The first step in studying the motion of a particle begins with a detailed examination of the terms and concepts of the particle's kinematic properties without delving into the causes of this motion. During this study, the particle is considered dimensionless (meaning its dimensions are negligible compared to the surroundings), and parameters of the particle's motion, such as its positions, displacements, direction of motion, velocity, acceleration, and path relative to a spatial-temporal reference frame, are determined.

*The **spatiotemporal reference** is used to study the precise motion and analysis of a particle. This reference allows for the temporal and spatial determination, observation, and examination of the body's motion, significantly facilitating the understanding and accurate analysis of motion and its monitoring.*

*An example of this concept in practical application is in athletics, such as a footrace, where specific points in time and space (the starting line and the moment of the race's starting) are chosen as **spatial and temporal references** to track the athletes' performance accurately, where the athletes cannot commence the race freely from any location and at any moment they desire.*

1) Reference Frame (or Reference):

The reference is the point of observation and monitoring used to study the motion of particles. In other words, it is a specific point in space (referred to as the origin and denoted as "O") chosen for the observation and monitoring of motion variables for other bodies.

One, two, or three directed axes (one axis if the motion occurs in a single direction, two axes if the motion is in a plane, i.e., two directions, and three directed axes if the motion occurs in three-dimensional space) can be assigned to the origin point to facilitate the monitoring and analysis of these kinetic variables of the moving particle during its motion.

The combination of directed axes and the origin point is referred to as the "coordinate system.

1) Coordinate systems

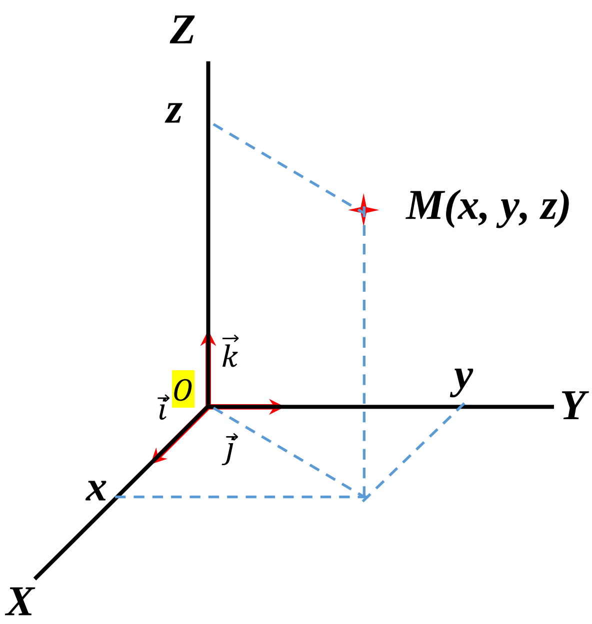
The Coordinate systems are reference frameworks used to represent numerical data geometrically. Every coordinate system includes a reference point known as the origin, from which one or more directed axes extend. In physics, coordinate systems are employed to represent the kinematic properties of particles, such as their position, *displacement, velocity, and acceleration, in a way that makes them easily analyzable and simplified.*

*There are several coordinate systems used in the study of mechanical phenomena and particle motion, such as **Cartesian**, **polar**, **cylindrical**, **spherical** and **intrinsic** coordinate systems. Choosing the appropriate system for studying particle motion depends on the nature of the motion. Therefore, you should select the system that allows you to simplify the analysis of particle motion and avoid unnecessary complexity in calculations.*

2-1) *Cartesian coordinate system*

This system comprises a fixed reference point, known as the origin (usually denoted as the point '0'), from which emanate three basic orthogonal axes oriented (commonly labeled as the x-axis, the y-axis, and the z-axis) at 90-degree angles to each other, and carrying three unit vectors(commonly labeled as $\vec{\imath}, \vec{\jmath}$, *and* $\vec{k}$*).*

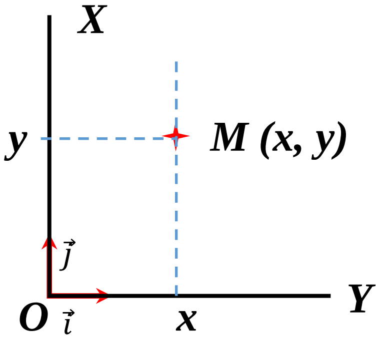
The coordinates of a point M in the Cartesian coordinate system are given as (x_M, y_M, z_M)*, where each of x, y, and z represents its projection along the three principal axes, namely, OX, OY, and OZ, respectively. This is illustrated in the following diagram:*



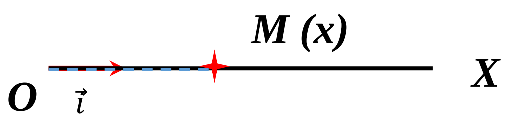
The position vector for any point can be written as follows:

$$\overrightarrow{OM} = \overrightarrow{r_M} = x_M\vec{\imath} + y_M\vec{\jmath} + z_M\vec{k}$$

Based on the nature of motion, we can use only the necessary coordinates to determine the position of a point during its motion. For example, if the motion occurs in the plane (OXY), the third coordinate (z) can be omitted as it typically does not have a significant role in this context. Similarly, if the motion is straight linear, it can be studied by considering only a single axis (OX). This approach simplifies the analysis by using the coordinates that are most relevant to the specific motion, making it more efficient and straightforward to describe and understand.



Motion in plane (XOY)



Straight Linear Motion along(OX)

2-2) *Polar coordinate system*

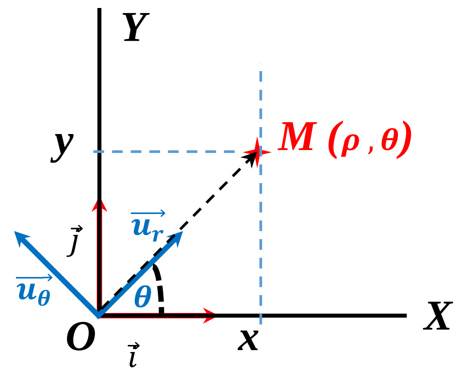
If the particle's motion occurs within the (OXY) plane, an alternative coordinate system can be employed the polar coordinate system with unit radial vectors ($\overrightarrow{u_r}$) and angular vectors ($\overrightarrow{u_\theta}$).

Radial unit vector ($\overrightarrow{u_r}$):*It possesses a length of 1 unit and aligns with the position vector OM.*

Angularunit vector ($\overrightarrow{u_\theta}$):*Also with a unit length of 1, it stands perpendicular to the unit radial vector ($\overrightarrow{u_r}$), indicating a counterclockwise direction to signify angular changes (θ).*

The angle θ is the angle enclosed between the position vector ($\overrightarrow{OM}$) and the(OX) axis.

The polar coordinates of point M represented by both Radial ***(ρ)****and Angular****(θ)****coordinates, and given as* ***M (ρ, θ).***

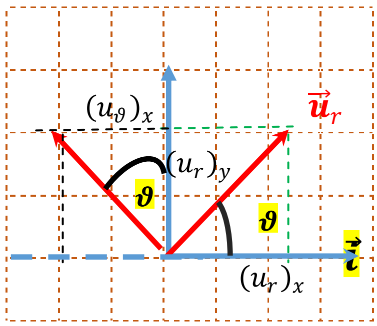


The position vector for the point M can be written in the polar coordinate system as follows:

$$\overrightarrow{OM} = \overrightarrow{r_M} = \rho \overrightarrow{u_r} \textit{where} \rho = \|\overrightarrow{OM}\| \quad \text{and } \overrightarrow{u_r} = \frac{\overrightarrow{r_M}}{\rho} = \frac{\overrightarrow{OM}}{\|\overrightarrow{OM}\|}$$

$$\rho = \|\overrightarrow{OM}\| = \sqrt{x^2 + y^2} \text{and} \tan\theta = \frac{y}{x}$$

2-2-1) Formulationof the unit vectors $\vec{u}_r$ and $\vec{u}_\theta$ in terms of $\vec{i}$ and $\vec{j}$



We can obtain the expressions for the unit vectors in polar coordinates in terms of the unit vectors in Cartesian coordinates through the process of orthogonal projection (meaning we find the components of the polar unit vectors in Cartesian coordinates) as follows:

$\vec{u}_r = (u_r)_x \times \vec{i} + (u_r)_y \times \vec{j}$

$(u_r)_x$ *and* $(u_r)_y$ *are the components of the polar unit vectors* $\vec{u}_r$ *in Cartesian coordinates*

$(u_r)_x = \|\vec{u}_r\| \times \cos\vartheta;\ (u_r)_y = \|\vec{u}_r\| \times \sin\vartheta\ \|\vec{u}_r\| = 1$

$$\vec{u}_r = \cos\vartheta \times \vec{i} + \sin\vartheta \times \vec{j}$$

$\vec{u}_\vartheta = -(u_\vartheta)_x \times \vec{i} + (u_\vartheta)_y \times \vec{j}$

$(u_\vartheta)_x$ *and* $(u_\vartheta)_y$ *are the components of the polar unit vectors* $\vec{u}_\vartheta$ *in Cartesian coordinates*

