ dx}{t}=p\int_\Omega |u|^{p-2}uvdx.$$

Now, we define

$$\begin{array}{rcl} A: L^p & \to & \mathbb{R}\\ v & \mapsto & A(v)=p\int_\Omega |u|^{p-2}uvdx. \end{array}$$

we prove that $A\in(L^p(\Omega))'$*, i.e. A is linear and continuous.*
A is linear, that is, let $v_1,v_2\in L^p(\Omega)$ *and let* $\alpha,\beta\in\mathbb{R}$

$$\begin{array}{rcl} A(\alpha v_1+\beta v_2) & = & p\int_\Omega |u|^{p-2}u(\alpha v_1+\beta v_2)dx\\ & = & p[\int_\Omega |u|^{p-2}u\alpha v_1dx+\int_\Omega |u|^{p-2}u\beta v_2dx]\\ & = & \alpha p\int_\Omega |u|^{p-2}uv_1dx+\beta p\int_\Omega |u|^{p-2}uv_2dx\\ & = & \alpha A(v_1)+\beta A(v_2). \end{array}$$

then A is linear. For continuity of A, let $u,v\in L^p(\Omega)$

$$\begin{array}{rcl} |A(v)|=|p\int_\Omega |u|^{p-2}uvdx| & \le & p\int_\Omega |u|^{p-1}|v|dx\\ & \le & p(\int_\Omega |u|^{(p-1)p'}dx)^{\frac{1}{p'}}(\int_\Omega |v|^pdx)^{\frac{1}{p}}\\ & \le & p(\int_\Omega |u|^{(p-1)(\frac{p}{p-1})}dx)^{\frac{p-1}{p}}(\int_\Omega |v|^pdx)^{\frac{1}{p}}\\ & = & \|u\|_{L^p}^{p-1}\|v\|_{L^p}. \end{array}$$

Then A is a continuous, so the functional I is G-differentiable and $\langle I'_G(u),v\rangle=p\int_\Omega |u|^{p-2}uvdx.$

Example 1.8 *In the previous example 1.7, if we set* $p=2$*, the functional*

$$\begin{array}{rcl} I: L^2(\Omega) & \to & \mathbb{R}\\ u & \mapsto & I(u)=\int_\Omega |u|^2\ dx=\|u\|_{L^2}^2 \end{array}$$

is Gâteaux differentiable and we have

$$\langle I'_G(u),v\rangle=2\int_\Omega uv\ dx.$$

Example 1.9 *The functional defined by*

$$\begin{array}{rcl} J: W_0^{1,p}(\Omega) & \to & \mathbb{R}\\ u & \mapsto & J(u)=\int_\Omega |\nabla u|^p\ dx=\|u\|_{W_0^{1,p}}^2 \end{array}$$

is Gâteaux differentiable and we have

$$\langle J'_G(u),v\rangle=p\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\ dx.$$

1.3.2 Fréchet differentiability

Definition 1.19 *Let E be a Banach space, $\Omega \subset E$ an open set and let $I : \Omega \longrightarrow \mathbb{R}$ be a functional, we say that I is Fréchet differentiable at $u \in \Omega$, if there exists $A_u \in E'$ such that*

$$\lim_{\|v\| \to 0} \frac{I(u+v) - I(v) - A_u v}{\|v\|} = 0 \tag{1.9}$$

Thus, for a Fréchet differentiable functional I, we have

$$I(u+v) - I(u) = A_u(v) + o(\|v\|)$$

as $\|v\| \to 0$.

If a functional I is differentiable at u, then A_u is a unique.

Definition 1.20 *Let $I : \Omega \to \mathbb{R}$ be differentiable at $u \in \Omega$. The unique element A_u such that (1.9) holds is called the Fréchet differential of I at u, and is denoted by $I'(u)$. Then we can write*

$$I(u+v) - I(u) = \langle I'(u), v \rangle + o(\|v\|)$$

as $\|v\| \to 0$.

Definition 1.21 *Let $\Omega \subset E$ be an open set.*

1. *If the functional I is differentiable at every $u \in \Omega$, we say that I is differentiable on Ω.*
2. *The map $I' : \Omega \to E'$ that sends $u \in \Omega$ to $I'(u) \in E'$ is called the Fréchet derivative of I. Note that I' is in general a nonlinear map.*
3. *If the derivative I' is continuous from Ω to E' we say that I is of class C^1 on Ω and we write $I \in C^1(\Omega)$.*

Remark 1.5
- *Notice that $I'(u)$ is defined on E, even if I is defined only in Ω.*
- *If I is Fréchet differentiable at $u \in \Omega$ then I is continuous at u.*
- *If I is Fréchet differentiable at $u \in \Omega$ then I is Gâteaux differentiable at $u \in \Omega$ and $I'(u) = I'_G(u)$.*

The converse of the third point of Remark 1.5 is not always true but we have the following result:

Proposition 1.3 *Suppose that $\Omega \subseteq E$ is on open set, such that I G-differentiable in Ω and that $I'_G : \Omega \in E \to E'$ is continuous, then I is also Fréchet differentiable at u, and we have $I'_G = I'(u)$.*

The proof of Proposition 1.3 is given in [3].

