All this means that at the moment our collection of functions we've defined is rather small, and doesn't give the richness we'd like when exploring examples of sequences. So for now we'll work with these familiar functions (trig, exponential, log)—we'll assume that they exist and have the properties we expect. You can do this on the problems sheets too. When we come to defining them later on, you can watch out to see that we don't have any circular arguments!

Notation. When we use logarithms, these will all be to the base e. We write $\log x$ for $\log_{\mathrm{e}}(x)$. We don't write $\ln x$.

For $a > 0$ and $x \in \mathbb{R}$, we define $a^x = \mathrm{e}^{x \log a}$. (Of course this relies on definitions of the exponential and logarithm functions, which will come later.)

Remark. Examples can be really useful. I don't mean worked examples (although these can also be really useful), I mean examples of objects that do or don't have certain properties. I'll include some examples in these notes and the accompanying videos. You'll find additional examples in Dr Hilary Priestley's lecture notes, on the Moodle page for Analysis I, and I encourage you to work through those too. I also encourage you to try your own examples (and non-examples), to help you to deepen your experience and understanding of the definitions and results we'll meet.

Example. Here are some informal examples of sequences.

- $\dfrac{3}{10}, \dfrac{33}{100}, \dfrac{333}{1000}, \dfrac{3333}{10000}, \ldots$ are approximations to $\frac{1}{3}$, each better than the previous.
- $\dfrac{14}{10}, \dfrac{141}{100}, \dfrac{1414}{1000}, \dfrac{14142}{10000}, \ldots$ are approximations to $\sqrt{2}$, each better than the previous.

- Take $\varepsilon > 0$. Then, by the Archimedean property, there is $N \geqslant 1$ such that $0 < \frac{1}{N} < \varepsilon$. Now for all $n \geqslant N$ we have $0 < \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon$. We see that apart from finitely many terms at the start, the terms of the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ all lie within distance ε of 0. This is the case for any positive real number ε.

- $1, -1, 2, -2, 3, -4, 4, -4, \ldots$ is another sequence, and intuitively it feels as though it does not tend to a limit.

- $7, 1.2, -5, 2, 324, -9235.32, \ldots$ is another sequence—there is no clear pattern to the terms (I just typed them wherever my fingers landed), but it is still a sequence.

What exactly is a sequence?

Definition. A *real sequence*, or *sequence* of real numbers, is a function $\alpha : \mathbb{N} \to \mathbb{R}$. We call $\alpha(n)$ the n^{th} term of the sequence.

We usually write a_n for $\alpha(n)$, and say that α defines the sequence (a_n) with terms $a_1, a_2, a_3, a_4, \ldots$. We might also write this as $(a_n)_{n \geqslant 1}$ or $(a_n)_{n=1}^{\infty}$.

Similarly, a *complex sequence* is formally a function $\alpha : \mathbb{N} \to \mathbb{C}$, and we write it as (a_n), where now $a_n \in \mathbb{C}$ for $n \geqslant 1$.

Remark.

- The order of the terms in a sequence matters!

- We write (a_n) for the sequence, and a_n for a term of the sequence.

- Much of the theory relating to sequences applies to both real and complex sequences. Sometimes, though, we'll need to focus only on real sequences—for example if we're using inequalities. In this case we'll carefully specify that we're working with real sequences. If we don't specify, and just say ‘sequences’, then it applies equally to real and complex sequences. We’ll also have a section (and corresponding video) at the end of this block concentrating on complex sequences.

Example.

- Let $a_n = (-1)^n$. Then the first few terms of the sequence are $-1, 1, -1, 1, -1, 1, \dots$.
- Let $a_n = \frac{\sin n}{2n+1}$. Then the first few terms of the sequence are
$$\frac{1}{3}\sin 1, \frac{1}{5}\sin 2, \frac{1}{7}\sin 3, \dots .$$
- Let
$$a_n = \begin{cases} 0 & \text{if } n \text{ is prime} \\ 1 + \frac{1}{n} & \text{otherwise.} \end{cases}$$
Then the first few terms of the sequence are $2, 0, 0, \frac{5}{4}, 0, \frac{7}{6}, 0, \frac{9}{8}, \dots$.
- Let $a_n = n$. Then the first few terms of the sequence are $1, 2, 3, 4, 5, \dots$.

Definition. We can make new sequences from old. Let (a_n), (b_n) be sequences and let c be a constant. Then we can define new sequences ‘termwise’: $(a_n + b_n)$, $(-a_n)$, $(a_n b_n)$, (ca_n), $(|a_n|)$. If $b_n \neq 0$ for all n, then we can also define a sequence $(\frac{a_n}{b_n})$.