$(u_\vartheta)_x = \|\vec{u}_\vartheta\| \times \sin\vartheta;\ (u_\vartheta)_y = \|\vec{u}_\vartheta\| \times \cos\vartheta\ \|\vec{u}_\vartheta\| = 1$

$$\vec{u}_\vartheta = -\sin\vartheta \times \vec{i} + \cos\vartheta \times \vec{j}$$

2-2-2) Formulation of the coordinates (x, y) in term (ρ , θ)

*The coordinates of point M can be expressed in either Cartesian or polar coordinates. The relationship between the Cartesian coordinates (**x, y**) and the polar coordinates (**ρ , θ**) for point M can be expressed as follows:*

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

Example01:

*Convert the coordinates of the position of M point from polar **M (4, π/6)** to the Cartesian coordinates*

$$\begin{matrix} x = \rho\cos\theta \\ y = \rho\sin\theta \end{matrix} \text{where} \rho = 4 \text{ and } \theta = \frac{\pi}{6}$$

$$\overrightarrow{OM} = 4\,\overrightarrow{u_r}$$

$$M\,(4, \pi/6) \Rightarrow \begin{cases} x = 4\,cos\left(\frac{\pi}{6}\right) = \frac{4\sqrt{3}}{2} = \sqrt{3} \\ y = 4\,sin\left(\frac{\pi}{6}\right) = 4 \times \frac{1}{2} = 2 \end{cases}$$

$$\overrightarrow{OM} = \sqrt{3}\vec{i} + 2\,\vec{j}$$

2-3) cylindrical coordinate system

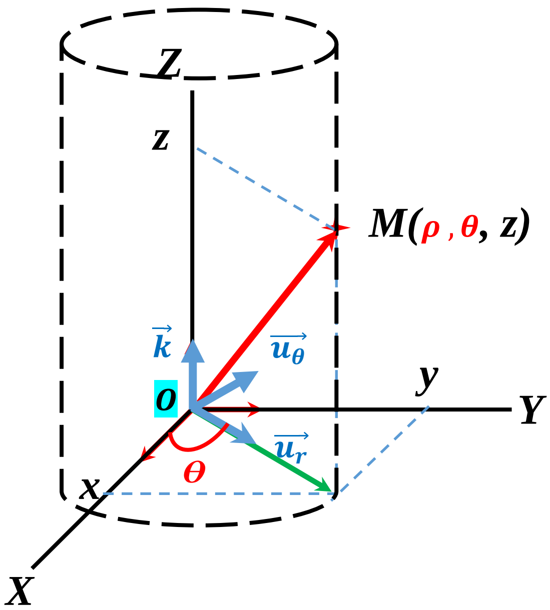
The cylindrical coordinate system is a three-dimensional reference system that isn't fixed in place (correlated to the position of the moving object)and uses three cylindrical coordinates to describe the positions of objects in space:

- ***Radial Coordinate (ρ)**: It represents the length of the vector resulting from projecting the position vector onto the plane formed by the unit radial vector ($\overrightarrow{u_r}$) and the unit azimuthal vector ($\overrightarrow{u_\theta}$).*
- ***Angular Coordinate (θ)**: This represents the angle between the radial line and the positive axis of the Cartesian coordinates (usually measured counterclockwise) and is depicted as an angle that starts from the positive Cartesian axis.*
- ***Coordinate (z)**: This indicates the vertical height of the point above the origin plane, which corresponds to the Cartesian coordinate plane.*

The cylindrical coordinate system relies on three fundamental unit vectors, which are:

- ***Radial unit vector*** **($\vec{u_r}$):** *It possesses a length of 1 unit and aligns with the position vector OM.*
- ***Angular unit vector*** **($\vec{u_\theta}$):** *Also with a unit length of 1, it stands perpendicular to the unit radial vector ($\vec{u_r}$), indicating a counterclockwise direction to signify angular changes (θ).*
- ***Unit Vertical Vector*** *($\vec{k}$): Also with a unit length of 1, it points vertically, and perpendicular to both ($\vec{u_r}$) and ($\vec{u_\theta}$) unit vectors.*

The cylindrical coordinates of point M represented by both Radial **(ρ)** *and Angular* **(θ)** *and* **(z)** *coordinates, and given as* **$M(\rho, \theta, z)$** *as shown in the following figure*

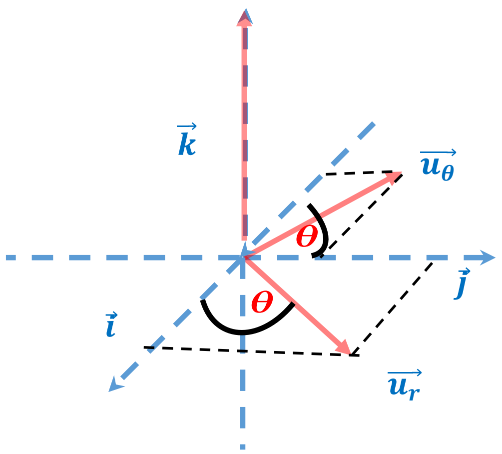


The position vector for the point M can be written in the polar coordinate system as follows:

$$\overrightarrow{OM} = \overrightarrow{r_M} = \rho\vec{u_r} + z\,\vec{k} \text{ where } \rho = \sqrt{x^2 + y^2} \quad , \|\overrightarrow{OM}\| = \sqrt{\rho^2 + z^2} \text{ and } \quad \tan\theta = \frac{y}{x}$$

2-2-1) Formulationof the unit vectors $\vec{u}_r$, $\vec{u}_\theta$, and $\vec{k}$ in terms of $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$

Using the same projection method employed to express the unit vectors in the polar coordinate system, we can obtain expressions for the unit vectors in the cylindrical coordinate system.



$$\begin{cases} \vec{u}_r = \cos\vartheta \times \vec{i} + \sin\vartheta \times \vec{j} \\ \vec{u}_\vartheta = -\sin\vartheta \times \vec{i} + \cos\vartheta \times \vec{j} \\ \vec{k} = \vec{k} \end{cases}$$

2-3-2) Formulation of the coordinates (x, y, z) in term (ρ , θ , z)

The relationship between the Cartesian coordinates (x, y, z) and the cylindrical coordinates (ρ , θ, z) for point M can be expressed as follows:

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \\ z = z \end{cases}$$

Example02:

Convert the coordinates of the position of M point from cartesian ***M (4, 3, 1)*** *to the cylindrical coordinates*

$$\overrightarrow{OM} = \rho \vec{u_r} + z\,\vec{k}$$

$$tan\,\theta = \frac{y}{x} = \frac{3}{4} = 0.75 = 36.86^0 = 0.643\ Rad$$

$$z = 1$$

$$\overrightarrow{OM} = 5\,\vec{u_r} + \vec{k}$$

2-4) Spherical coordinate system

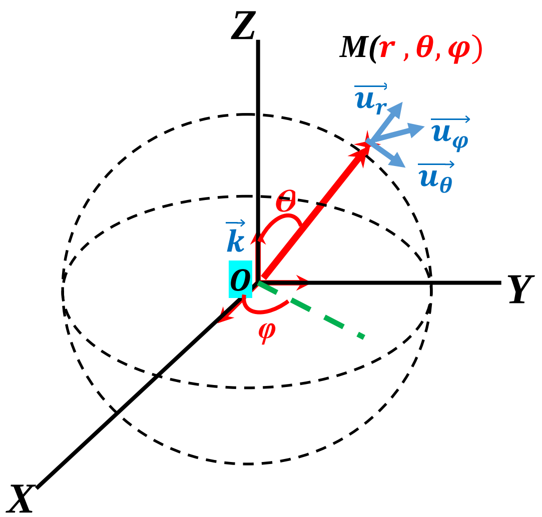
The spherical coordinate system is a three-dimensional reference system used to express the positions of points in three-dimensional space. This system relies on three main coordinates:

- ***Radial Coordinate (r)***: *Represents the distance between the point and the origin (the pole) and is always positive.*
- ***Polar Angle Coordinate (θ)***: *Indicates the angle enclosed between the radial line and the vertical axis (usually ranging from 0 to 180 degrees).*
- ***Azimuthal Angle Coordinate (φ)***: *Represents the angle enclosed between the radial line and the horizontal plane (usually ranging from 0 to 360 degrees).*

The position vector for the point M can be written in the spherical coordinate system as follows:

$$\overrightarrow{OM} = \overrightarrow{r_M} = r\overrightarrow{u_r} \text{ where } r = \|\overrightarrow{OM}\| \quad \text{and } \overrightarrow{u_r} = \frac{\overrightarrow{r_M}}{r} = \frac{\overrightarrow{OM}}{\|\overrightarrow{OM}\|}$$

$$r = \|\overrightarrow{OM}\| = \sqrt{x^2 + y^2 + z^2} \text{ and } \tan\theta = \frac{y}{x} \text{ and } \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$$



2-4-1) Formulationof the unit vectors $\vec{u}_r$, $\vec{u}_\theta$, and $\vec{u}_\varphi$ in terms of $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$