Remark 1.6 *The importance of Proposition 1.3 reside in the fact that calculate the Gâteau derivative and then to prove that it is continuous is often technically easier to directly prove the Fréchet differentiability. Also we deduce that the functional is of class C^1.*

Example 1.10 *We prove that the functional $J : L^p(]0,1[) \to \mathbb{R}$ $(p > 2)$ defined by*

$$u \to J(u) = \int_0^1 |u|^p dx.$$

is a Fréchet differentiable on $L^p(]0,1[)$. Indeed; we already prove that J is G-differentiable, with

$$\langle J'_G(u), v\rangle = p\int_0^1 |u|^{p-1} uv\, dx, \quad \forall u, v \in L^p(]0,1[). \tag{1.10}$$

it remains to prove that

$$\begin{array}{rcl} J'_G : L^p(\Omega) & \to & (L^p(\Omega))' \\ u & \mapsto & J'_G(u) \end{array}$$

where $J'_G(u)$ is defined as in (1.10).

Let $(u_n) \subset L^p(\Omega)$ such that

$$u_n \to u \;\; in\; L^p(\Omega),$$

we prove that $J'_G(u_n) \to J'_G(u)$ in $(L^p(\Omega))'$. By definition we have,

$$\|J'_G(u_n) - J'_G(u)\|_{(L^p(\Omega))'} = \sup_{\|v\|_{L^p}=1} |\langle J'_G(u_n) - J'_G(u), v\rangle|$$

let $v \in L^p(\Omega)$, such that $\|v\|_{L^p} = 1$, we have

$$\begin{aligned} |\langle J'_G(u_n) - J'_G(u), v\rangle| &= |\langle J'_G(u_n), v\rangle - \langle J'_G(u), v\rangle| \\ &= p|\int_0^1 |u_n|^{p-2}u_n v dx - \int_0^1 |u|^{p-2}uv\, dx| \\ &\le p|\int_0^1 |u_n|^{p-2}u_n v dx - \int_0^1 |u|^{p-2}uv dx| \\ &\le p[\int_0^1 ||u_n|^{p-2}u_n - |u|^{p-2}u|^{\frac{p}{p-1}} dx](\int_0^1 |v|^p dx)^{\frac{1}{p}} \\ &\le p\int_0^1 ||u_n|^{p-2}u_n - |u|^{p-2}u|^{\frac{p}{p-1}} dx. \end{aligned}$$

We set $w_n = ||u_n|^{p-2}u_n - |u|^{p-2}u|^{\frac{p}{p-1}}$, we know that $u_n \to u$ in $L^p(0,1)$, there is a subsequence $u_{nk} = u_n$, such that $u_n(x) \to u(x)$ a.e. in $]0,1[$, so we have

$$w_n(x) \to 0 a.e.\; in]0,1[. \tag{1.11}$$

Moreover there exists $g \in L^p$ such that

$$|u_n(x)| \le g(x) \;\; a.e.\;\; in\;]0,1[.$$

Thus,

$$|u_n(x)|^{p-1} \le |g(x)|^{p-1} \quad \in L^{\frac{p}{p-1}}(0,1) \subset L^1(0,1). \tag{1.12}$$

According (1.11) and (1.12), by the dominated convergence theorem we have $w_n \to 0$ in $L^1(\Omega)$; that is

$$\int_0^1 ||u_n|^{p-2}u_n - |u|^{p-2}u|^{\frac{p}{p-1}} dx \to 0,$$

as $n \to +\infty$. Consequently

$$|\langle J'_G(u_n) - J'_G(u), v\rangle| \to 0.$$

as $n \to +\infty$, which means that $J'_G(u_n) \to J'_G(u)$ in $\left(L^p(0,1)\right)'$, so J'_G is continuous functional. In conclusion, the functional J is Fréchet differentiable and it is of class C^1.

1.3.3 Critical points

Definition 1.22 *Let Ω be an open set of Banach space E, assume that $I : \Omega \to \mathbb{R}$ is differentiable. We say that $u \in \Omega$ is a critical point of I, if*

$$I'(u) = 0.$$

Remark 1.7

1. *As $I'(u)$ is an element of the dual space E', then $I'(u) = 0$ means that $\langle I'(u), v\rangle$ for all $v \in E$.*
2. *If u is not critical point, then we say that u is regular point of I.*
3. *If $I(u) = c$ and $I'(u) = 0$ we say that u is a critical point for I at level c and we say that c is a critical level for I. If c is not a critical level, then c is called a regular level for I.*
4. *The equation $I'(u) = 0$ is called the Euler, or Euler-Lagrange equation associated to the functional I.*

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