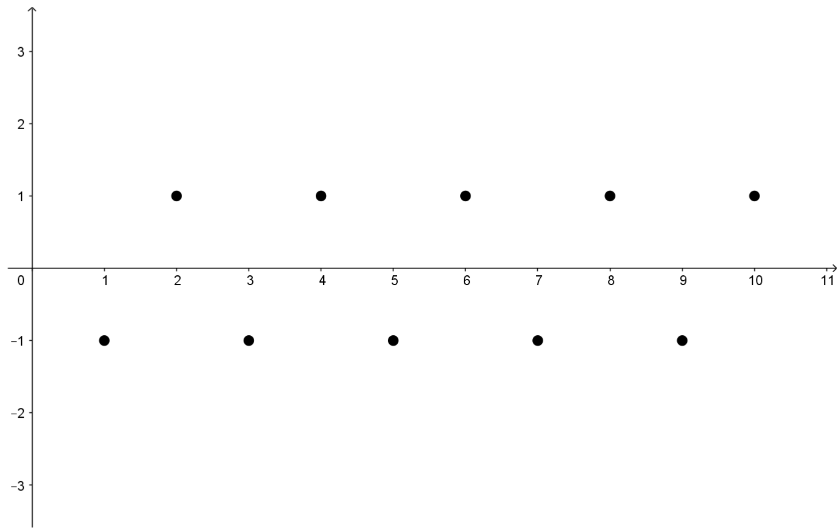
Example. Let $a_n = (-1)^n$ and $b_n = 1$ for $n \geqslant 1$.

Then the first few terms of $(a_n + b_n)$ are $0, 2, 0, 2, 0, 2, \dots$; and $(-a_n) = ((-1)^{n+1})$; and $(|a_n|) = (b_n)$.

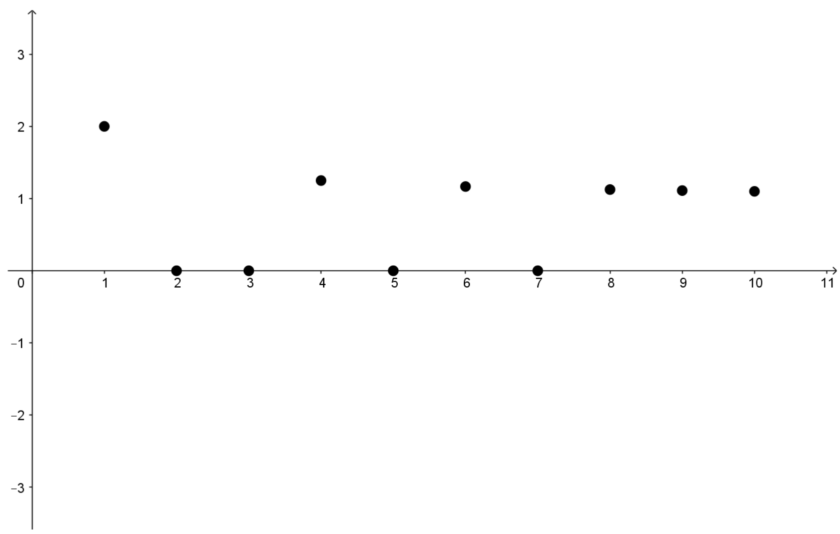
16 Convergence of a sequence

Before we see a formal definition of convergence, let’s consider some examples informally. Here’s one way I like to visualise a sequence. These are examples from the previous section. Each graph plots the points (n, a_n) for $1 \leqslant n \leqslant 10$.

