On the Noetherian properties of reduction system of words

1. Introduction

A reduction system of words is a pair $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ where Σ is an alphabet and $\mathcal{R}$ is a non-empty finite binary on Σ^*, we write $xly \underset{\mathcal{R}}{\Rightarrow} xmy$ whenever $x, y \in \Sigma^*$ and $(l, m) \in \mathcal{R}$. We write $u \overset{*}{\underset{\mathcal{R}}{\Rightarrow}} v$ if there exists a words $u_0, u_1, ..., u_n \in \Sigma^*$ such that,

$$u_0 = u, u_i \underset{\mathcal{R}}{\Rightarrow} u_{i+1}, \forall 0 \leq i \leq n-1 \text{ and } u_n = v.$$

If $n = 0$, we get $u = v$, and if $n = 1$, we get $u \underset{\mathcal{R}}{\Rightarrow} v$. Where $\overset{*}{\underset{\mathcal{R}}{\Rightarrow}}$ is the reflexive transitive closure of $\underset{\mathcal{R}}{\Rightarrow}$ [7].

The reduction system of words $(\Sigma, \mathcal{R})$ is Noetherian if there does not exist an infinite chain $w_1 \underset{\mathcal{R}}{\Rightarrow} w_2 \underset{\mathcal{R}}{\Rightarrow} w_3 \underset{\mathcal{R}}{\Rightarrow} ...$ in Σ^*.

However, this property is said to be undecidable. It is not possible to find an algorithm taking as input a reduction system of words and rendering true if and only if this reduction system of words is Noetherian [8, 10].

2. Preliminaries

We formally define an alphabet as a non-empty finite set. A word over an alphabet Σ is a finite sequence of symbols of Σ. Although one writes a sequence as $(\sigma_1, \sigma_2, ..., \sigma_n)$, in the present context, we prefer to write it as $\sigma_1\sigma_2...\sigma_n$. The set of all words on the alphabet Σ is denoted by $\Sigma^*$and is equipped with the associative operation defined by the concatenation of two sequences. The concatenation of two sequences $\alpha_1\alpha_2...\alpha_n$ and $\beta_1\beta_2...\beta_m$ is the sequence $\alpha_1\alpha_2...\alpha_n\beta_1\beta_2...\beta_m$ [1, 4].

The concatenation is an associative operation. The string consisting of zero letters is called the empty word, written ϵ. Thus, $\epsilon, \alpha, \beta, \alpha\alpha\beta\alpha, \alpha\alpha\alpha\beta\alpha$ are words over the alphabet $\{\alpha, \beta\}$. Thus the set Σ^* of words is equipped with the structure of a monoid. The monoid Σ^* is called the free monoid on Σ. The length of a word w, denoted $|w|$, is the number of letters in w when each letter is counted as many times as it occurs. Again by definition, $|\epsilon| = 0$. For example $|\alpha\alpha\beta\alpha| = 4$ and $|\alpha\alpha\alpha\beta\alpha| = 5$. Let w be a word over an alphabet Σ. For $\sigma \in \Sigma$, the number of occurrences of σ in w shall be denoted by $|w|_\sigma$. For example $|\alpha\alpha\beta\alpha|_\beta = 1$ and $|\alpha\alpha\alpha\beta\alpha|_\alpha = 4$.

A mapping $h : \Sigma^* \longrightarrow \Delta^*$, where Σ and Δ are alphabets, satisfying the condition

$$h(uv) = h(u)h(v), \text{ for all words } u \text{ and } v,$$

is called a morphism. To define a morphism h, it suffices to list all the words $h(\sigma)$, where a ranges over all the (finitely many) letters of Σ. If M is a monoid, then any mapping $f : \Sigma \longrightarrow M$ extends to a unique morphism $h : \Sigma^* \longrightarrow M$. For instance, if M is the additive monoid $\mathbb{N}$, and f is defined by $f(\sigma) = 1$ for each $\sigma \in \Sigma$, then $h(u)$ is the length $|u|$ of the word u [6, 7].