The relationship between the three basic unit vectors in the spherical coordinate system ($\vec{u}_r$, $\vec{u}_\theta$, and $\vec{u}_\varphi$)*and the Cartesian unit vectors* $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ *is given as follows:*

$$\begin{cases} \vec{u}_r = \sin\theta\cos\varphi \times \vec{\imath} + \sin\theta\sin\varphi \times \vec{\jmath} + \cos\theta\,\vec{k} \\ \vec{u}_\vartheta = \cos\theta\cos\varphi \times \vec{\imath} + \cos\theta\sin\varphi \times \vec{\jmath} - \sin\theta\,\vec{k} \\ \vec{u}_\varphi = -\sin\varphi \times \vec{\imath} + \cos\varphi \times \vec{\jmath} \end{cases}$$

2-4-2) Formulation of the coordinates (x, y, z) in term (r , θ , φ)

The relationship between the three spherical coordinates (r , θ , φ)and the Cartesian coordinates (x, y, z)is given as follows:

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

2-5) - Intrinsic coordinates system

The intrinsic coordinate system at for any point from the path is denoted by the base ($\vec{u}_T$,$\vec{u}_N$) where :

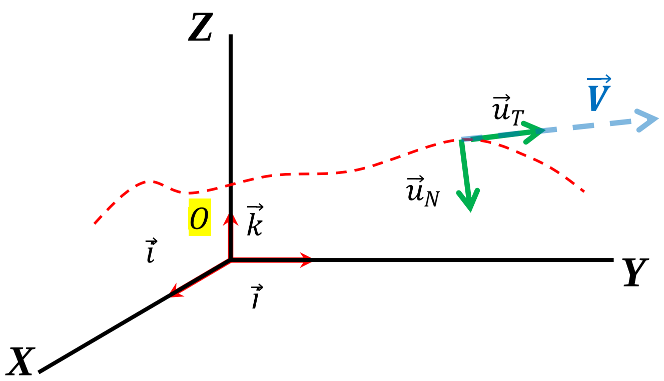
($\vec{u}_T$) represent the normal unit vector and oriented towards the center of curvature.

($\vec{u}_N$ represent the tangent unit vector and oriented in the direction of motion.

The velocity vector is tangent to the trajectory and in the same direction as the movement, it carries the unit vector $\vec{u}_T$ and is written:

$$\vec{v} = \|\vec{v}\|\,\vec{u}_T = \frac{dS}{dt}\,\vec{u}_T$$

where dS is the curvilinear abscissa



1) Kinematic features of a particle

The kinetic properties of a moving particle can be studied by focusing attention on its kinetic features during its movement. This includes tracking its change in position relative to a frame of reference, examining its path (trajectory) of movement, measuring its velocity, and estimating its acceleration.

1.1) Position and vector position

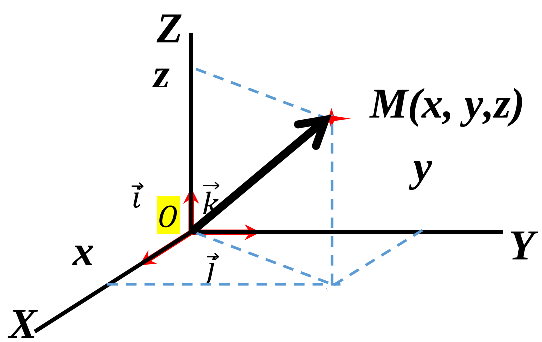
The position of a moving particle at a specific moment (t) is determined with respect to a specified reference frame (origin) through its coordinates. If the motion occurs in one dimension, the position of the moving entity is expressed with a single coordinate. In the case of motion in a plane, it is represented by two coordinates. Meanwhile, three coordinates are employed to express motion in three-dimensional space.

The position vector*for the moving particle 'M' at the moment 't' is a vector extending from the origin to the location of the point. It is symbolized by '$\overrightarrow{OM}$' and is analytically formulated as follows:*$\overrightarrow{OM} = \overrightarrow{r_M} = x\ \ \vec{\imath} + y\ \ \vec{\jmath} + z\ \ \vec{k}$

The position vector*is characterized by:*

- ***Starting point (the tail) of the vector:** The origin point 'O'*
- ***Ending point (Head or Tip) of the vector:** The location of the moving particle 'M'.*
- ***Direction of the vector:** From point 'O' to position 'M'.*
- ***Support of the vector:** The straight line connecting the origin 'O' to the position 'M'.*
- ***The magnitude of the vector:***$\|\overrightarrow{OM}\| = \sqrt{x^{2} + y^{2} + z^{2}}$

The position vector $\overrightarrow{OM}$ is represented in the Cartesian coordinate system as shown in the figure



1.2) The Displacement Vector

The displacement vector for a moving particle between two moments, $\boldsymbol{t_i}$and $\boldsymbol{t_f}$, is a vector that represents the particle's transition from position M_i to position M_f. It is symbolized by $\overrightarrow{M_iM_f}$and extends from the initial position to the final position. It can be expressed as the difference between the position vectors, as follows:

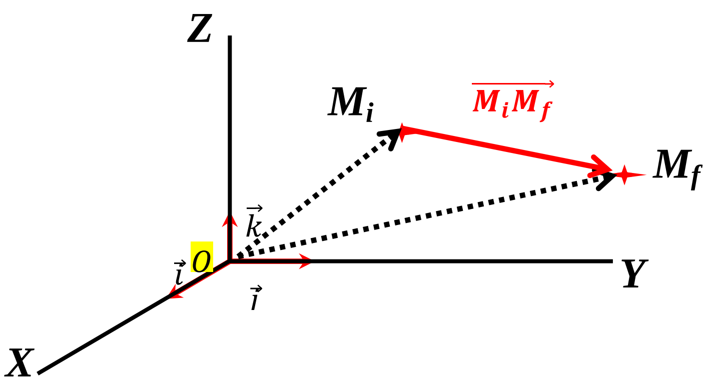
$$\overrightarrow{\boldsymbol{M_iM_f}} = \overrightarrow{\boldsymbol{OM_f}} - \overrightarrow{\boldsymbol{OM_i}}$$

$$\overrightarrow{\boldsymbol{M_iM_f}} = (\boldsymbol{x_f} - \boldsymbol{x_i})\vec{\boldsymbol{\imath}} + (\boldsymbol{y_f} - \boldsymbol{y_i})\vec{\boldsymbol{\jmath}} + (\boldsymbol{z_f} - \boldsymbol{z_i})\vec{\boldsymbol{k}}$$

The displacement vector is characterized by:

- ***Starting point (Tail) of the vector:*** *The position of the particle 'M_i' at moment 't_i.'*
- ***Head (Tip) of the vector:*** *The position of the moving particle 'M_f' at moment 't_f.'*
- ***Direction of the vector:*** *From position M_i to position M_f*
- ***Support of the vector:*** *The straight line connecting position M_i to position M_f*
- ***The magnitude of the vector:*** $\left\|\overrightarrow{\boldsymbol{M_iM_f}}\right\| = \sqrt{(\boldsymbol{x_f} - \boldsymbol{x_i})^2 + (\boldsymbol{y_f} - \boldsymbol{y_i})^2 + (\boldsymbol{z_f} - \boldsymbol{z_i})^2}$

The position vector $\overrightarrow{M_iM_f}$ is represented in the Cartesian coordinate system as shown in the figure



1.3)Velocity vector

Generally, velocity represents the change in distance travelled by a moving object over a specific time duration. By means of calculating velocity, we can classify it into two forms: average velocity and instantaneous velocity.

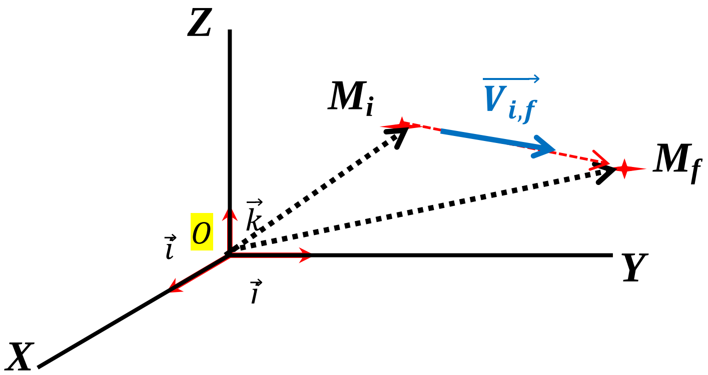
1.3.1) Average velocity

Average velocity, denoted as $\overrightarrow{\boldsymbol{V_a}}$ *or* $\overrightarrow{\boldsymbol{V_{i,f}}}$*, serves as a vector quantity to describe the velocity over an extended time interval between two distinct moments* $\boldsymbol{t_i}$ *and* $\boldsymbol{t_f}$*. Mathematically, it is articulatedas the ratio of the displacement vector* $\overrightarrow{\boldsymbol{M_iM_f}}$ *to the time interval* $\Delta \boldsymbol{t} = \boldsymbol{t_f} - \boldsymbol{t_i}$*during this displacement.*

