Chapter 2
Minimization Techniques: Compact Problems

Throughout this chapter we show how techniques based on minimization arguments can be used to establish existence results for various types of problems.

Our aim is not to describe the most general results, but to give a series of examples, and to show how simple techniques can be refined to treat more complex cases.

2.1 Coercive Problems

We begin with the following problem, that will provide our guideline through the whole chapter. We want to find a (weak) solution to

$$\begin{cases} -\Delta u + q(x)u = f(u) + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

In this section, the general framework is specified by the assumptions

($\mathbf{h_1}$) $\Omega \subset \mathbb{R}^N$ is bounded and open, $q \in L^\infty(\Omega)$ and $q(x) \geq 0$ a.e. in Ω.
($\mathbf{h_2}$) $h \in L^2(\Omega)$.

We equip $H_0^1(\Omega)$ with the scalar product

$$(u|v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega q(x)uv \, dx, \tag{2.2}$$

and we denote by $\|\cdot\|$ the induced norm, equivalent to the standard one.

Remark 2.1.1 In assumption ($\mathbf{h_1}$), the requirement $q(x) \geq 0$ a.e. is used only to obtain $\lambda_1(-\Delta + q(x)) > 0$, which guarantees that (2.2) is indeed a scalar product and that the induced norm is equivalent to the standard norm of $H_0^1(\Omega)$, see Remark 1.7.5 and Exercise 9 in Chap. 1. Therefore in all the results of this chapter, and similarly in all the subsequent chapters, the assumption $q \geq 0$ could be replaced be the "abstract" condition

$$\lambda_1(-\Delta + q(x)) > 0, \tag{2.3}$$

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and everything would work perfectly well with no changes in the proofs. The point is, precisely, that (2.3) is abstract, and nobody knows for which general q's it is satisfied. We prefer, in this book, to assume an explicit sign condition on q, rather than an indirect one on λ_1. The reader should however keep in mind this clarification.

We begin by assuming the following hypothesis on the nonlinearity f.

($\mathbf{h_3}$) $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded.

Setting $F(t) = \int_0^t f(s)\,ds$, the computations carried out in Example 1.3.20 show that the functional $I : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega q(x)u^2\,dx - \int_\Omega F(u)\,dx - \int_\Omega hu\,dx$$
$$= \frac{1}{2}\|u\|^2 - \int_\Omega F(u)\,dx - \int_\Omega hu\,dx$$

is differentiable on $H_0^1(\Omega)$. Its critical points are the weak solutions of (2.1).

Note that unless F is concave, which we do not assume, the functional I needs not be convex.

Theorem 2.1.2 *Under the assumptions* ($\mathbf{h_1}$)–($\mathbf{h_3}$), *Problem* (2.1) *admits at least one solution.*

Remark 2.1.3 The leading idea of the proof is that since f is bounded, the term $\int_\Omega F(u)\,dx$ should grow *at most linearly* with respect to $\|u\|$, as well as the last term. If this is true, the functional I can be seen as an "at most linear" perturbation of the quadratic term $\|u\|^2$. This suggests the existence of a global minimum. Let us see how all this really works.

Proof We break it into two steps. We make repeated use of Hölder and Sobolev inequalities.

Step 1. *The functional I is coercive.* Note first that since f is bounded, then

$$|F(t)| \le M|t|$$

for some $M > 0$ and all $t \in \mathbb{R}$. Hence

$$\left|\int_\Omega F(u)\,dx\right| \le M\int_\Omega |u|\,dx \le C\|u\|,$$

where the last inequality comes from the continuity of the embedding of $H_0^1(\Omega)$ into $L^1(\Omega)$. This confirms the idea of the linear growth as in the preceding remark. Thus

$$I(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(u)\,dx - \int_\Omega hu\,dx \ge \frac{1}{2}\|u\|^2 - C\|u\| - |h|_2|u|_2$$
$$\ge \frac{1}{2}\|u\|^2 - C\|u\|,$$

which shows that I is coercive.

Step 2. *The infimum of I is attained.* Set

$$m = \inf_{u \in H_0^1(\Omega)} I(u).$$

Step 1 shows that $m > -\infty$, although one does not really need this: it will follow automatically from the fact that it is attained.

Let $\{u_k\}_k \subset H_0^1(\Omega)$ be a minimizing sequence for I; from Step 1 we immediately see that $\{u_k\}_k$ is bounded in $H_0^1(\Omega)$, and therefore we can assume that there is a subsequence, still denoted u_k, such that

- $u_k \rightharpoonup u$ in $H_0^1(\Omega)$;
- $u_k \to u$ in $L^2(\Omega)$;
- $u_k(x) \to u(x)$ a.e. in Ω;
- there exists $w \in L^2(\Omega)$ such that $|u_k(x)| \le w(x)$ a.e. in Ω and for all k.