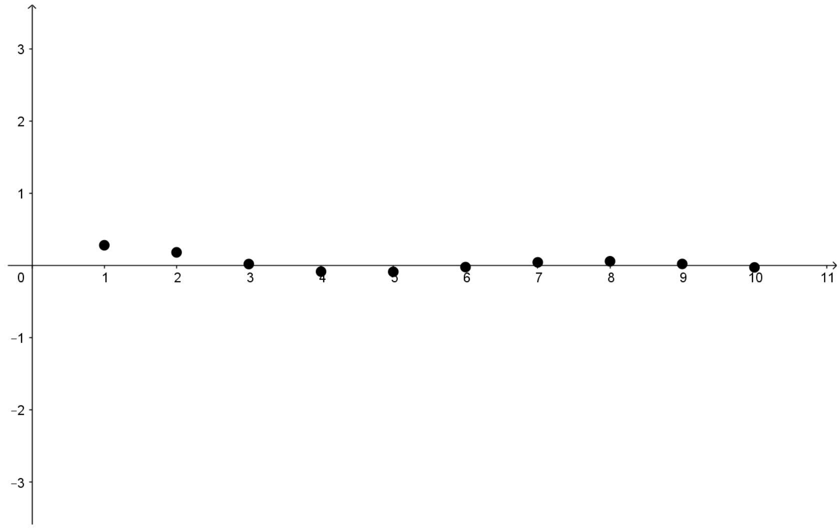
- $a_n = (-1)^n$



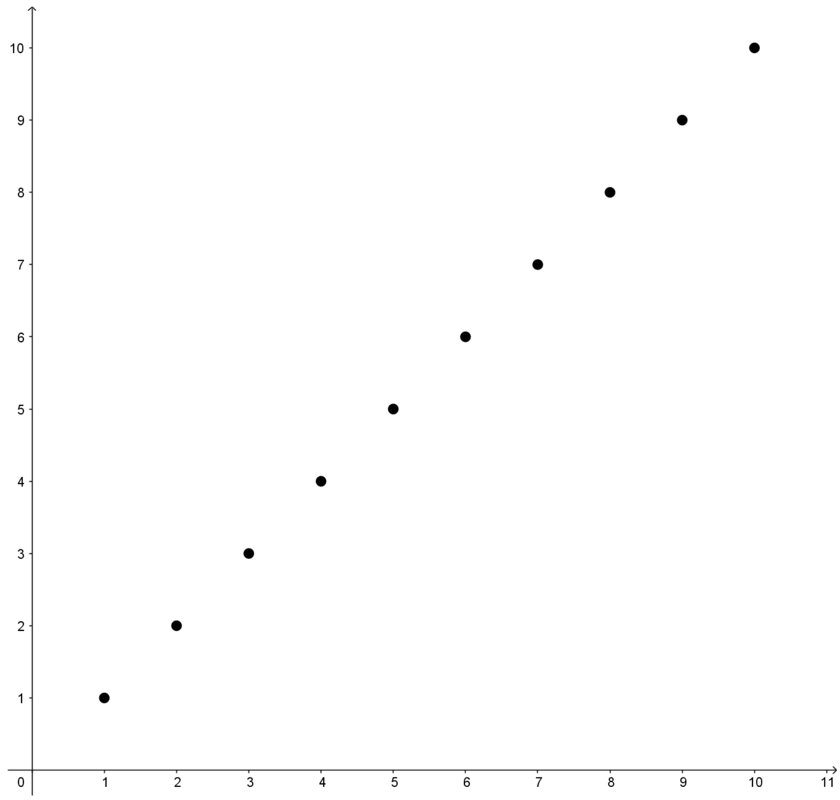
- $a_n = \begin{cases} 0 & \text{if } n \text{ is prime} \\ 1 + \frac{1}{n} & \text{otherwise} \end{cases}$



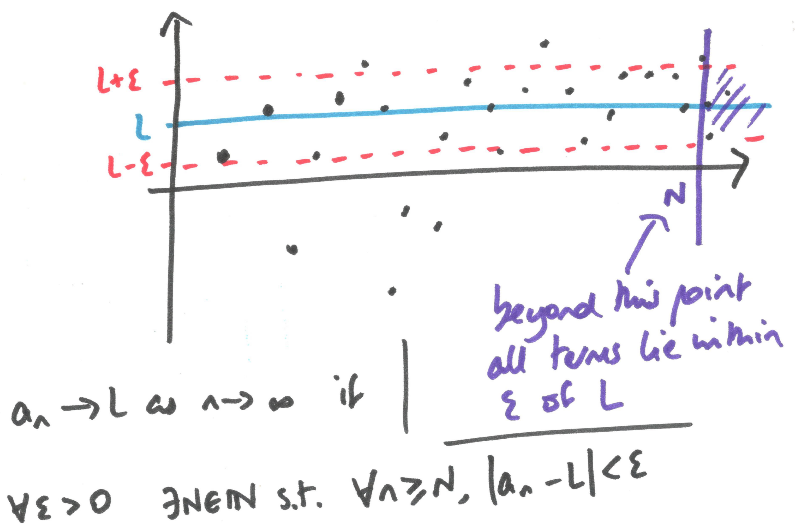
- $a_n = \frac{\sin n}{2n+1}$



- $a_n = n$



Here is an unofficial picture of the definition of convergence.



Definition. Let (a_n) be a real sequence, let $L \in \mathbb{R}$. We say that (a_n) *converges* to L as $n \to \infty$ if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n \geqslant N, \ |a_n - L| < \varepsilon.$$

In this case we write $a_n \to L$ as $n \to \infty$, and we say that L is the *limit* of (a_n).

Remark.

- We might also say that (a_n) *tends to* L as $n \to \infty$, and we might also write that $\lim_{n\to\infty} a_n = L$.
- N can depend on ε, and almost always will.
- The 'order of the quantifiers' matters. We wrote "$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \dots$". This order allows N to depend on ε. If we wrote "$\exists N \in \mathbb{N}$ such that $\forall \varepsilon > 0 \dots$" that would be something quite different.

  We could replace $n \geqslant N$ in the definition by $n > N$, and $|a_n - L| < \varepsilon$ by $|a_n - L| \leqslant \varepsilon$, without changing the definition. (Check this!) But it's crucial that we have $\varepsilon > 0$ not $\varepsilon \geqslant 0$. (Check this!)

- I put 'the' limit in the definition. We'll see later that if it exists then it's unique.

Definition. Let (a_n) be a real sequence. We say that (a_n) *converges*, or *is convergent*, if there is $L \in \mathbb{R}$ such that $a_n \to L$ as $n \to \infty$. If (a_n) does not converge, then we say that it *diverges*, or *is divergent*.

Intuitively, the first thousand or million terms of a sequence shouldn't affect whether it converges. We'll prove a result that makes this precise, but first we need a quick definition.

Definition. Let (a_n) be a sequence. A *tail* of (a_n) is a sequence (b_n), where for some natural number k we have $b_n = a_{n+k}$ for $n \geqslant 1$. That is, (b_n) is the sequence obtained by deleting the first k terms of (a_n).

Lemma 23 (Tails Lemma)**.** *Let (a_n) be a sequence.*

(i) If (a_n) converges to a limit L, then every tail of (a_n) also converges, and to this same limit L.

(ii) If a tail $(b_n) = (a_{n+k})$ of (a_n) converges, then (a_n) converges.

Proof. (i) Take a tail of (a_n): take $k \geqslant 1$ and let $b_n = a_{n+k}$ for $n \geqslant 1$.

Assume that (a_n) converges to a limit L.

Take $\varepsilon > 0$.

Then there is N such that if $n \geqslant N$ then $|a_n - L| < \varepsilon$.

Now if $n \geqslant N$ then $n + k \geqslant N$ so $|a_{n+k} - L| < \varepsilon$, that is, $|b_n - L| < \varepsilon$.