The vector of the average velocity $(\overrightarrow{\boldsymbol{V_{i,f}}})$ *shares the same direction and support as the displacement vector* $(\overrightarrow{\boldsymbol{M_iM_f}})$*, while its magnitude is determined by the ratio of the displacement vector's length to the time interval.*

$$\overrightarrow{\boldsymbol{V_{i,f}}} = \frac{\overrightarrow{\boldsymbol{M_iM_f}}}{\Delta t} = \frac{\overrightarrow{\boldsymbol{OM_f}} - \overrightarrow{\boldsymbol{OM_i}}}{\boldsymbol{t_f} - \boldsymbol{t_i}} = \frac{(\boldsymbol{x_f} - \boldsymbol{x_i})}{\boldsymbol{t_f} - \boldsymbol{t_i}}\vec{\boldsymbol{\imath}} + \frac{(\boldsymbol{y_f} - \boldsymbol{y_i})}{\boldsymbol{t_f} - \boldsymbol{t_i}}\vec{\boldsymbol{\jmath}} + \frac{(\boldsymbol{z_f} - \boldsymbol{z_i})}{\boldsymbol{t_f} - \boldsymbol{t_i}}\vec{\boldsymbol{k}}$$

$$\left\|\overrightarrow{\boldsymbol{V_{i,f}}}\right\| = \sqrt{\left(\frac{\boldsymbol{x_f} - \boldsymbol{x_i}}{\boldsymbol{t_f} - \boldsymbol{t_i}}\right)^2 + \left(\frac{\boldsymbol{y_f} - \boldsymbol{y_i}}{\boldsymbol{t_f} - \boldsymbol{t_i}}\right)^2 + \left(\frac{\boldsymbol{z_f} - \boldsymbol{z_i}}{\boldsymbol{t_f} - \boldsymbol{t_i}}\right)^2}$$



1.3.2) Instantaneous velocity

The instantaneous velocity, represented by $\vec{\boldsymbol{V}}$*, is a vector describing the velocity of an particle at a specific moment "*$\boldsymbol{t}$*" or within an Infinitesimally small time interval* $(\boldsymbol{\Delta t})$ *approaching (tending to) zero. Mathematically, the vector of instantaneous velocity is defined as the limiting valueof the ratio of the displacement vector to the time interval as* $\boldsymbol{\Delta t}$ *approaches (tends to) zero.*

$$\vec{\boldsymbol{V}}(\boldsymbol{t}) = \lim_{\Delta t \to 0} \frac{\overrightarrow{\boldsymbol{M_iM_f}}}{\Delta \boldsymbol{t}}$$

In mathematical calculations, the limit process is expressed by the derivativeof the position vector with respect to time.

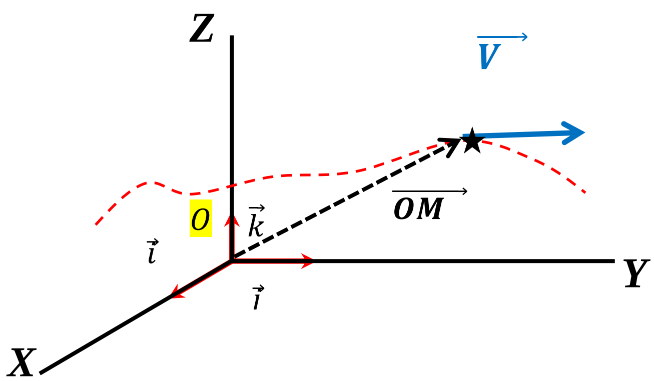
$$\vec{V}(t) = \frac{d(\overrightarrow{OM})}{dt} = \frac{d(x\ \ \vec{i} + y\ \ \vec{j} + z\ \ \vec{k})}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

$$\|\vec{V}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

The vector of instantaneous velocity represents a tangential vector to the path at the location where the instantaneous velocity is calculated.

Note: The vector of average velocity between two moments t_i and t_f equals the vector of instantaneous velocity at the midpoint of this interval $\frac{t_i+t_f}{2}$

$$\vec{V}\left(\frac{t_i + t_f}{2}\right) = \overrightarrow{V_{i,f}}$$



1.4)Acceleration vector

Acceleration is defined as the rate of change in velocity with respect to time.Based on how acceleration is calculated, we can distinguish between two types: average acceleration and instantaneous acceleration.

1.4.1) Average acceleration

Average acceleration, denoted as $\overrightarrow{a_a}$or $\overrightarrow{a_{i,f}}$, describes the change in velocity over a specific time interval, typically between two distinct momentst_i and t_f.Mathematically, the Average acceleration is expressed as the ratio of the change in velocity ($\overrightarrow{\Delta V_{i,f}} = \overrightarrow{V_f} - \overrightarrow{V_i}$) to the *change in time ($\Delta t = t_f - t_i$). The vector of average acceleration ($\overrightarrow{a_{i,f}}$) shares the same direction and support as the vector of velocity change ($\overrightarrow{\Delta V_{i,f}}$), While its magnitude is the ratio of the magnitude of the velocity change vector ($\|\overrightarrow{\Delta V_{i,f}}\|$) to the duration of the time interval$\Delta t = t_f - t_i$) .*

$$\overrightarrow{a_{i,f}} = \frac{\overrightarrow{\Delta V_{i,f}}}{\Delta t} = \frac{\overrightarrow{V_f} - \overrightarrow{V_i}}{t_f - t_i} = \frac{(V_{xf} - V_{xi})}{t_f - t_i}\vec{\imath} + \frac{(V_{yf} - V_{yi})}{t_f - t_i}\vec{\jmath} + \frac{(V_{zf} - V_{zi})}{t_f - t_i}\vec{k}$$

$$\|\overrightarrow{a_{i,f}}\| = \frac{\|\overrightarrow{\Delta V_{i,f}}\|}{t_f - t_i}$$

1.4.2) Instantaneous acceleration

The instantaneous acceleration vector, denoted by $\vec{a}$, signifies the acceleration at a precise moment$t$, specifically within an infinitesimally small time interval (approaching or tending to zero).

The acceleration vector is mathematically defined as the limit of the rate of change in velocity over an infinitesimally small period of time, approaching or tending to zero.

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\overrightarrow{V_{i,f}}}{\Delta t}$$

In mathematical terms, when the time interval becomes extremely small, the limit of a function represents its derivative. Consequently, we can express the formula for instantaneous acceleration as follow

$$\vec{a}(t) = \frac{d(\vec{V})}{dt} = \frac{d(V_x\ \vec{\imath} + V_y\ \vec{\jmath} + V_z\ \vec{k})}{dt} = \frac{dV_x}{dt}\vec{\imath} + \frac{dV_y}{dt}\vec{\jmath} + \frac{dV_z}{dt}\vec{k}$$

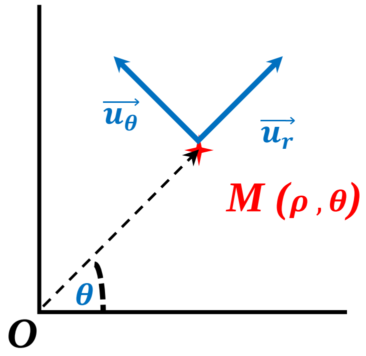
$$\|\vec{a}(t)\| = \sqrt{\left(\frac{dV_x}{dt}\right)^2 + \left(\frac{dV_y}{dt}\right)^2 + \left(\frac{dV_z}{dt}\right)^2}$$

Note: The vector of average acceleration between two moments t_i and t_f equals the vector of instantaneous acceleration at the midpoint of this interval $\frac{t_i+t_f}{2}$

$$\vec{a}\left(\frac{t_i + t_f}{2}\right) = \overrightarrow{a_{i,f}}$$

1.5) Velocity and acceleration in polar coordinate system

The polar coordinates of point M represented by both Radial ***(ρ)*** *and Angular* ***(θ)*** *coordinates, and given as* ***M (ρ, θ),*** *andthe position vector for the point M can be written in the polar coordinate system as follows:*$\overrightarrow{OM} = \rho\overrightarrow{\boldsymbol{u_r}}$ ***where*** $\rho = \|\overrightarrow{OM}\|$ *and* $\overrightarrow{\boldsymbol{u_r}} = \frac{\overrightarrow{r_M}}{\rho} = \frac{\overrightarrow{OM}}{\|\overrightarrow{OM}\|}$