Notice now that since F is continuous we have $F(u_k(x)) \to F(u(x))$ a.e. in Ω, and due to the growth properties of F, we also have

$$|F(u_k(x))| \le M\,|u_k(x)| \le M\,w(x)$$

a.e. in Ω and for all k. Since Ω is bounded, $w \in L^1(\Omega)$, and by dominated convergence we obtain $F(u_k) \to F(u)$ in $L^1(\Omega)$; in particular,

$$\int_\Omega F(u_k)\,dx \to \int_\Omega F(u)\,dx.$$

We also have, of course,

$$\int_\Omega hu_k\,dx \to \int_\Omega hu\,dx \quad \text{and} \quad \|u\|^2 \le \liminf_k \|u_k\|,$$

by weak lower semicontinuity of the norm. Thus

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega F(u)\,dx - \int_\Omega hu\,dx \\ &\le \liminf_k \frac{1}{2}\|u_k\|^2 - \lim_k \int_\Omega F(u_k)\,dx - \lim_k \int_\Omega hu_k\,dx \\ &= \liminf_k \left(\frac{1}{2}\|u_k\|^2 - \int_\Omega F(u_k)\,dx - \int_\Omega h\,u_k\,dx \right) = \liminf_k I(u_k) = m. \end{aligned}$$

But $u \in H_0^1(\Omega)$, so that $I(u) \ge m$, which shows that $I(u) = m$. Therefore u is a global minimum for I, and hence it is a critical point, namely a solution to (2.1). □

Remark 2.1.4 Analyzing the preceding proof one sees that what we actually did is to show that I is coercive and weakly lower semicontinuous on $H_0^1(\Omega)$. These are exactly the assumptions that one needs in the (generalized) Weierstrass Theorem to deduce the existence of a global minimum, see Remark 1.5.7.

The boundedness of f in the previous result has been used to show that the nonlinear term $\int_\Omega F(u)\,dx$ does not destroy the growth properties of I inherited by the term $\|u\|^2$. This occurred because, as we have seen, the nonlinear term grows at most linearly. Now this is not really necessary: it is enough that this term grows *less than quadratically*. Let us see what kind of assumptions we can use in this sense in the next two results.

We begin by replacing the boundedness condition ($\mathbf{h_3}$) by the growth assumption

($\mathbf{h_4}$) $f : \mathbb{R} \to \mathbb{R}$ is continuous and there exist $\sigma \in (0, 1)$ and $a, b > 0$ such that

$$|f(t)| \le a + b|t|^\sigma \quad \forall t \in \mathbb{R}.$$

Thus f is no longer bounded, but is allowed to grow *sublinearly* ($\sigma < 1$). It follows that F grows at most *subquadratically*, in the sense that for some $a_1, b_1 > 0$,

$$|F(t)| \le a_1 + b_1|t|^{\sigma+1} \quad \forall t \in \mathbb{R}, \tag{2.4}$$

with $\sigma + 1 < 2$.

Theorem 2.1.5 *Under the assumptions* ($\mathbf{h_1}$), ($\mathbf{h_2}$) *and* ($\mathbf{h_4}$), *Problem* (2.1) *admits at least one solution.*

Proof Working as in the preceding proof we first show that I is coercive. Using the fact that $\sigma + 1 < 2$ we have

$$\left| \int_\Omega F(u)\, dx \right| \le a_1|\Omega| + b_1 \int_\Omega |u|^{\sigma+1}\, dx \le C_1 + C_2\|u\|^{1+\sigma},$$

thanks to the continuity of the embedding $H_0^1(\Omega) \hookrightarrow L^{\sigma+1}(\Omega)$. Then

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega F(u)\, dx - \int_\Omega hu\, dx \ge \frac{1}{2}\|u\|^2 - C_1 - C_2\|u\|^{\sigma+1} - |h|_2|u|_2 \\ &\ge \frac{1}{2}\|u\|^2 - C_2\|u\|^{\sigma+1} - C_3\|u\| - C_1, \end{aligned}$$

and coercivity follows.