So (b_n) converges and $b_n \to L$ as $n \to \infty$.

(ii) Assume that $(b_n) = (a_{n+k})$ converges.

Then there is $L \in \mathbb{R}$ such that $b_n \to L$ as $n \to \infty$.

Take $\varepsilon > 0$.

Then there is N such that if $m \geqslant N$ then $|b_m - L| < \varepsilon$, that is, $|a_{m+k} - L| < \varepsilon$.

Now if $n \geqslant N + k$ then $n = m + k$ where $m \geqslant N$, and so $|a_n - L| < \varepsilon$.

So (a_n) converges and $a_n \to L$ as $n \to \infty$.

□

Example. We'll see later (soon!) that there are other ways to prove convergence, not only directly from the definition. But for now we've only got the definition (and the Tails Lemma), so let's get some practice using what we've got so far.

- **Claim.** $\frac{1}{n} \to 0$ as $n \to \infty$.

  Proof. Take $\varepsilon > 0$.

  Then there is $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$ (by the Archimedean property).

  For $n \geqslant N$ we have $|\frac{1}{n} - 0| = \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon$.

  So $\frac{1}{n} \to 0$ as $n \to \infty$. □

- **Claim.** Let $a_n = 1 + (-1)^n \frac{1}{\sqrt{n}}$ for $n \geqslant 1$. Then $a_n \to 1$ as $n \to \infty$.

  Proof. Take $\varepsilon > 0$.

  Aim: want N such that if $n \geqslant N$ then $|a_n - 1| < \varepsilon$

  that is, $|(1 + (-1)^n \frac{1}{\sqrt{n}}) - 1| < \varepsilon$

  that is, $\frac{1}{\sqrt{n}} < \varepsilon$, that is, $\frac{1}{\varepsilon} < \sqrt{n}$.

Take $N = \lceil \frac{1}{\varepsilon^2} \rceil + 1$.

Here $\lceil x \rceil$ *denotes the* ceiling function*: it is defined to be the smallest integer greater than or equal to* x*. Informally, if* x *is an integer then take that value; otherwise, round up to the next integer.*

If $n \geqslant N$, then

$$\begin{aligned}
&n > \frac{1}{\varepsilon^2} \\
&\text{so } \sqrt{n} > \frac{1}{\varepsilon} \\
&\text{so } \frac{1}{\sqrt{n}} < \varepsilon \\
&\text{so } |a_n - 1| < \varepsilon.
\end{aligned}$$

So $a_n \to 1$ as $n \to \infty$. □

- **Claim.** Let $a_n = \dfrac{n\cos(n^3+1)}{5n^2+1}$ for $n \geqslant 1$. Then $a_n \to 0$ as $n \to \infty$.

Proof. Take $\varepsilon > 0$.

Aim: want N such that if $n \geqslant N$ then $|a_n - 0| < \varepsilon$,
that is, $\left|\dfrac{n\cos(n^3+1)}{5n^2+1}\right| < \varepsilon$

but $|\cos(n^3+1)| \leqslant 1$ so it's enough to ensure that $\left|\dfrac{n}{5n^2+1}\right| < \varepsilon$

and $5n^2 + 1 \geqslant 5n^2$ so it's enough to ensure that $\left|\frac{n}{5n^2}\right| < \varepsilon$

that is, $\frac{1}{5n} < \varepsilon$, that is, $n > \frac{1}{5\varepsilon}$.

Take $N = \lceil \frac{1}{\varepsilon} \rceil + 1$.

If $n \geqslant N$, then $n > \frac{1}{5\varepsilon}$ so

$$|a_n| = \left|\frac{n\cos(n^3+1)}{5n^2+1}\right| \leqslant \frac{1}{5n} < \varepsilon.$$

So $a_n \to 0$ as $n \to \infty$. $\square$

Remark. Here are some top tips!

- We don't need the smallest possible N. It's (almost always) not even interesting to know what it is. So make your life easier! If an inequality (in the right direction) helps, then go for it.
- Be careful to make sure that the logic flows in the right direction, and that you've set out the logic explicitly. Hopefully the examples we've just seen help you to have ideas of how to do this.
- The definition officially says $N \in \mathbb{N}$, but we don't really care whether N is a natural number. If we have a value that works, then we can always choose a natural number larger than it.
- We think of ε as a small positive real number, but we are obliged to prove it for *all* $\varepsilon > 0$. But if we can prove it for say $0 < \varepsilon < 1$ then that's enough—if N works for a certain ε then it works for all larger values too. So you can work with a smaller range of ε, such as $0 < \varepsilon < 1$, if that is most convenient (but it would be a good idea to mention briefly why this is sufficient).
- It's really worth becoming comfortable with inequalities and modulus. In the examples, it was nicer to use the absolute values to write things like $|a_n - L| < \varepsilon$, rather than $-\varepsilon < a_n - L < \varepsilon$. If you prefer the second at the moment, then I recommend practising to get used to the first!

Working directly from the definition is often painful or impractical. Our next goal is to prove a result that will give a more convenient strategy for proving convergence in some circumstances.

17 Limits: first key results

The next result is extremely useful in practice! We'll see a more general version later, but even this version is strong enough to be useful.