The expression for the instantaneous **velocity** *vector in* **polar** *coordinates is given as follows:*

$$\vec{V}(t) = \frac{d(\overrightarrow{OM})}{dt} = \frac{d(\rho\overrightarrow{u_r})}{dt} = \frac{d\ \ \rho}{dt}\overrightarrow{u_r} + \rho\frac{d\overrightarrow{u_r}}{dt}$$

$$\vec{u}_r = \cos\theta \times \vec{i} \ + \sin\theta \times \vec{j}$$

$$\vec{u}_\vartheta = -\sin\theta \times \vec{i} \ + \cos\theta \times \vec{j}$$

$$\frac{d\ (\vec{u}_r)}{dt} = \frac{d(\cos\theta \times \vec{\imath} \ + \sin\theta \times \vec{\jmath})}{dt} = -\frac{d\theta}{dt}\sin\theta \times \vec{\imath} \ + \frac{d\theta}{dt}\cos\theta \times \vec{\jmath}$$

where$\frac{d\ \vec{\imath}}{dt} = \vec{0}\ \frac{d\ \vec{\jmath}}{dt} = \vec{0}\ \frac{d\vartheta}{dt} = \dot{\theta}$

$$\frac{d\ (\vec{u}_r)}{dt} = \frac{d\theta}{dt}(-\sin\theta \times \vec{\imath} \ + \cos\theta \times \vec{\jmath}) = \dot{\theta}\vec{u}_\theta$$

$$\frac{d\ (\vec{u}_\theta)}{dt} = \frac{d(-\sin\theta \times \vec{\imath} \ + \cos\theta \times \vec{\jmath})}{dt} = -\frac{d\theta}{dt}\cos\theta \times \vec{\imath} \ - \frac{d\theta}{dt}\sin\theta \times \vec{\jmath}$$

where$\frac{d\ \vec{\imath}}{dt} = \vec{0}\ \frac{d\ \vec{\jmath}}{dt} = \vec{0}\ \frac{d\theta}{dt} = \dot{\theta}$

$$\frac{d\ (\vec{u}_\theta)}{dt} = -\frac{d\theta}{dt}(\cos\theta \times \vec{\imath} \ + \sin\theta \times \vec{\jmath}) = -\dot{\theta}\vec{u}_r$$

$$\vec{V}(t) = \frac{d\ \ \rho}{dt}\overrightarrow{u_r} + \rho\frac{d\theta}{dt}\vec{u}_\theta = \dot{\rho}\overrightarrow{u_r} + \rho\dot{\theta}\vec{u}_\theta$$

$$\|\vec{V}(t)\| = \sqrt{(\dot{\rho})^2 + (\rho\dot{\theta})^2}$$

The expression for the instantaneous ***acceleration*** *vector in* ***polar*** *coordinates is given as follows:*

$$\vec{a}(t) = \frac{d(\vec{V})}{dt} = \frac{d\ \ (\dot{\rho}\vec{u_r} + \rho\dot{\theta}\vec{u}_\theta)}{dt}$$

$$= \frac{d\ \ \dot{\rho}}{dt}\vec{u_r} + \dot{\rho}\frac{d\vec{u_r}}{dt} + \frac{d\rho}{dt}\dot{\theta}\vec{u}_\theta + \rho\frac{d\dot{\theta}}{dt}\vec{u}_\theta + \rho\dot{\theta}\frac{d\vec{u}_\theta}{dt}$$

$$= \ddot{\rho}\vec{u_r} + \dot{\rho}\dot{\theta}\vec{u}_\theta + \dot{\rho}\dot{\theta}\vec{u}_\theta + \rho\ddot{\theta}\vec{u}_\theta - \rho\dot{\theta}\dot{\theta}\vec{u}_r$$

$$\vec{a}(t) = (\ddot{\rho} - \rho\dot{\theta}^2)\,\vec{u}_r + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\vec{u}_\theta$$

$$\|\vec{a}(t)\| = \sqrt{= (\ddot{\rho} - \rho\dot{\theta}^2)^2 + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})^2}$$

1.6) Velocity and acceleration in cylindrical coordinate system

$$\overrightarrow{OM} = \rho\vec{u_r} + z\,\vec{k}$$

$$\vec{V}(t) = \frac{d\ \ \rho}{dt}\vec{u_r} + \rho\frac{d\theta}{dt}\vec{u}_\theta + \frac{dz}{dt}\vec{k} = \dot{\rho}\vec{u_r} + \rho\dot{\theta}\vec{u}_\theta + \dot{z}\vec{k}$$

$$\|\vec{V}(t)\| = \sqrt{(\dot{\rho})^2 + (\rho\dot{\theta})^2 + \dot{z}^2}$$

$$\vec{a}(t) = (\ddot{\rho} - \rho\dot{\theta}^2)\vec{u}_r + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\vec{u}_\theta + \ddot{z}\,\vec{k}$$

$$\|\vec{a}(t)\| = \sqrt{= (\ddot{\rho} - \rho\dot{\theta}^2)^2 + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})^2 + \ddot{z}^2}$$

1.6) - Velocity and acceleration in spherical coordinate system

$$\overrightarrow{OM} = r\,\vec{u_r}$$

$$\vec{V}(t) = \frac{d(r\vec{u_r})}{dt} = \frac{dr}{dt}\vec{u_r} + r\,\frac{d\vec{u_r}}{dt}$$

$$\frac{d\vec{u_r}}{dt} = \frac{d}{dt}(\sin\theta\cos\varphi\ \vec{i} + \sin\theta\ \sin\varphi\ \vec{j} + \cos\theta\,\vec{k})$$

$$\frac{d\vec{u_r}}{dt} = \cos\theta\ \frac{d\theta}{dt}\cos\varphi\ \vec{i} - \sin\theta\sin\varphi\frac{d\varphi}{dt}\ \vec{i} + \cos\theta\ \frac{d\theta}{dt}\sin\varphi\ \vec{j} + \sin\theta\ \cos\varphi\frac{d\varphi}{dt}\ \vec{j} - \frac{d\theta}{dt}\sin\theta\,\vec{k}$$

$$\frac{d\vec{u_r}}{dt} = \frac{d\theta}{dt}\left(\cos\theta\ \ \cos\varphi\ \vec{i} + \cos\theta\ \sin\varphi\ \vec{j} - \sin\theta\,\vec{k}\right) + \sin\theta\frac{d\varphi}{dt}(-\sin\varphi\ \ \vec{i} + \ \cos\varphi\ \vec{j})$$

$$\frac{d\overrightarrow{u_r}}{dt} = \frac{d\theta}{dt}(\vec{u}_\vartheta) + \sin\theta\frac{d\varphi}{dt}(\vec{u}_\varphi) = \dot{\theta}\,\vec{u}_\vartheta + \dot{\varphi}\sin\theta\;\vec{u}_\varphi$$

$$\vec{V}(t) = \dot{r}\overrightarrow{u_r} + r\,\dot{\theta}\,\vec{u}_\vartheta + r\,\dot{\varphi}\sin\theta\;\vec{u}_\varphi$$

$$\vec{a}(t) = \frac{d(\vec{V})}{dt} = \frac{d(\dot{r}\overrightarrow{u_r} + r\,\dot{\theta}\,\vec{u}_\vartheta + r\,\dot{\varphi}\sin\theta\;\vec{u}_\varphi)}{dt}$$

$$\vec{a}(t) = \frac{d\dot{r}}{dt}\overrightarrow{u_r} + \frac{d\overrightarrow{u_r}}{dt}\dot{r} + \frac{dr}{dt}\dot{\theta}\,\vec{u}_\vartheta + \frac{d\dot{\theta}}{dt}r\,\vec{u}_\vartheta + \frac{d\vec{u}_\vartheta}{dt}r\,\dot{\theta} + \frac{dr}{dt}\dot{\varphi}\sin\theta\;\vec{u}_\varphi + \frac{d\dot{\varphi}}{dt}r\,\sin\theta\;\vec{u}_\varphi + \frac{d\sin\theta}{dt}r\;\dot{\varphi}\,\vec{u}_\varphi + \frac{d\vec{u}_\varphi}{dt}r\,\dot{\varphi}\sin\theta$$

To simplify this equation we need to find the time derivatives of the following unit vectors:

- $\frac{d\overrightarrow{u_r}}{dt} = \dot{\theta}\,\vec{u}_\vartheta + \dot{\varphi}\sin\theta\;\vec{u}_\varphi$ *(It was previously calculated)*
- $\frac{d\vec{u}_\vartheta}{dt} = \frac{d}{dt}(\cos\theta\cos\varphi\,\vec{i} + \cos\theta\,\sin\varphi\,\vec{j} - \sin\theta\,\vec{k})$

$$\frac{d\vec{u}_\vartheta}{dt} = \frac{d\cos\theta}{dt}\cos\varphi\,\vec{i} + \frac{d\cos\varphi}{dt}\cos\theta\,\vec{i} + \frac{d\cos\theta}{dt}\sin\varphi\,\vec{j} + \frac{d\sin\varphi}{dt}\cos\theta\;\vec{j} - \frac{d\sin\theta}{dt}\vec{k}$$

$$\frac{d\vec{u}_\vartheta}{dt} = -\frac{d\theta}{dt}\sin\theta\cos\varphi\,\vec{i} - \sin\varphi\,\frac{d\varphi}{dt}\cos\theta\,\vec{i} - \frac{d\theta}{dt}\sin\theta\,\sin\varphi\,\vec{j} + \cos\varphi\,\frac{d\varphi}{dt}\cos\theta\;\vec{j} - \cos\theta\,\frac{d\theta}{dt}\,\vec{k}$$

$$\frac{d\vec{u}_\vartheta}{dt} = (-\sin\theta\cos\varphi\,\vec{i} - \sin\theta\,\sin\varphi\,\vec{j} - \cos\theta\,\vec{k})\frac{d\theta}{dt} + (-\sin\varphi\,\cos\theta\,\vec{i} + \cos\varphi\,\cos\theta\;\vec{j})\,\frac{d\varphi}{dt}$$

$$\frac{d\vec{u}_\vartheta}{dt} = -\frac{d\theta}{dt}\overrightarrow{u_r} + \frac{d\varphi}{dt}\cos\theta\,\vec{u}_\varphi = -\dot{\theta}\overrightarrow{u_r} + \dot{\varphi}\cos\theta\,\vec{u}_\varphi$$

