The first step in studying the motion of a particle begins with a detailed examination of the terms and concepts of the particle's kinematic properties without delving into the causes of this motion. During this study, the particle is considered dimensionless (meaning its dimensions are negligible compared to the surroundings), and parameters of the particle's motion, such as its positions, displacements, direction of motion, velocity, acceleration, and path relative to a spatial-temporal reference frame, are determined.

The ***spatiotemporal reference*** *is used to study the precise motion and analysis of a particle. This reference allows for the temporal and spatial determination, observation, and examination of the body's motion, significantly facilitating the understanding and accurate analysis of motion and its monitoring.*

An example of this concept in practical application is in athletics, such as a footrace, where specific points in time and space (the starting line and the moment of the race's starting) are chosen as ***spatial and temporal references*** *to track the athletes' performance accurately, where the athletes cannot commence the race freely from any location and at any moment they desire.*

1) Reference Frame (or Reference):

The reference is the point of observation and monitoring used to study the motion of particles. In other words, it is a specific point in space (referred to as the origin and denoted as "O") chosen for the observation and monitoring of motion variables for other bodies.

One, two, or three directed axes (one axis if the motion occurs in a single direction, two axes if the motion is in a plane, i.e., two directions, and three directed axes if the motion occurs in three-dimensional space) can be assigned to the origin point to facilitate the monitoring and analysis of these kinetic variables of the moving particle during its motion.

The combination of directed axes and the origin point is referred to as the "coordinate system.

1) Coordinate systems

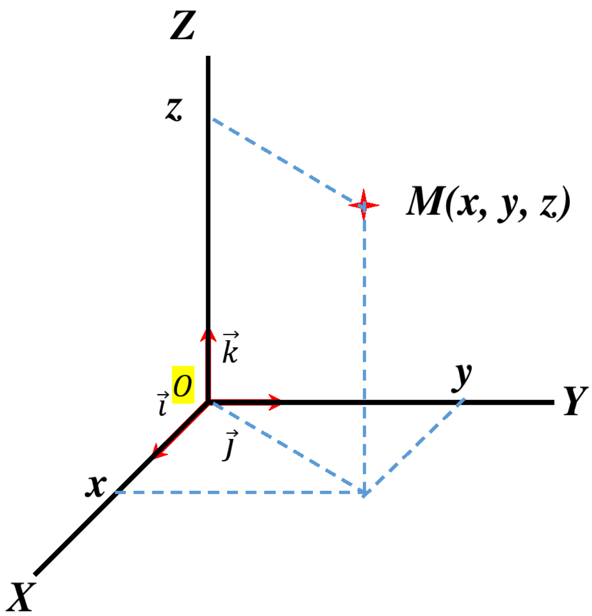
The Coordinate systems are reference frameworks used to represent numerical data geometrically. Every coordinate system includes a reference point known as the origin, from which one or more directed axes extend. In physics, coordinate systems are employed to represent the kinematic properties of particles, such as their position, *displacement, velocity, and acceleration, in a way that makes them easily analyzable and simplified.*

There are several coordinate systems used in the study of mechanical phenomena and particle motion, such as ***Cartesian***, ***polar***, ***cylindrical***, *and* ***spherical*** *coordinate systems. Choosing the appropriate system for studying particle motion depends on the nature of the motion. Therefore, you should select the system that allows you to simplify the analysis of particle motion and avoid unnecessary complexity in calculations.*

2-1) *Cartesian coordinate system*

This system comprises a fixed reference point, known as the origin (usually denoted as the point '0'), from which emanate three basic orthogonal axes oriented (commonly labeled as the x-axis, the y-axis, and the z-axis) at 90-degree angles to each other, and carrying three unit vectors (commonly labeled as $\vec{\imath}, \vec{\jmath}$, *and* $\vec{k}$*).*