Proposition 24 (Sandwiching, first version)**.** *Let (a_n) and (b_n) be real sequences with $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant 1$. If $b_n \to 0$ as $n \to \infty$, then $a_n \to 0$ as $n \to \infty$.*

Remark. You might like to draw yourself a diagram to develop your intuition for what this result says.

Proof. Idea: if N works for b_n then it works for a_n too.

Assume that $0 \leqslant a_n \leqslant b_n$ for all n, and that $b_n \to 0$ as $n \to \infty$.

Take $\varepsilon > 0$.

Since $b_n \to 0$, there exists N such that if $n \geqslant N$ then $|b_n| < \varepsilon$.

Now if $n \geqslant N$ then $0 \leqslant a_n \leqslant b_n < \varepsilon$, so $|a_n| < \varepsilon$.

So $a_n \to 0$ as $n \to \infty$. □

Example.

- **Claim.** $\frac{1}{2^n} \to 0$ as $n \to \infty$.

  Proof. We have $2^n \geqslant n$ for $n \geqslant 1$ (can prove this by induction),

  so $0 \leqslant \frac{1}{2^n} \leqslant \frac{1}{n}$ for $n \geqslant 1$, and $\frac{1}{n} \to 0$,

  so by Sandwiching $\frac{1}{2^n} \to 0$ as $n \to \infty$. □

•

Claim. Let $a_n = \dfrac{n\cos(n^3+1)}{5n^2+1}$ for $n \geqslant 1$ (we saw this example earlier). Then $a_n \to 0$ as $n \to \infty$.

Proof. Idea: apply Sandwiching to $(|a_n|)$.

We have

$$0 \leqslant \left|\frac{n\cos(n^3+1)}{5n^2+1}\right| \leqslant \frac{1}{5n} \leqslant \frac{1}{n}$$

for $n \geqslant 1$,

and $\frac{1}{n} \to 0$ as $n \to \infty$,

so, by Sandwiching, $|a_n| \to 0$ as $n \to \infty$.

But (looking back at the definition) we see that $|a_n| \to 0$ if and only if $a_n \to 0$. □

Here are two key sequences; it will be useful later to have studied them. (You can also think of them as further worked examples.)

Lemma 25. *(i) Take $c \in \mathbb{R}$ with $|c| < 1$. Then $c^n \to 0$ as $n \to \infty$.*

(ii) Let $a_n = \frac{n}{2^n}$ for $n \geqslant 1$. Then $a_n \to 0$ as $n \to \infty$.

Proof. (i) Write $|c| = \frac{1}{1+y}$ where $y > 0$.

Take $\varepsilon > 0$.

Let $N = \lceil \frac{1}{y\varepsilon} \rceil + 1$. (When writing this proof, we might leave this line blank and fill it in at the end!)

Take $n \geqslant N$.

By Bernoulli's inequality (since $y > 0$ and $n \geqslant 1$) we have $(1+y)^n \geqslant 1 + ny$, so

$$|c^n| = \frac{1}{(1+y)^n} \leqslant \frac{1}{1+ny} \leqslant \frac{1}{Ny} < \varepsilon.$$

So $c^n \to 0$ as $n \to \infty$.

(ii) Note that if $n \geqslant 2$ then $2^n = (1+1)^n \geqslant \binom{n}{2}$ (by the binomial theorem).

Take $\varepsilon > 0$.

Let $N = \lceil 2 + \frac{2}{\varepsilon} \rceil$.

For $n \geqslant N$, we have

$$|a_n - 0| = \frac{n}{2^n} \leqslant \frac{n}{\binom{n}{2}} = \frac{2}{n-1} \leqslant \frac{2}{N-1} < \varepsilon.$$

So $a_n \to 0$ as $n \to \infty$.

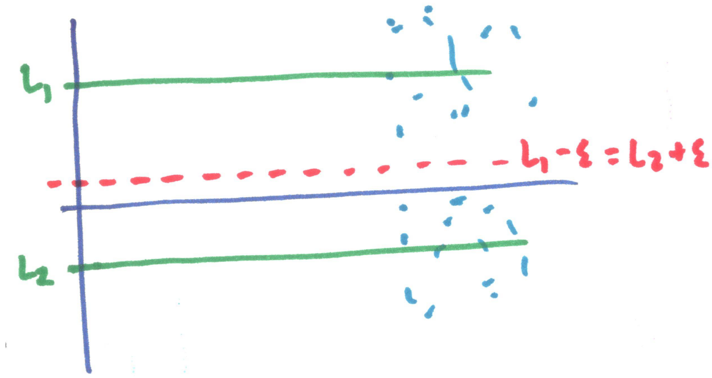
$\square$

As promised earlier, let's show that if a sequence converges, then its limit is unique.

Theorem 26 (Uniqueness of limits)**.** *Let (a_n) be a convergent sequence. Then the limit is unique.*

Proof. Assume that $a_n \to L_1$ and $a_n \to L_2$ as $n \to \infty$. Aim: $L_1 = L_2$.

Idea: contradiction. If $L_1 \neq L_2$, then eventually all the terms are really close to L_1, and also to L_2, and that's not possible.



Suppose, for a contradiction, that $L_1 \neq L_2$.

Let $\varepsilon = \dfrac{|L_1 - L_2|}{2} > 0$.

Since $a_n \to L_1$ as $n \to \infty$, there is N_1 such that if $n \geqslant N_1$ then $|a_n - L_1| < \varepsilon$.