- $\frac{d\vec{u}_\varphi}{dt} = \frac{d(-\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j})}{dt}$

$$\frac{d\vec{u}_\varphi}{dt} = \frac{d(-\sin\varphi\;\vec{i} + \cos\varphi\;\vec{j})}{dt}$$

$$\frac{d\vec{u}_\varphi}{dt} = -\frac{d\varphi}{dt}\cos\varphi\;\vec{i} - \frac{d\varphi}{dt}\sin\varphi\;\vec{j} == -\dot{\varphi}\cos\varphi\;\vec{i} - \dot{\varphi}\sin\varphi\;\vec{j}$$

We multiply both sides of the equation $\vec{u}_r = \sin\theta\cos\varphi\ \vec{\imath} + \sin\theta\ \sin\varphi\ \vec{\jmath} + \cos\theta\,\vec{k}$ *by* $\sin\theta$ *, and we get:*

$$\sin\theta \times \vec{u}_r = \sin\theta \times (\sin\theta\cos\varphi\ \vec{\imath} + \sin\theta\ \sin\varphi\ \vec{\jmath} + \cos\theta\,\vec{k}) \qquad \textbf{Eq(a)}$$

Also, We multiply both sides of the equation $\vec{u}_\vartheta = \cos\theta\cos\varphi\,\vec{\imath} + \cos\theta\ \sin\varphi\ \vec{\jmath} - \sin\theta\,\vec{k}$ *by* $\cos\theta$ *, and we get:*

$$\cos\theta \times \vec{u}_\vartheta = \cos\theta \times (\cos\theta\cos\varphi\,\vec{\imath} + \cos\theta\ \sin\varphi\ \vec{\jmath} - \sin\theta\,\vec{k}) \qquad \textbf{Eq(b)}$$

The sum of the equations Eq(a) and Eq(b) gives:

$$\sin\theta \times \vec{u}_r + \cos\theta \times \vec{u}_\vartheta = \sin\theta^2\cos\varphi\,\vec{\imath} + \sin\theta^2\sin\varphi\,\vec{\jmath} + \cos\theta^2\cos\varphi\,\vec{\imath} + \cos\theta^2\sin\varphi\,\vec{\jmath}$$

$$\sin\theta \times \vec{u}_r + \cos\theta \times \vec{u}_\vartheta = (\sin\theta^2 + \cos\theta^2)\cos\varphi\,\vec{\imath} + (\sin\theta^2 + \cos\theta^2)\sin\varphi\,\vec{\jmath}$$

$$\sin\theta \times \vec{u}_r + \cos\theta \times \vec{u}_\vartheta = \cos\varphi\,\vec{\imath} + \sin\varphi\,\vec{\jmath}$$

and therefore we can write ,

$$\frac{d\vec{u}_\varphi}{dt} = -\dot{\varphi}\cos\varphi\ \vec{\imath} - \dot{\varphi}\sin\varphi\ \vec{\jmath} = -\dot{\varphi}\left(\cos\varphi\,\vec{\imath} + \sin\varphi\ \vec{\jmath}\right)$$

$$\frac{d\vec{u}_\varphi}{dt} == -\dot{\varphi}\,(\sin\theta \times \vec{u}_r + \cos\theta \times \vec{u}_\vartheta)$$

$$\begin{cases} \dfrac{d\vec{u}_r}{dt} = \dot{\theta}\,\vec{u}_\vartheta + \dot{\varphi}\sin\theta\ \vec{u}_\varphi \\ \dfrac{d\vec{u}_\vartheta}{dt} = -\dot{\theta}\vec{u}_r + \dot{\varphi}\cos\theta\,\vec{u}_\varphi \\ \dfrac{d\vec{u}_\varphi}{dt} = -\dot{\varphi}\,(\sin\theta \times \vec{u}_r + \cos\theta \times \vec{u}_\vartheta) \end{cases}$$

1.7) - Velocity and acceleration in Intrinsic coordinates system

The velocity vector is tangent to the path in the same direction as the movement, it carries the unit vector $\vec{u}_T$ *and is written:*

$$\vec{v} = \|\vec{v}\|\,\vec{u}_T = \frac{dS}{dt}\,\vec{u}_T$$

where dS *is the curvilinear abscissa*

The acceleration of the point M in is written:

$$\vec{a} = a_T\,\vec{u}_T + a_N\,\vec{u}_N$$

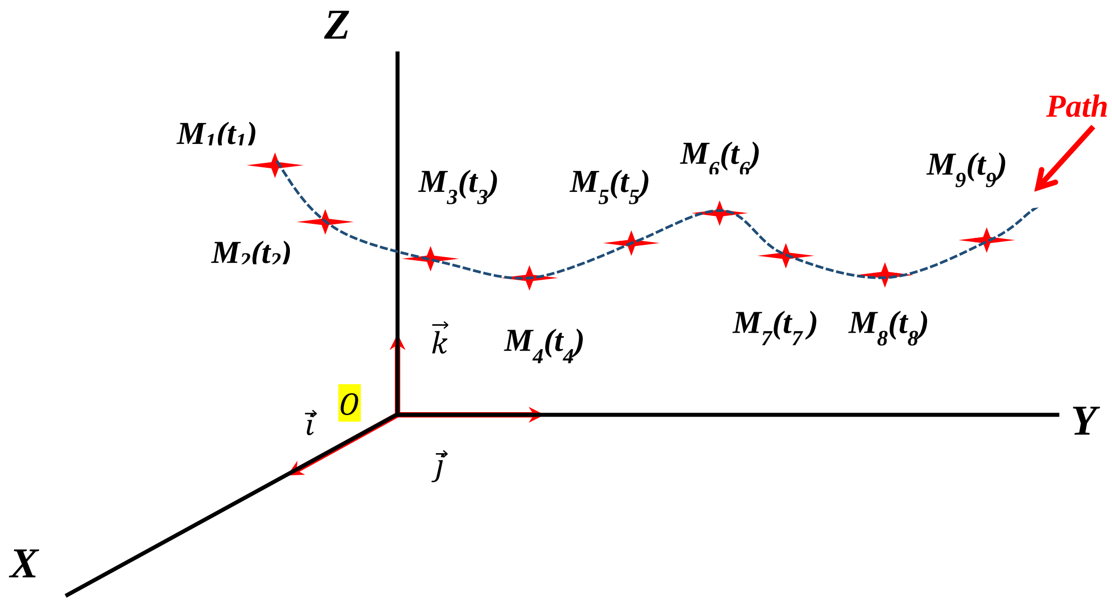
where:

$\boldsymbol{a_T}$ *is the tangential component of the acceleration :* $a_T = \frac{dv}{dt}$

$\boldsymbol{a_N}$ *is the normal component of the acceleration :* $a_N = \frac{v^2}{R}$ *here, R represent the curvature radius.*

Motion Path (trajectory)

The trajectory is a set of successive positions occupied by a moving particle at consecutive (successive) moments. In other words, the trajectory is the geometric locus of consecutive positions of the moving particle. It represents the connected line passing through all the positions of the moving particle during its motion in chronological order. By examining the shape of the trajectory followed by the moving particle, we can determine the type of its motion, whether it is straight, curved, or circular.



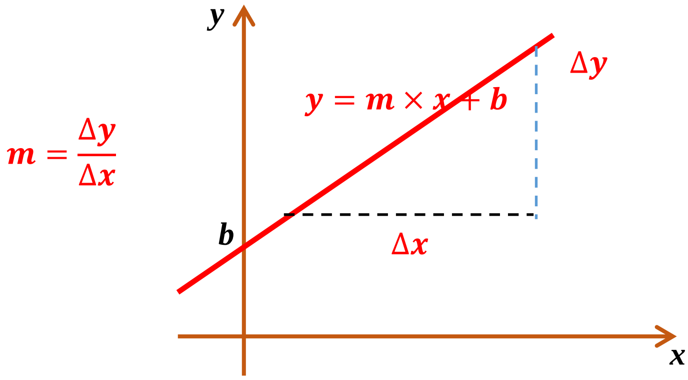
1.8.1) Path (trajectory) Equation

The path equation for a moving particle is a function that defines the relationship between the coordinates of this particle during its movement. Examples of some standard forms of the path equations:

- ***Straight Path Equation:***
  In two-dimensional Cartesian coordinates, a straight path can be represented by the equation:

$$y = m \times x + b$$

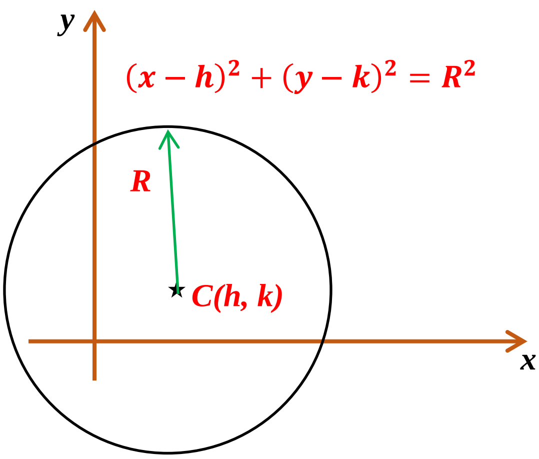
where m *is the slope of the line, and* b *is the y-intercept.*



- ***Circular Path Equation:***

The equation for a circle with radius R *centered at the point* $C(h,k)$

$$(x - h)^2 + (y - k)^2 = R^2$$



- ***Parabolic Path Equation:***

The general equation for a parabola in standard form is:

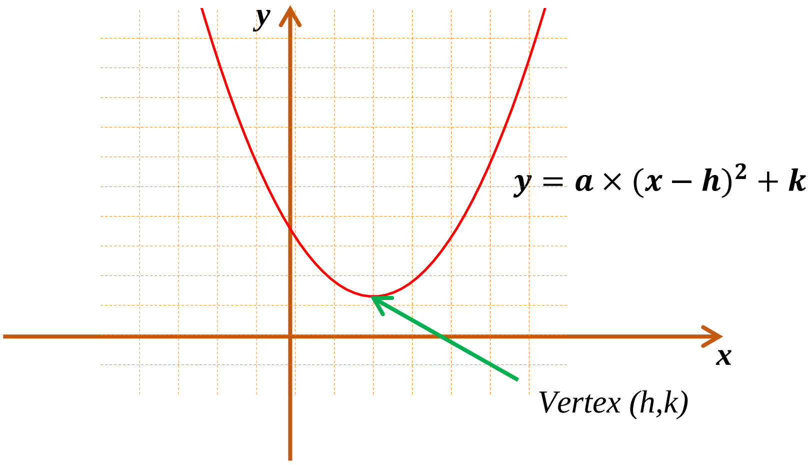
$$y = a \times x^2 + b \times x + c$$

This is a quadratic equation, where a, b, *and* c *are constants.*

The vertex form of a parabola is:

$$y = a \times (x - h)^2 + k$$

Where (h,k) is the vertex of the parabola.

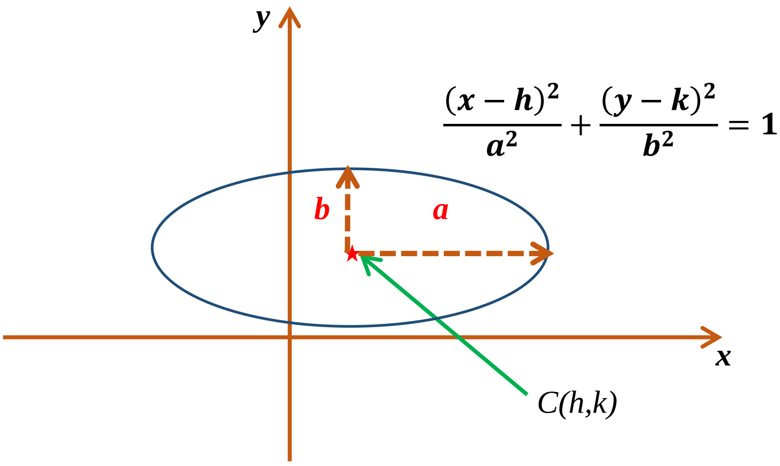


- ***Elliptical Path Equation:***

  The general equation for a hyperbola is given by:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

*where, **(h, k)** is the center of the hyperbola, and **a** and **b** are constants determining the shape of the hyperbola.*



Example01:

The position of a moving particle in the Cartesian coordinate system is described over time by the following equations: $x = 3(1 + cos(2t)), y = 3(2 + sin(2t))$

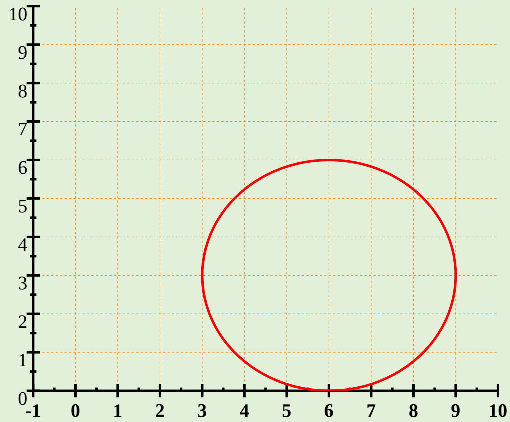
1- Determine the Path (trajectory) Equation and plot it.

$$\begin{cases} x = 3\big(1 + cos(2t)\big) \\ y = 2(2 + sin(2t)) \end{cases} \Rightarrow \begin{cases} x = 3 + 3\,cos(2t) \\ y = 6 + 3\,sin(2t) \end{cases} \Rightarrow \begin{cases} \frac{x-3}{3} = cos(2t) \\ \frac{y-6}{3} = sin(2t) \end{cases}$$

$$\begin{cases} \left(\frac{x-3}{3}\right)^2 = \left(cos(2t)\right)^2 \\ \left(\frac{y-6}{3}\right)^2 = \left(sin(2t)\right)^2 \end{cases} \Rightarrow \left(\frac{x-3}{3}\right)^2 + \left(\frac{y-6}{3}\right)^2 = 1$$

$$(x-3)^2 + (y-6)^2 = 3^2$$

The trajectory takes on a circular form as the derived equation represents a circle with a radius of 3, centered at the coordinates (3, 6)



9) Motion types

The nature of particle motion is determined by the shape of its path (straight, circular, or curved) and the variation in its velocity (constant, non-uniform, or uniformly varying).

9-1) Rectilinear motion (RM)

Rectilinear motion is the type of movement that occurs along a straight path, described by a single coordinate and studied in a one-dimensional coordinate system (dimension). The expressions for the kinematics characteristics (the position, velocity and acceleration) of a moving particle are given as follows:

$$\begin{cases} \overrightarrow{OM} = x(t)\,\vec{\imath} \\ \vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx(t)}{dt}\vec{\imath} = \dot{x}(t)\,\vec{\imath} \\ \vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2x(t)}{d^2t}\vec{\imath} = \ddot{x}(t)\vec{\imath} \end{cases}$$

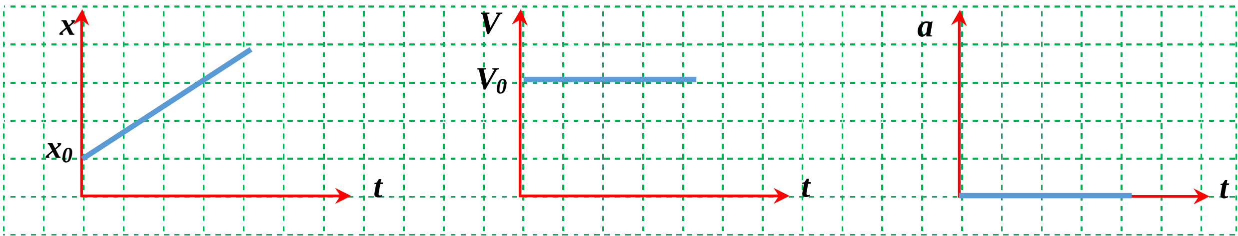
9-1-1) Uniform rectilinear motion (URM)

Uniform rectilinear motion occurs along a straight path with constant magnitude and direction of velocity (uniform velocity). This means that during this motion, the particle travels equal distances in equal time intervals. The expressions for the position, velocity, and acceleration of a moving particle are given as follows:

$$\begin{cases} \vec{a} = \dfrac{d\vec{V}}{dt} = \vec{0} \\ \vec{V} = \displaystyle\int \vec{a}\,dt = constant\ \vec{\imath} = V_0\vec{\imath} \\ \overrightarrow{OM} = \displaystyle\int \vec{V}\,dt = (V_0 \times t + x_0)\ \vec{\imath} \end{cases} \Rightarrow \begin{cases} \overrightarrow{OM} = x(t)\vec{\imath} = (V_0 \times t + x_0)\ \vec{\imath} \\ \vec{V} = \dfrac{d\overrightarrow{OM}}{dt} = constant\ \vec{\imath} = V_0\vec{\imath} \\ \vec{a} = \dfrac{d\vec{V}}{dt} = \vec{0} \end{cases}$$

The time equations of motion are given:

$$x(t) = V_0 \times t + x_0$$



9-1-2) Uniformly varied rectilinear motion (UVRM)