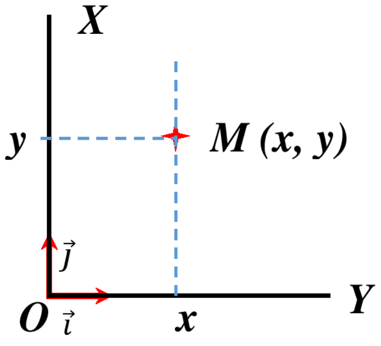
The coordinates of a point M in the Cartesian coordinate system are given as (x_M, y_M, z_M) *, where each of x, y, and z represents its projection along the three principal axes, namely, OX, OY, and OZ, respectively. This is illustrated in the following diagram:*



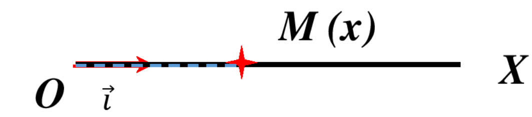
The position vector for any point can be written as follows:

$$\overrightarrow{OM} = \overrightarrow{r_M} = x_M\vec{\imath} + y_M\vec{\jmath} + z_M\vec{k}$$

Based on the nature of motion, we can use only the necessary coordinates to determine the position of a point during its motion. For example, if the motion occurs in the plane (OXY), the third coordinate (z) can be omitted as it typically does not have a significant role in this context. Similarly, if the motion is straight linear, it can be studied by considering only a single axis (OX). This approach simplifies the analysis by using the coordinates that are most relevant to the specific motion, making it more efficient and straightforward to describe and understand.



Motion in plane (XOY)



Straight Linear Motion along(OX)

2-2) *Polar coordinate system*

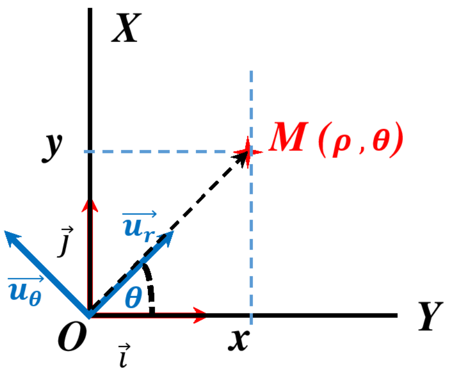
If the particle's motion occurs within the (OXY) plane, an alternative coordinate system can be employed the polar coordinate system with unit radial vectors ($\vec{u_r}$) and angular vectors ($\vec{u_\theta}$).

Radial unit vector ($\vec{u_r}$): *It possesses a length of 1 unit and aligns with the position vector OM.*

Angular unit vector ($\vec{u_\theta}$): *Also with a unit length of 1, it stands perpendicular to the unit radial vector ($\vec{u_r}$), indicating a counterclockwise direction to signify angular changes (θ).*

The angle θ is the angle enclosed between the position vector ($\overrightarrow{OM}$) and the (OX) axis.

*The polar coordinates of point M represented by both Radial **(ρ)** and Angular **(θ)** coordinates, and given as **M (ρ, θ).***

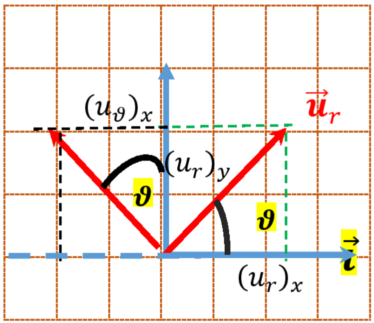


The position vector for the point M can be written in the polar coordinate system as follows:

$$\overrightarrow{OM} = \overrightarrow{r_M} = \rho\overrightarrow{u_r} \quad \textit{where}\ \rho = \|\overrightarrow{OM}\| \quad \textit{and}\ \overrightarrow{u_r} = \frac{\overrightarrow{r_M}}{\rho} = \frac{\overrightarrow{OM}}{\|\overrightarrow{OM}\|}$$

$$\boldsymbol{\rho} = \|\overrightarrow{\boldsymbol{OM}}\| = \sqrt{\boldsymbol{x^2+y^2}} \qquad \textit{and} \quad \boldsymbol{\tan\theta} = \frac{\boldsymbol{y}}{\boldsymbol{x}}$$

2-2-1) Formulation of the unit vectors $\vec{u}_r$ and $\vec{u}_\theta$ in terms of $\vec{\imath}$ and $\vec{\jmath}$