Let now $\{u_k\}_k \subset H_0^1(\Omega)$ be a minimizing sequence for I. As in the proof of Theorem 2.1.2 above, $\{u_k\}_k$ is bounded and therefore, up to subsequences, it converges weakly to some $u \in H_0^1(\Omega)$ and satisfies the same properties as in the preceding case. Then, reasoning as we did above, we obtain again

$$\int_\Omega F(u_k)\, dx \to \int_\Omega F(u)\, dx,$$

so that

$$I(u) \le \liminf_k \left(\frac{1}{2}\|u_k\|^2 - \int_\Omega F(u_k)\, dx - \int_\Omega h\, u_k\, dx \right) = \liminf_k I(u_k) = \inf_{H_0^1(\Omega)} I.$$

The function u is a global minimum, hence a critical point of I, and we have found a solution of (2.1). □

In our quest for more general assumptions we now try to go one step further: precisely, can we allow a *linear* growth for f, and then a *quadratic* growth for F? The answer is in the affirmative, provided we supply a quantitative control of the linear growth. This control is formulated in terms of the first eigenvalue $\lambda_1 = \lambda_1(-\Delta + q)$ in the following assumption.

($\mathbf{h_5}$) $f : \mathbb{R} \to \mathbb{R}$ is continuous and there exist $a > 0$ and $b \in (0, \lambda_1)$ such that

$$|f(t)| \le a + b|t| \quad \forall t \in \mathbb{R}.$$

Integrating, it follows immediately that

$$|F(t)| \le a|t| + \frac{b}{2}|t|^2 \quad \forall t \in \mathbb{R}.$$

Notice the difference with respect to (1.9): this is because we now want to keep the coefficient in front of $|t|^2$ as small as possible.

Theorem 2.1.6 *Under the assumptions* ($\mathbf{h_1}$), ($\mathbf{h_2}$) *and* ($\mathbf{h_5}$), *Problem* (2.1) *admits at least one solution.*

Proof To control the term $\int_\Omega F(u)\,dx$ we use the characterization of the first eigenvalue, Theorem 1.7.6. We have

$$\left| \int_\Omega F(u)\,dx \right| \le a \int_\Omega |u|\,dx + \frac{b}{2} \int_\Omega |u|^2\,dx \le C\|u\| + \frac{b}{2\lambda_1}\|u\|^2,$$

so that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega F(u)\,dx - \int_\Omega hu\,dx \ge \frac{1}{2}\|u\|^2 - C\|u\| - \frac{1}{2}\frac{b}{\lambda_1}\|u\|^2 - |h|_2|u|_2 \\ &\ge \frac{1}{2}\left(1 - \frac{b}{\lambda_1}\right)\|u\|^2 - C_1\|u\|. \end{aligned}$$

Since $b < \lambda_1$, the functional is coercive.

The remaining part of the proof works exactly as in the preceding theorems. □

Remark 2.1.7 In the literature, the growth conditions contained in assumptions ($\mathbf{h_4}$) and ($\mathbf{h_5}$) are often written

$$\limsup_{t \to \pm\infty} \frac{|f(t)|}{|t|^\sigma} < +\infty \quad \text{and} \quad \limsup_{t \to \pm\infty} \frac{|f(t)|}{|t|} < \lambda_1$$

respectively.

Remark 2.1.8 It is interesting to inspect what happens if we allow $b \ge \lambda_1$ in ($\mathbf{h_5}$). In this case the functional I is *no longer coercive* and may be unbounded from below. In some cases, as for example if we take $f(t) = \lambda_k t$ $(k \ge 1)$, Problem (2.1) has *no solution* for some h (see Theorem 1.7.8). Later we will see how to deal with nonlinearities that grow more than quadratically.

We now examine a variant of Problem (2.1), with the aim of showing how the variational information can be of help in establishing existence results. Consider

$$\begin{cases} -\Delta u + q(x)u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

If $f(0) = 0$, a frequent case in the applications, then the problem admits $u \equiv 0$ as a solution (called the trivial solution).

Without further assumptions, it may very well be that the trivial solution is the *only* solution. For example, if $f(t)t \leq 0$ for all t, then any weak solution satisfies

$$\|u\|^2 = \int_\Omega f(u)u\,dx \leq 0,$$

and hence $u \equiv 0$.

In the next result we show a condition that prevents this fact.

Theorem 2.1.9 *Let* ($\mathbf{h_1}$) *hold. Assume moreover that* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies*

$$f(0) = 0 \quad \textit{and} \quad \limsup_{t\to\pm\infty} \frac{|f(t)|}{|t|} < \lambda_1.$$

Then Problem (2.5) *admits at least one solution* (*which may be trivial*).

If in addition f *also satisfies*

$$\liminf_{t\to 0^+} \frac{f(t)}{t} > \lambda_1, \tag{2.6}$$

then Problem (2.5) *admits at least one nontrivial solution.*

Proof The first part is a special case of Theorem 2.1.6. We now show that under condition (2.6) the solution found in the first part is not identically zero. We use a *level argument*, as follows.