Also, since $a_n \to L_2$ as $n \to \infty$, there is N_2 such that if $n \geqslant N_2$ then $|a_n - L_2| < \varepsilon$.

For $n \geqslant \max\{N_1, N_2\}$ we have $|a_n - L_1| < \varepsilon$ and $|a_n - L_2| < \varepsilon$, so

$$\begin{aligned}
|L_1 - L_2| &= |(L_1 - a_n) + (a_n - L_2)| \\
&\leqslant |L_1 - a_n| + |a_n - L_2| \text{ by the triangle inequality} \\
&< 2\varepsilon = |L_1 - L_2|.
\end{aligned}$$

This is a contradiction.

So $L_1 = L_2$. $\square$

18 Limits: modulus and inequalities

Proposition 27. *Let (a_n) be a convergent sequence. Then $(|a_n|)$ also converges. Moreover, if $a_n \to L$ as $n \to \infty$ then $|a_n| \to |L|$ as $n \to \infty$.*

Proof. Say $a_n \to L$ as $n \to \infty$.

Take $\varepsilon > 0$.

Then there is N such that if $n \geqslant N$ then $|a_n - L| < \varepsilon$.

Now if $n \geqslant N$ then, by the Reverse Triangle Inequality, we have

$$||a_n| - |L|| \leqslant |a_n - L| < \varepsilon.$$

So $(|a_n|)$ converges, and $|a_n| \to |L|$ as $n \to \infty$. $\square$

Remark. We could instead have proved Proposition 27 using the Sandwiching Lemma, since $a_n \to L$ as $n \to \infty$ if and only if $|a_n - L| \to 0$ as $n \to \infty$ (check this using the definition of convergence).

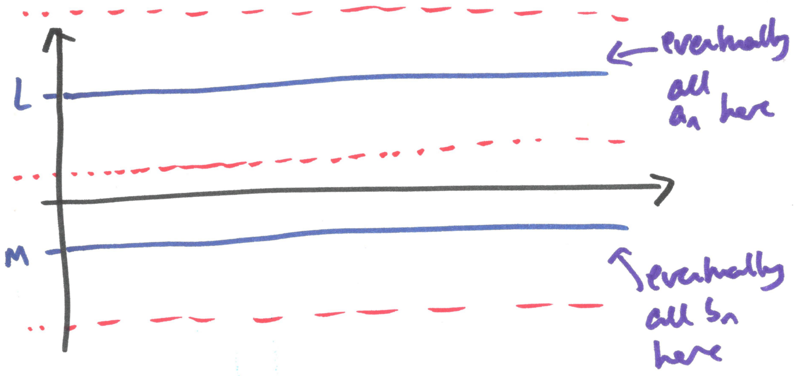
Now let's think about inequalities. If (a_n) is a convergent sequence and $a_n > 0$ for all n, then what can we say about the limit? It's not the case that the limit must be positive. For example, if $a_n = \frac{1}{n}$ then $a_n > 0$ for all n but $a_n \to 0$. But it's hard to see how a sequence of positive terms could have a negative limit.

Proposition 28 (Limits preserve weak inequalities)**.** *Let (a_n) and (b_n) be real sequences, and assume that $a_n \to L$ and $b_n \to M$ as $n \to \infty$, and that $a_n \leqslant b_n$ for all n. Then $L \leqslant M$.*

Remark.

- This includes the special case where $a_n = 0$ for all n: Proposition 28 says that if $b_n \geqslant 0$ for all n, and $b_n \to M$ as $n \to \infty$, then $M \geqslant 0$. (This is because the constant sequence $0, 0, 0, \ldots$ certainly converges to 0.)
- A common mistake is to use the non-result that limits preserve strict inequalities. As we've seen, this is not true. Please try not to do this!

Proof. Suppose, for a contradiction, that it is not the case that $L \leqslant M$, so (by trichotomy) $L > M$.



Let $\varepsilon = \frac{1}{2}(L - M) > 0$.

Since $a_n \to L$ as $n \to \infty$, there is N_1 such that if $n \geqslant N_1$ then $|a_n - L| < \varepsilon$.

Since $b_n \to M$ as $n \to \infty$, there is N_2 such that if $n \geqslant N_2$ then $|b_n - M| < \varepsilon$.

Now for $n \geqslant \max\{N_1, N_2\}$ we have $a_n > L - \varepsilon$ and $b_n < M + \varepsilon$,

so $L - \varepsilon < a_n \leqslant b_n < M + \varepsilon$,

so $L - M < 2\varepsilon = L - M$. This is a contradiction. □

We saw a sandwiching result earlier. Here is a generalisation.

Proposition 29 (Sandwiching). *Let (a_n), (b_n) and (c_n) be real sequences with $a_n \leqslant b_n \leqslant c_n$ for all $n \geqslant 1$. If $a_n \to L$ and $c_n \to L$ as $n \to \infty$, then $b_n \to L$ as $n \to \infty$.*

Proof. Take $\varepsilon > 0$.

Since $a_n \to L$ as $n \to \infty$, there is N_1 such that if $n \geqslant N_1$ then $|a_n - L| < \varepsilon$.

Since $c_n \to L$ as $n \to \infty$, there is N_2 such that if $n \geqslant N_2$ then $|c_n - L| < \varepsilon$.