A binary reation on Σ^* is a subset $\mathcal{R} \subseteq \Sigma^* \times \Sigma^*$. If $(x, y) \in \mathcal{R}$, we say that x is related to y by $\mathcal{R}$, denoted $x\mathcal{R}y$. The relation $I_{\Sigma^*} = \{(x, x), x \in \Sigma^*\}$ is called the identity relation. The relation $(\Sigma^*)^2$ is called the complete relation.

Let $\mathcal{R} \subseteq \Sigma^* \times \Sigma^*$ and $S \subseteq \Sigma^* \times \Sigma^*$ binary relations. The composition of $\mathcal{R}$ and S is a binary relation $S \circ \mathcal{R} \subseteq \Sigma^* \times \Sigma^*$ defined by

$$x(\mathcal{S} \circ \mathcal{R})z \Longleftrightarrow \exists y \in \Sigma^* \text{ such that } x\mathcal{R}y \text{ and } y\mathcal{S}z.$$

A binary relation $\mathcal{R}$ on a set Σ^* is said to be

- reflexive if $x\mathcal{R}x$ for all x in Σ^*.
- transitive if $x\mathcal{R}y$ and $y\mathcal{R}z$ imply $x\mathcal{R}z$.

Let $\mathcal{R}$ be a relation on a set Σ^*. The reflexive closure of $\mathcal{R}$ is the smallest reflexive relation $\mathcal{R}^r$ on Σ^* that contains $\mathcal{R}$, that is,

- $\mathcal{R} \subseteq \mathcal{R}^r$
- if S is a reflexive relation on Σ^* and $\mathcal{R} \subseteq \mathcal{S}$, then $\mathcal{R}^r \subseteq S$.

The transitive closure of $\mathcal{R}$ is the smallest transitive relation $\mathcal{R}^+$ on Σ^* that contains $\mathcal{R}$; that is,

- $\mathcal{R} \subseteq \mathcal{R}^+$
- if S is a transitive relation on Σ^* and $\mathcal{R} \subseteq S$, then $\mathcal{R}^+ \subseteq S$.

The reflexive transitive closure of $\mathcal{R}$ is the smallest reflexive transitive relation $\mathcal{R}^*$ on Σ^* that contains $\mathcal{R}$; that is,

- $\mathcal{R} \subseteq \mathcal{R}^*$
- if S is a reflexive transitive relation on Σ^* and $\mathcal{R} \subseteq S$, then $\mathcal{R}^* \subseteq S$.

Let $\mathcal{R}$ be a relation on a set Σ^*. Then

$$\mathcal{R}^0 = \mathcal{R} \cup I_{\Sigma^*},\ \mathcal{R}^+ = \bigcup_{k=1}^{k=+\infty} \mathcal{R}^k,\ \mathcal{R}^* = \bigcup_{k=0}^{k=+\infty} \mathcal{R}^k \text{ [6]}.$$

Where $\mathcal{R}^k = \mathcal{R}^0 \circ \mathcal{R}^{k-1}$, $\mathcal{R}^0$ is the identity relation, and $\circ$ denote composition of relations.

Let $\mathcal{R} \subseteq \Sigma^* \times \Sigma^*$be a finite set. We define the binary relation $\underset{\mathcal{R}}{\Rightarrow}$ as follows, where $u, v \in \Sigma^* : u \underset{\mathcal{R}}{\Rightarrow} v$ if there exist $x, y \in \Sigma^*$ and $(l, m) \in \mathcal{R}$ with $u = xly$ and $v = xmy$. The structure $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ is a reduction system of words and the relation $\underset{\mathcal{R}}{\Rightarrow}$ is the reduction relation. If $u \in \Sigma^*$ and there is no $v \in \Sigma^*$ suth that $u \underset{\mathcal{R}}{\Rightarrow} v$, then u is irreducible; otherwise, u is reducible. The set of all irreducible elements of Σ^* with respect to $\underset{\mathcal{R}}{\Rightarrow}$ is denoted $IRR\left(\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)\right)$ [2]. Let $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ be a reduction system of words, we write $u \overset{*}{\underset{\mathcal{R}}{\Rightarrow}} v$ if there words $u_0, u_1, ..., u_n \in \Sigma^*$ such that,

$$u_0 = u, u_i \underset{\mathcal{R}}{\Rightarrow} u_{i+1}, \forall 0 \leq i \leq n-1 \text{ and } u_n = v.$$