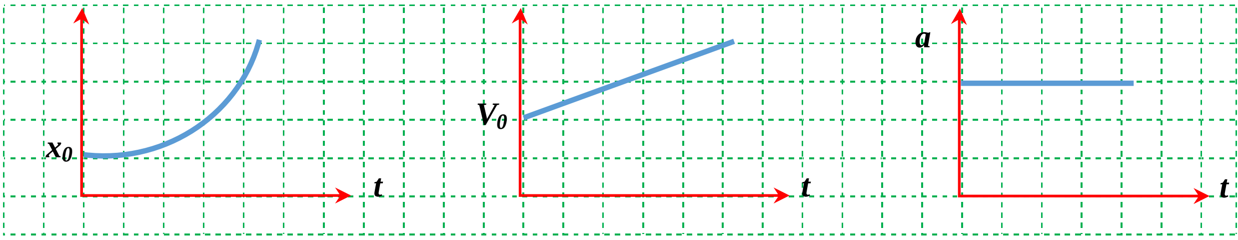
Uniformly Varied Rectilinear Motion occurs when an object moves along a straight path with a uniformly changing velocity, implying a constant acceleration. In simpler terms, during this motion, the particle's velocity undergoes a consistent change at a steady rate over equal time intervals. The expressions for the position, velocity, and acceleration of a moving particle are as follows:

$$\begin{cases} \vec{a} = \dfrac{d\vec{V}}{dt} = onstant\ \vec{\imath} = \ a\,\vec{\imath} \\ \vec{V} = \displaystyle\int \vec{a}\,dt = (\,a \times t + V_0)\,\vec{\imath} \\ \overrightarrow{OM} = \displaystyle\int \vec{V}\,dt = (\frac{1}{2}a \times t^2 + V_0 \times t + x_0)\ \vec{\imath} \end{cases} \Rightarrow \begin{cases} \overrightarrow{OM} = (\dfrac{1}{2}a \times t^2 + V_0 \times t + x_0)\,\vec{\imath} \\ \vec{V} = \dfrac{d\overrightarrow{OM}}{dt} = (\,a \times t + V_0)\,\vec{\imath} \\ \vec{a} = \dfrac{d\vec{V}}{dt} = a\,\vec{\imath} \end{cases}$$

The time equations of motion are given:

$$x(t) = (\tfrac{1}{2}a \times t^2 + V_0 \times t + x_0)$$

x **V**

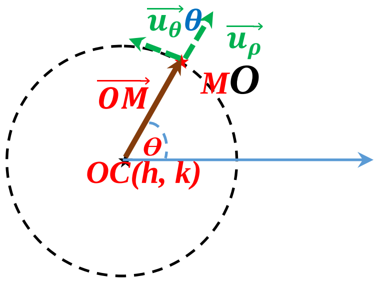


According to the direction of both the velocity and acceleration vectors of the particle's movement, we can distinguish two cases:

- ***Accelerated motion:*** *If acceleration and velocity are in the same direction* $\Rightarrow$ $\vec{v}.\vec{a} > 0$
- ***Retarded motion:*** *If acceleration and velocity are in opposite directions* $\Rightarrow$ $\vec{v}.\vec{a} < 0$

9-2) Circular motion

Circular motion is motion that takes place along a circular path that has a fixed radius ***R****. This movement can be studied using the polar coordinate system.*



The expressions for the position, velocity and acceleration vectors of a moving particle are given as follows:

$$\overrightarrow{OM} = \rho\vec{u_\rho} = R\,\vec{u_\rho}$$

$$\vec{V}(t) = \frac{d(\overrightarrow{OM})}{dt} = \frac{d(R\vec{u_\rho})}{dt} = R\,\frac{d\vec{u_\rho}}{dt}$$

$$\frac{d\,(\vec{u}_\rho)}{dt} = \dot{\theta}\vec{u}_\theta$$

$$\frac{d\,(\vec{u}_\theta)}{dt} = -\dot{\theta}\vec{u}_\rho$$

$$\vec{V}(t) = R\,\dot{\theta}\vec{u}_\theta \text{ where } \dot{\theta} = \frac{d\theta}{dt}$$

$$\|\vec{V}(t)\| = \sqrt{(R\dot{\theta})^2} = R\dot{\theta}$$

$\dot{\theta}$ *is he angular velocity*

for the instantaneous **acceleration** *vector in* **polar** *coordinates is given as follows:*

$$\vec{a}(t) = \frac{d(\vec{V})}{dt} = \frac{d(R\dot{\theta}\vec{u}_\theta)}{dt}$$

$$= R\frac{d\dot{\theta}}{dt}\vec{u}_\theta + R\dot{\theta}\frac{d\vec{u}_\theta}{dt}$$

$$= R\ddot{\theta}\vec{u}_\theta - R\dot{\theta}\dot{\theta}\vec{u}_\rho$$

$$\vec{a}(t) = (-R\dot{\theta}^2)\,\vec{u}_\rho + (R\ddot{\theta})\vec{u}_\theta \;= (a_r)\,\vec{u}_\rho + (a_\theta)\vec{u}_\theta$$

$$\|\vec{a}(t)\| = \sqrt{= \left(R\dot{\theta}^2\right)^2 + (R\ddot{\theta})^2}$$

$\ddot{\theta}$ *is the angular acceleration.*

Note: *For the intrinsic coordinates, we perform the same steps, only we replace the unit vectors as follows:* $\vec{u_\rho} = -\vec{u_N}$ **and** $\vec{u_\theta} = \vec{u_T}$

$$\vec{a}(t) = a_N\vec{u}_N + a_T\vec{u}_T$$

$$\begin{cases} a_{N=}R\dot{\theta}^2 \\ a_T = R\ddot{\theta} \end{cases}$$

9-2-1) Uniform circular motion UCM

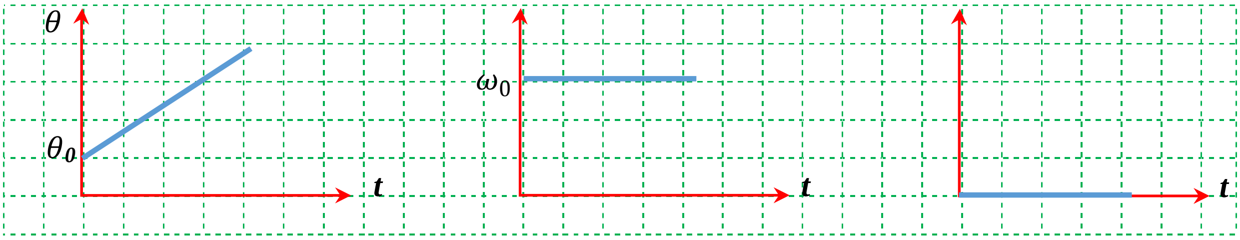
The Uniform circular motion occurs along a circular path with a constant angular velocity ($\dot{\theta} = \frac{d\theta}{dt} = constant = \omega_0$). *This means that the particle displaces at constant angles during equal time intervals.*

$$\begin{cases} \ddot{\theta} = \dfrac{d\dot{\theta}}{dt} = 0 \\ \dot{\theta} = \displaystyle\int \ddot{\theta}dt = constant = \omega_0 \end{cases} \Rightarrow \begin{cases} \overrightarrow{OM} = \rho\vec{u_\rho} = R\,\vec{u_\rho} \\ \vec{V}(t) = R\,\omega_0\vec{u}_\theta \\ \vec{a}(t) = (-R\dot{\theta}^2)\,\vec{u}_\rho \end{cases}$$

In uniform circular motion, the velocity vector stays consistently magnitude and tangential to the object's path at its position. Simultaneously, the acceleration is radial, pointing towards the centre of the circle, with a zero tangential component.

The time equations of motion are given:

$$\begin{cases} \ddot{\theta} = 0 \\ \dot{\theta} = \dfrac{d\theta}{dt} = \omega_0 \\ \theta(t) = \omega_0 \times t + \theta_0 \end{cases}$$



2-2-2) Uniformly varied circular motion (UVCM)

Uniformly varied circular motion (UVCM) is movement that takes place on a circular path with angular velocities that vary regularly (constant angular acceleration). In other words, the angular velocity changes at a constant rate over equal time intervals.

$$\begin{cases} \ddot{\theta} = \dfrac{d\dot{\theta}}{dt} = constant \\ \dot{\theta} = \int \ddot{\theta} dt = \ddot{\theta} \times t + \dot{\theta}_0 \end{cases} \Rightarrow \begin{cases} \overrightarrow{OM} = \rho \overrightarrow{\boldsymbol{u}_\rho} = \boldsymbol{R}\, \overrightarrow{\boldsymbol{u}_\rho} \\ \vec{\boldsymbol{V}}(\boldsymbol{t}) = \boldsymbol{R}\, \dot{\theta} \vec{\boldsymbol{u}}_\theta \\ \vec{\boldsymbol{a}}(\boldsymbol{t}) = (-R\dot{\theta}^2)\, \vec{\boldsymbol{u}}_\rho + (R\ddot{\boldsymbol{\theta}}) \vec{\boldsymbol{u}}_\theta \end{cases}$$

The time equations of motion are given:

$$\begin{cases} \ddot{\theta} = constant \\ \dot{\theta} = \ddot{\theta} \times t + \dot{\theta}_0 \\ \theta(t) = \dfrac{1}{2}\ddot{\theta} \times t^2 + \dot{\theta}_0 \times t + \theta_0 \end{cases}$$

During uniformly varied circular motion, the acceleration vector has both radial and tangential components