First notice that by (2.6), there exists $\beta > \lambda_1$ and $\delta > 0$ such that

$$f(t) \geq \beta t \quad \forall t \in [0, \delta],$$

which implies that

$$F(t) \geq \frac{1}{2}\beta t^2 \quad \forall t \in [0, \delta].$$

Let $\varphi_1 > 0$ be the first eigenfunction of $-\Delta + q(x)$, and take $\varepsilon > 0$ so small that $\varepsilon\varphi_1(x) < \delta$ for almost every x; this is possible because $\varphi_1 \in L^\infty(\Omega)$, see Theorem 1.7.3.

Then

$$F(\varepsilon\varphi_1(x)) \geq \frac{1}{2}\beta\varepsilon^2\varphi_1^2(x)$$

a.e. in Ω. This implies that

$$\begin{aligned} I(\varepsilon\varphi_1) &= \frac{1}{2}\|\varepsilon\varphi_1\|^2 - \int_\Omega F(\varepsilon\varphi_1)\,dx \leq \frac{1}{2}\varepsilon^2\|\varphi_1\|^2 - \frac{1}{2}\beta\varepsilon^2\int_\Omega \varphi_1^2\,dx \\ &= \frac{1}{2}\varepsilon^2\lambda_1\int_\Omega \varphi_1^2\,dx - \frac{1}{2}\beta\varepsilon^2\int_\Omega \varphi_1^2\,dx = \frac{\varepsilon^2}{2}(\lambda_1 - \beta)\int_\Omega \varphi_1^2\,dx < 0, \end{aligned}$$

since $\beta > \lambda_1$. Let u be the solution that minimizes I. Then

$$I(u) = \min_{v \in H_0^1(\Omega)} I(v) \le I(\varepsilon\varphi_1) < 0.$$

As $I(0) = 0$, u cannot be the trivial solution. □

Remark 2.1.10 It is possible to show that the preceding problem admits a *nonnegative* solution. Indeed it is enough to proceed as in Example 1.7.10.

Since $(\mathbf{h_3})$ implies $(\mathbf{h_4})$ that implies $(\mathbf{h_5})$, it is clear Theorem 2.1.6 implies Theorem 2.1.5 that in turn implies Theorem 2.1.2. As a further example we examine now another case in which we can apply the scheme of the previous results and that leads to a theorem that is independent of the preceding ones. Consider the assumption

$(\mathbf{h_6})$ $f : \mathbb{R} \to \mathbb{R}$ is continuous and there exist $a, b > 0$ such that

$$|f(t)| \le a + b|t|^{2^*-1} \quad \forall t \in \mathbb{R}.$$

Moreover

$$f(t)t \le 0 \quad \forall t \in \mathbb{R}.$$

By integration one easily sees that there exist $a_1, b_1 > 0$ such that

$$|F(t)| \le a_1 + b_1|t|^{2^*} \quad \forall t \in \mathbb{R}$$

and that

$$F(t) \le 0 \quad \forall t \in \mathbb{R}$$

Notice that $-F$ is allowed to have *critical* growth, but F is not. Moreover the sign condition $f(t)t \le 0$ prevents, as we have seen, the existence of nontrivial solutions when $h \equiv 0$. In spite of this, Problem (2.1) is solvable.

Theorem 2.1.11 *Under the assumptions* $(\mathbf{h_1})$, $(\mathbf{h_2})$ *and* $(\mathbf{h_6})$, *Problem* (2.1) *admits at least one solution.*

Proof By Example 1.3.20, the usual functional I is differentiable on $H_0^1(\Omega)$. Coercivity is simple consequence of the sign of F:

$$I(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(u)\,dx - \int_\Omega hu\,dx \ge \frac{1}{2}\|u\|^2 - |h|_2|u|_2 \ge \frac{1}{2}\|u\|^2 - C\|u\|.$$

The proof then proceeds exactly as in the previous theorems. □

Remark 2.1.12 This last existence result is similar to the one obtained in Theorem 1.6.6. However here we do not assume the monotonicity of f, so that the functional needs not be convex. This implies that we have to prove the weakly lower semicontinuity, as in the other theorems of this section, and we do not have a uniqueness result.

Remark 2.1.13 If $|F|$ grows more than critically, the functional I is no longer well defined on $H_0^1(\Omega)$, because a function in $H_0^1(\Omega)$ need not be in $L^p(\Omega)$ if $p > 2^*$. This means that the integral of $F(u)$ may be divergent for some u.