If $n = o$, we get $u = v$, and if $n = 1$, we get $u \underset{\mathcal{R}}{\Rightarrow} v$. Where $\overset{*}{\underset{\mathcal{R}}{\Rightarrow}}$ is the reflexive transitive closure of $\underset{\mathcal{R}}{\Rightarrow}$.

We say that $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ is Noetherian if there does not exist an infinite sequence of words $w_i \in \Sigma^* \ (i \in \mathbb{N})$ with $w_0 \underset{\mathcal{R}}{\Rightarrow} w_1 \underset{\mathcal{R}}{\Rightarrow} w_2 \underset{\mathcal{R}}{\Rightarrow} \ldots$. For example $(\mathbb{N}, >)$ is Noetherian [5].

Theorem 1 *Let* $\left(\Sigma_1, \underset{\mathcal{R}_1}{\Rightarrow}\right)$ *be a reduction system of words. Then the following two statements are equivalent :*

1. $\left(\Sigma_1, \underset{\mathcal{R}_1}{\Rightarrow}\right)$ is Noetherian;

2. There exists another reduction system of words $\left(\Sigma_2, \underset{\mathcal{R}_2}{\Rightarrow}\right)$ that is Noetherian and the morphism $\psi : \Sigma_1^* \longrightarrow \Sigma_2^*$ such that $\psi(\underset{\mathcal{R}_1}{\Rightarrow}) \subseteq \overset{+}{\underset{\mathcal{R}_2}{\Rightarrow}}$. $\overset{+}{\underset{\mathcal{R}_2}{\Rightarrow}}$ is the transitive closure of $\underset{\mathcal{R}_2}{\Rightarrow}$.

Proposition 2 *Let* $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ *be a reduction system of words and* $\varphi : \Sigma^* \longrightarrow \mathbb{N}$ *the morphism of monoids. Consider the mapping* $P : \Sigma^* \longrightarrow \mathbb{N}$ *defined by :*

$P(w) = \sum_{i=1}^{i=|w|} n^i \times \varphi(w(i))$, $n \in \mathbb{N} - \{0\}$ *where* $w\,(i)$ *is the* $i-th$ *letter of* w.

If for all $(l, m) \in \mathcal{R}$, $\begin{cases} |l| = |m| & (C_1) \\ \text{and} & \\ P(l) > P(m) & (C_2) \end{cases}$, then $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ is Noetherian.

Proof. First, we show that, we have : $\forall x, y \in \Sigma^* : P(xy) = P(x) + n^{|x|} \times P(y)$. We have

$$P(xy) = \sum_{i=1}^{i=|xy|} n^i \times \varphi((xy)\,(i)) = \sum_{i=1}^{i=|x|} n^i \times \varphi((xy)\,(i)) + \sum_{i=|x|+1}^{i=|x|+|y|} n^i \times \varphi((xy)\,(i))$$

$$= \sum_{i=1}^{i=|x|} n^i \times \varphi((x)\,(i)) + \sum_{i=1}^{i=|y|} n^{|x|+i} \times \varphi((xy)\,(|x|+i))$$

$$= \sum_{i=1}^{i=|x|} n^i \times \varphi((x)\,(i)) + \sum_{i=1}^{i=|y|} n^{|x|+i} \times \varphi((y)\,(i)) = P(x) + n^{|x|} \times P(y).$$