We can obtain the expressions for the unit vectors in polar coordinates in terms of the unit vectors in Cartesian coordinates through the process of orthogonal projection (meaning we find the components of the polar unit vectors in Cartesian coordinates) as follows:

$\vec{\boldsymbol{u}}_r = (u_r)_x \times \vec{\boldsymbol{\imath}} + (u_r)_y \times \vec{\boldsymbol{\jmath}}$

$(u_r)_x$*and* $(u_r)_y$ *are the components of the polar unit vectors* $\vec{\boldsymbol{u}}_r$ *in Cartesian coordinates*

$(u_r)_x = \|\vec{\boldsymbol{u}}_r\| \times \cos\vartheta\ ;\ (u_r)_y = \|\vec{\boldsymbol{u}}_r\| \times \sin\vartheta \qquad \|\vec{\boldsymbol{u}}_r\| = 1$

$$\vec{\boldsymbol{u}}_r = \cos\vartheta \times \vec{\boldsymbol{\imath}} + \sin\vartheta \times \vec{\boldsymbol{\jmath}}$$

$\vec{\boldsymbol{u}}_\vartheta = -(u_\vartheta)_x \times \vec{\boldsymbol{\imath}} + (u_\vartheta)_y \times \vec{\boldsymbol{\jmath}}$

$(u_\vartheta)_x$*and* $(u_\vartheta)_y$ *are the components of the polar unit vectors* $\vec{\boldsymbol{u}}_\vartheta$ *in Cartesian coordinates*

$(u_\vartheta)_x = \|\vec{\boldsymbol{u}}_\vartheta\| \times \sin\vartheta\ ;\ (u_\vartheta)_y = \|\vec{\boldsymbol{u}}_\vartheta\| \times \cos\vartheta \qquad \|\vec{\boldsymbol{u}}_\vartheta\| = 1$

$$\vec{\boldsymbol{u}}_\vartheta = -\sin\vartheta \times \vec{\boldsymbol{\imath}} + \cos\vartheta \times \vec{\boldsymbol{\jmath}}$$

2-2-2) Formulation of the coordinates (x, y) in term (ρ , θ)

*The coordinates of point M can be expressed in either Cartesian or polar coordinates. The relationship between the Cartesian coordinates (**x, y**) and the polar coordinates (**ρ , θ**) for point M can be expressed as follows:*

$$\begin{cases} x = \boldsymbol{\rho} \cos \boldsymbol{\theta} \\ y = \boldsymbol{\rho} \sin \boldsymbol{\theta} \end{cases}$$

Example01:

*Convert the coordinates of the position of M point from polar **M (4, π/6)** to the Cartesian coordinates*

$$\begin{matrix} x = \boldsymbol{\rho} \cos \boldsymbol{\theta} \\ y = \boldsymbol{\rho} \sin \boldsymbol{\theta} \end{matrix} \quad where\ \boldsymbol{\rho} = \mathbf{4}\ and\ \boldsymbol{\theta} = \frac{\pi}{6}$$

$$\overrightarrow{OM} = 4\,\overrightarrow{\boldsymbol{u_r}}$$

$$M\,(4, \pi/6) \;\Rightarrow \begin{cases} x = 4\,cos\left(\dfrac{\pi}{6}\right) = \dfrac{4\sqrt{3}}{2} = \sqrt{3} \\ y = 4\,sin\left(\dfrac{\pi}{6}\right) = 4 \times \dfrac{1}{2} = 2 \end{cases}$$