Then for $n \geqslant \max\{N_1, N_2\}$ we have $L - \varepsilon \leqslant a_n \leqslant b_n \leqslant c_n \leqslant L + \varepsilon$,

so $|b_n - L| < \varepsilon$.

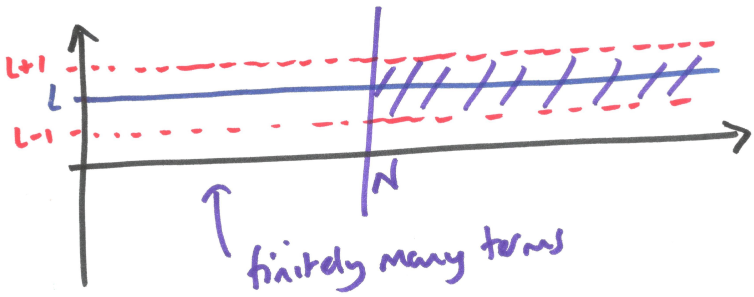
So $b_n \to L$ as $n \to \infty$. □

19 Bounded and unbounded sequences

Definition. Let (a_n) be a sequence. We say that (a_n) is *bounded* if the set $\{a_n : n \geqslant 1\}$ is bounded, that is, there is M such that $|a_n| \leqslant M$ for all $n \geqslant 1$. If (a_n) is not bounded then we say that it is *unbounded*.

Proposition 30 (A convergent sequence is bounded)**.** *Let (a_n) be a convergent sequence. Then (a_n) is bounded.*

Remark. Proposition 30 tells us that if (a_n) is unbounded then (a_n) diverges.



Proof.

Assume that $a_n \to L$ as $n \to \infty$.

Then (taking $\varepsilon = 1$) there is N such that if $n \geqslant N$ then $|a_n - L| < 1$ so

$$|a_n| = |(a_n - L) + L| \leqslant |a_n - L| + |L| < 1 + |L|.$$

Let $M = \max\{|a_1|, |a_2|, \dots, |a_N|, |L| + 1\}$.

Then $|a_n| \leqslant M$ for all $n \geqslant 1$. $\square$

Remark.
- As remarked earlier, if (a_n) is unbounded then (a_n) diverges. So, for example, (2^n) diverges.
- Unboundedness is not the same as divergence. The converse of Proposition 30 is not true. A bounded sequence can diverge. For example, let $a_n = (-1)^n$. Then $|a_n| \leqslant 1$ for all $n \geqslant 1$, so (a_n) is bounded.

  Claim. $((-1)^n)$ does not converge.

Proof. Suppose, for a contradiction, that $(-1)^n \to L$ as $n \to \infty$.

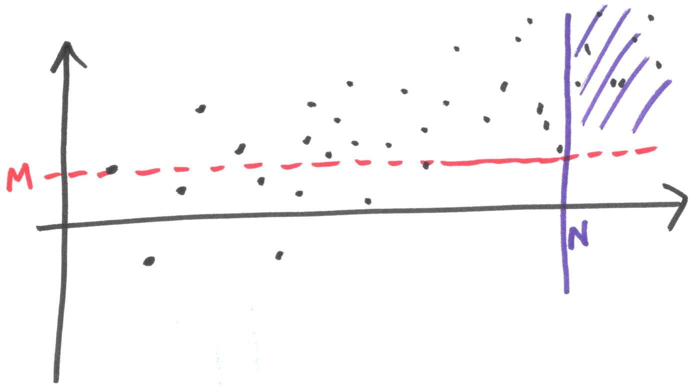
Then (taking $\varepsilon = 1$) there is N such that if $n \geqslant N$ then $|(-1)^n - L| < 1$.

In particular ($n = 2N$) we have $|L - 1| < 1$ so $L > 0$,

and ($n = 2N + 1$) we have $|L + 1| < 1$ so $L < 0$.

This is a contradiction. $\square$

What would it mean to say that a sequence tends to infinity?



$\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}$ s.t. $\forall n \geqslant N \quad a_n > M$

Definition. Let (a_n) be a real sequence. We say that (a_n) tends to infinity as $n \to \infty$ if

$$\forall M \in \mathbb{R}\ \exists N \in \mathbb{N} \text{ such that } \forall n \geqslant N,\ a_n > M.$$

In this case we write $a_n \to \infty$ as $n \to \infty$.

Similarly, we say that (a_n) tends to negative infinity as $n \to \infty$ if

$$\forall M \in \mathbb{R}\ \exists N \in \mathbb{N} \text{ such that } \forall n \geqslant N,\ a_n < M.$$

In this case we write $a_n \to -\infty$ as $n \to \infty$.

Remark. This is a separate definition from our earlier definition of convergence, and ∞ is definitely not a real number. Results about convergence to a real number L cannot just be applied by 'taking $L = \infty$'—this would be highly illegal!

Example.

- Let $a_n = n^2 - 6n$ for $n \geqslant 1$.

  Claim. $a_n \to \infty$ as $n \to \infty$.

  Proof. Fix $M > 0$. (It suffices to prove the result for $M > 0$.)

  We want N such that if $n \geqslant N$ then $n^2 - 6n \geqslant M$

  but $n^2 - 6n = (n-3)^2 - 9$

  so we are done if $(n-3)^2 \geqslant M + 9$

  that is, we are done if $n - 3 \geqslant \sqrt{M+9}$

  Let $N = \lceil 4 + \sqrt{M+9} \rceil$.