Let $(l, m) \in \mathcal{R}$ and $x, y \in \Sigma^*$, we show that $P(xly) > P(xmy)$.
We have $P(xly) = P(x(ly)) = P(x) + n^{|x|} \times P(l\,\mathrm{y}) = P(x) + n^{|x|}\left(P(l) + n^{|l|} \times P(y)\right)$
$= P(x) + n^{|x|} \times P(l) + n^{|x|+|l|} \times P(y)$. A similar argument, we have $P(xmy) = P(x(my))$
$= P(x) + n^{|x|} \times P(m\,\mathrm{y}) = P(x) + n^{|x|}\left(P(m) + n^{|m|} \times P(y)\right)$
$= P(x) + n^{|x|} \times P(m) + n^{|x|+|m|} \times P(y)$. According to the conditions (C_1), (C_2) described above, we have $P(xly) > P(xmy)$. Consequently $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ is Noetherian. ■

Example 3 *Consider the reduction system of words* $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ *with* $\Sigma = \{\alpha, \beta, \gamma\}$ *and* $\mathcal{R} = \{(\beta\alpha, \alpha\beta)\,;(\gamma\beta, \beta\gamma)\}$. *Let the morphism* $\varphi : \Sigma^* \longrightarrow \mathbb{N}$, *defined by* $\varphi\,(\alpha) = 3$,

$\varphi(\beta) = 2, \varphi(\gamma) = 1$ and the mapping $P : \Sigma^* \longrightarrow \mathbb{N}$, where $P(w) = \sum_{i=1}^{i=|w|} 2^i \times \varphi(w(i))$.
For the condition (C_1), we have $|\beta\alpha| = |\alpha\beta| = 2$ and $|\gamma\beta| = |\beta\gamma| = 2$. For the condition (C_2), we show that $P(\beta\alpha) > P(\alpha\beta)$ and $P(\gamma\beta) > P(\beta\gamma)$.
We have $P(\beta\alpha) = \sum_{i=1}^{i=2} 2^i \times \varphi(\beta\alpha(i)) = 2 \times \varphi(\beta) + 2^2 \times \varphi(\alpha) = 16$.
Similarly, $P(\alpha\beta) = \sum_{i=1}^{i=2} 2^i \times \varphi(\alpha\beta(i)) = 2 \times \varphi(\alpha) + 2^2 \times \varphi(\beta) = 14$, then $P(\beta\alpha) > P(\alpha\beta)$.
We have $P(\gamma\beta) = \sum_{i=1}^{i=2} 2^i \times \varphi(\gamma\beta(i)) = 2 \times \varphi(\gamma) + 2^2 \times \varphi(\beta) = 10$.
Similarly, $P(\beta\gamma) = \sum_{i=1}^{i=2} 2^i \times \varphi(\beta\gamma(i)) = 2 \times \varphi(\beta) + 2^2 \times \varphi(\gamma) = 8$, then $P(\gamma\beta) > P(\beta\gamma)$.
Consequently $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ is Noetherian.

Proposition 4 *Let $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ be a reduction system of words and $\varphi : (\Sigma^*, \cdot) \longrightarrow (\mathbb{N}, +)$ with a morphism of monoids, the mapping $P : \Sigma^* \longrightarrow \mathbb{N}$ defined by : $P(w) = \sum_{i=1}^{i=|w|} i \times \varphi(w(i))$, where $w(i)$ is the $i-th$ letter of w.*

$$\text{If for all } (l, m) \in \mathcal{R}, \begin{cases} |l| = |m| & (C_1) \\ \quad\text{and} & \\ P(l) > P(m) & (C_2) \\ \quad\text{and} & \\ \varphi(l) > \varphi(m) & (C_3) \end{cases}, \text{ then } \left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right) \text{ is Noetherian.}$$