$$\overrightarrow{OM} = \sqrt{\mathbf{3}}\,\vec{\imath} + 2\,\vec{\jmath}$$

2-3) cylindrical coordinate system

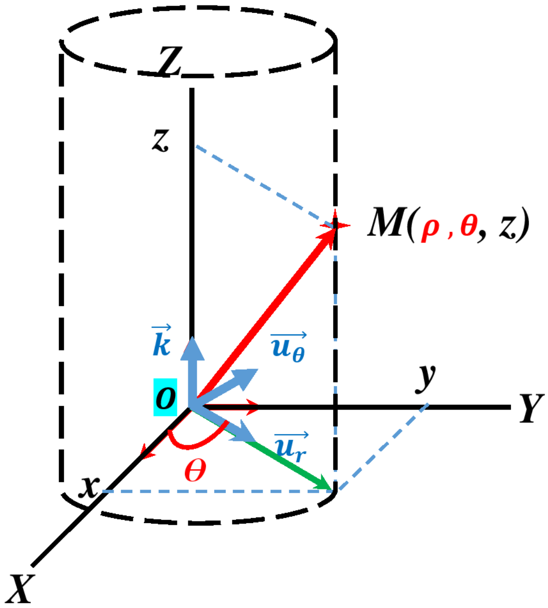
The cylindrical coordinate system is a three-dimensional reference system that isn't fixed in place (correlated to the position of the moving object) and uses three cylindrical coordinates to describe the positions of objects in space:

- ***Radial Coordinate (ρ)**: It represents the length of the vector resulting from projecting the position vector onto the plane formed by the unit radial vector ($\overrightarrow{\boldsymbol{u_r}}$) and the unit azimuthal vector ($\overrightarrow{\boldsymbol{u_\theta}}$).*
- ***Angular Coordinate (θ)**: This represents the angle between the radial line and the positive axis of the Cartesian coordinates (usually measured counterclockwise) and is depicted as an angle that starts from the positive Cartesian axis.*
- ***Coordinate (z)**: This indicates the vertical height of the point above the origin plane, which corresponds to the Cartesian coordinate plane.*

The cylindrical coordinate system relies on three fundamental unit vectors, which are:

- **Radial unit vector ($\overrightarrow{u_r}$):** *It possesses a length of 1 unit and aligns with the position vector OM.*
- **Angular unit vector ($\overrightarrow{u_\theta}$):** *Also with a unit length of 1, it stands perpendicular to the unit radial vector ($\overrightarrow{u_r}$), indicating a counterclockwise direction to signify angular changes (θ).*
- **Unit Vertical Vector ($\vec{k}$):** *Also with a unit length of 1, it points vertically, and perpendicular to both ($\overrightarrow{u_r}$) and ($\overrightarrow{u_\theta}$) unit vectors.*

*The cylindrical coordinates of point M represented by both Radial **(ρ)** and Angular **(θ)** and (z) coordinates, and given as **M (ρ, θ, z)** as shown in the following figure*

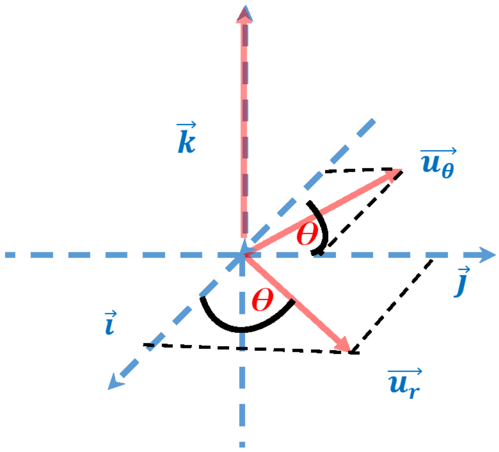


The position vector for the point M can be written in the polar coordinate system as follows:

$$\overrightarrow{OM} = \overrightarrow{r_M} = \rho\overrightarrow{u_r} + z\,\vec{k} \quad \text{where} \quad \rho = \sqrt{x^2 + y^2} \quad ,\|\overrightarrow{OM}\| = \sqrt{\rho^2 + z^2} \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

2-2-1) Formulation of the unit vectors $\vec{u}_r$, $\vec{u}_\theta$, and $\vec{k}$ in terms of $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$

Using the same projection method employed to express the unit vectors in the polar coordinate system, we can obtain expressions for the unit vectors in the cylindrical coordinate system.



$$\begin{cases} \vec{u}_r = \cos\vartheta \times \vec{i} + \sin\vartheta \times \vec{j} \\ \vec{u}_\vartheta = -\sin\vartheta \times \vec{i} + \cos\vartheta \times \vec{j} \\ \vec{k} = \vec{k} \end{cases}$$

2-3-2) Formulation of the coordinates (x, y, z) in term (ρ , θ , z)

The relationship between the Cartesian coordinates (x, y, z) and the cylindrical coordinates (ρ , θ, z) for point M can be expressed as follows:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

Example02:

Convert the coordinates of the position of M point from cartesian ***M (4, 3, 1)*** *to the cylindrical coordinates*

$$\overrightarrow{OM} = \rho\vec{u_r} + z\,\vec{k}$$

$$\tan\theta = \frac{y}{x} = \frac{3}{4} = 0.75 = 36.86^0 = 0.643\ Rad$$

$$z = 1$$

$$\overrightarrow{OM} = 5\,\vec{u_r} + \vec{k}$$

2-4) Spherical coordinate system

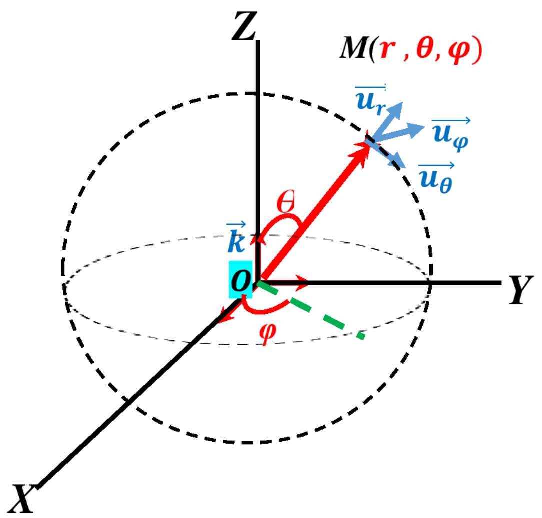
The spherical coordinate system is a three-dimensional reference system used to express the positions of points in three-dimensional space. This system relies on three main coordinates:

- ***Radial Coordinate (r):*** *Represents the distance between the point and the origin (the pole) and is always positive.*
- ***Polar Angle Coordinate (θ):*** *Indicates the angle enclosed between the radial line and the vertical axis (usually ranging from 0 to 180 degrees).*
- ***Azimuthal Angle Coordinate (φ):*** *Represents the angle enclosed between the radial line and the horizontal plane (usually ranging from 0 to 360 degrees).*

The position vector for the point M can be written in the spherical coordinate system as follows:

$$\overrightarrow{OM} = \overrightarrow{r_M} = r\overrightarrow{u_r} \quad \text{where } r = \|\overrightarrow{OM}\| \quad \text{and } \overrightarrow{u_r} = \frac{\overrightarrow{r_M}}{r} = \frac{\overrightarrow{OM}}{\|\overrightarrow{OM}\|}$$

$$r = \|\overrightarrow{OM}\| = \sqrt{x^2 + y^2 + z^2} \quad \text{and} \quad \tan\theta = \frac{y}{x} \text{ and } \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$$



2-4-1) Formulation of the unit vectors $\vec{u}_r$, $\vec{u}_\theta$, and $\vec{u}_\varphi$ in terms of $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$

The relationship between the three basic unit vectors in the spherical coordinate system ($\vec{u}_r$, $\vec{u}_\theta$, and $\vec{u}_\varphi$) and the Cartesian unit vectors $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ is given as follows:

$$\begin{cases} \vec{u}_r = \sin\theta\ \cos\varphi \times \vec{\imath} + \sin\theta \sin\varphi \times \vec{\jmath} + \cos\theta\ \vec{k} \\ \vec{u}_\vartheta = \cos\theta\ \cos\varphi \times \vec{\imath} + \cos\theta \sin\varphi \times \vec{\jmath} - \sin\theta\ \vec{k} \\ \vec{u}_\varphi = -\sin\varphi \times \vec{\imath} + \cos\varphi \times \vec{\jmath} \end{cases}$$

2-4-2) Formulation of the coordinates (x, y, z) in term (r, θ, φ)

The relationship between the three spherical coordinates (r, θ, φ) *and the Cartesian coordinates* (x, y, z) *is given as follows:*

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$$