Proof. First, we show that, we have : $\forall x, y \in \Sigma^* : P(xy) = P(x) + P(y) + |x| \times \varphi(y)$.
We have $P(xy) = \sum_{i=1}^{i=|xy|} i \times \varphi(xy(i)) = \sum_{i=1}^{i=|x|} i \times \varphi(xy(i)) + \sum_{i=|x|+1}^{i=|x|+|y|} i \times \varphi(xy(i))$
$= \sum_{i=1}^{i=|x|} i \times \varphi(x(i)) + \sum_{i=1}^{i=|y|} (|x| + i) \times \varphi((xy)(|x| + i))$
$= \sum_{i=1}^{i=|x|} i \times \varphi((x)(i)) + \sum_{i=1}^{i=|y|} (|x| + i) \times \varphi((y)(i))$
$= P(x) + P(y) + |x| \times \varphi(y)$.
Let $(l, m) \in \mathcal{R}$ and $x, y \in \Sigma^*$, we show that $P(xly) > P(xmy)$.
We have, $P(xly) = P(x(ly)) = P(x) + P(l\,\mathrm{y}) + |x| \times \varphi(l\,\mathrm{y})$
$= P(x) + P(l) + P(y) + |l| \times \varphi(\mathrm{y}) + |x| \times (\varphi(l) + \varphi(\mathrm{y}))$
$= [P(x) + P(y) + |x| \times \varphi(y)] + [P(l) + |l| \times \varphi(\mathrm{y}) + |x| \times \varphi(l)]$.
On the other hand, $P(xmy) = [P(x) + P(y) + |x| \times \varphi(y)] + [P(m) + |m| \times \varphi(\mathrm{y}) + |x| \times \varphi(m)]$.
According to the conditions (C_1), (C_2), (C_1) described above,we have $P(xly) > P(xmy)$.
Finally $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ is Noetherian. ■

Example 5 *Let* $\Sigma = \{\alpha, \beta, \gamma\}$ *and* $\mathcal{R} = \{(\beta\alpha, \beta\gamma)\,;(\alpha\beta, \alpha\gamma)\}$*. We define the morphism of monoids* $\varphi : \Sigma^* \longrightarrow \mathbb{N}$*, by* $\varphi(\alpha) = 2, \varphi(\beta) = 1, \varphi(\gamma) = 0$*. We consider the mapping* $P : \Sigma^* \longrightarrow \mathbb{N}$*, where* $P(w) = \sum_{i=1}^{i=|w|} i \times \varphi(w(i))$*.*
For the condition (C_1)*, we have* $|\beta\alpha| = |\beta\gamma| = 2$ *and* $|\alpha\beta| = |\alpha\gamma| = 2$*.*
For the condition (C_2)*, we show that* $P(\beta\alpha) > P(\beta\gamma)$ *and* $P(\alpha\beta) > P(\alpha\gamma)$*.*
We have $P(\beta\alpha) = \sum_{i=1}^{i=2} i \times \varphi(\beta\alpha(i)) = 1 \times \varphi(\beta) + 2 \times \varphi(\alpha) = 5$*.*
A similar argument, we have $P(\beta\gamma) = \sum_{i=1}^{i=2} i \times \varphi(\beta\gamma(i)) = 1 \times \varphi(\beta) + 2 \times \varphi(\gamma) = 1$*,*
then $P(\beta\alpha) > P(\beta\gamma)$*. And* $P(\alpha\beta) = \sum_{i=1}^{i=2} i \times \varphi(\alpha\beta(i)) = 1 \times \varphi(\alpha) + 2 \times \varphi(\beta) = 4$*.*
Similarly, $P(\alpha\gamma) = \sum_{i=1}^{i=2} i \times \varphi(\alpha\gamma(i)) = 1 \times \varphi(\alpha) + 2 \times \varphi(\gamma) = 2.$*, then* $P(\alpha\beta) > P(\alpha\gamma)$*.*
For the condition (C_3)*, we show that* $\varphi(\beta\alpha) > \varphi(\beta\gamma)$ *and* $\varphi(\alpha\beta) > \varphi(\alpha\gamma)$*.*
We have $\varphi(\beta\alpha) = 3, \varphi(\beta\gamma) = 1, \varphi(\alpha\beta) = 3, \varphi(\alpha\gamma) = 2$*.*
Consequently $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ *is Noetherian.*

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