  If $n \geqslant N$, then $n - 3 \geqslant \sqrt{M+9} > 0$,

  so $(n-3)^2 \geqslant M + 9$,

  so $n^2 - 6n \geqslant M$.

  So $a_n \to \infty$ as $n \to \infty$. $\square$

- Let $a_n = \begin{cases} 0 & \text{if } n \text{ prime} \\ n & \text{otherwise.} \end{cases}$

  Then (a_n) does not tend to infinity, because there are infinitely many primes: for any $N \in \mathbb{N}$, there is a prime n with $n > N$, and then $a_n = 0$.

Lemma 31. *(i) If $\alpha < 0$, then $n^\alpha \to 0$ as $n \to \infty$.*

(ii) If $\alpha > 0$, then $n^\alpha \to \infty$ as $n \to \infty$.

Proof. (i) Take $\varepsilon \in (0, 1)$. We have

$$\begin{aligned}
&n^\alpha < \varepsilon \\
\Leftrightarrow\ &\mathrm{e}^{\alpha \log n} < \varepsilon \\
\Leftrightarrow\ &\alpha \log n < \log \varepsilon \\
\Leftrightarrow\ &\log n > \frac{1}{\alpha} \log \varepsilon \text{ (note } \alpha < 0) \\
\Leftrightarrow\ &n > \mathrm{e}^{\frac{1}{\alpha} \log \varepsilon}
\end{aligned}$$

so we can take $N = 1 + \lceil \mathrm{e}^{\frac{1}{\alpha} \log \varepsilon} \rceil$.

(ii) Take $M > 0$. We have

$$\begin{aligned}
&n^\alpha > M \\
\Leftrightarrow\ &\mathrm{e}^{\alpha \log n} > M \\
\Leftrightarrow\ &\alpha \log n > \log M \\
\Leftrightarrow\ &\log n > \frac{1}{\alpha} \log M \text{ (note } \alpha > 0) \\
\Leftrightarrow\ &n > \mathrm{e}^{\frac{1}{\alpha} \log M}
\end{aligned}$$

so we can take $N = 1 + \lceil \mathrm{e}^{\frac{1}{\alpha} \log M} \rceil$.

□

Lemma 32. *Let $c \in \mathbb{R}^{>0}$.*

(i) If $c < 1$, then $c^n \to 0$ as $n \to \infty$.

(ii) If $c = 1$, then $c^n \to 1$ as $n \to \infty$.

(iii) If $c > 1$, then $c^n \to \infty$ as $n \to \infty$.

Proof. (i) This was Lemma 25.

(ii) This is clear from the definition of convergence.

(iii) Exercise. (You could adapt the argument from (i), or use logarithms.)

□

20 Complex sequences

A lot of the theory we have just seen applies equally to complex sequences, but there are some differences. Let's spell out the definition of convergence explicitly.

Definition. Let (z_n) be a complex sequence, let $L \in \mathbb{C}$. We say that (z_n) *converges* to L as $n \to \infty$ if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n \geqslant N,\ |z_n - L| < \varepsilon.$$

Remark.

- If (z_n) tends to a limit, then this limit is unique, exactly as in Theorem 26.
- We can have a sort of sandwiching for complex sequences, if we use the modulus. If (z_n) and (w_n) are complex sequences, and $|w_n| \leqslant |z_n|$ for all $n \geqslant 1$, and $z_n \to 0$ as $n \to \infty$, then $w_n \to 0$ as $n \to \infty$.

Given a complex sequence (z_n), there are two associated real sequences $(\mathrm{Re}(z_n))$ and $(\mathrm{Im}(z_n))$. The next result relates convergence of (z_n) to convergence of $(\mathrm{Re}(z_n))$ and $(\mathrm{Im}(z_n))$.

Theorem 33 (Convergence of complex sequences)**.** *Let (z_n) be a complex sequence. Write $z_n = x_n + \mathrm{i}y_n$ with x_n, $y_n \in \mathbb{R}$, so that (x_n) and (y_n) are real sequences. Then (z_n) converges if and only if both (x_n) and (y_n) converge. Moreover, in the case where (z_n) converges, we have $\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + \mathrm{i} \lim_{n\to\infty} y_n$.*

Proof. Exercise. □

Example.

- Let $z_n = \frac{\mathrm{i}^n}{n}$. Then $|z_n| = \frac{1}{n} \to 0$ as $n \to \infty$ so $z_n \to 0$ as $n \to \infty$.
- Let $z_n = (1+\mathrm{i})^n$. The sequence is

$$1+\mathrm{i}, 2\mathrm{i}, -2+2\mathrm{i}, -4, -4-4\mathrm{i}, -8\mathrm{i}, 8-8\mathrm{i}, 16, \dots .$$

  The real parts are $1, 0, -2, -4, -4, 0, 8, 16, \dots$—this sequence doesn't converge, and hence neither does (z_n).

21 Subsequences

We can make a good informal guess as to what we mean by a subsequence.

Let $(a_n)_{n\geqslant 1}$ be a sequence. Then a subsequence is a sequence $(b_r)_{r\geqslant 1}$, where each b_r is in (a_n), and the terms are in the right order.

Example. Let $a_n = n$ for $n \geqslant 1$. The following are subsequences of (a_n).

- $2, 4, 6, 8, \dots$ — the subsequence (a_{2n})
- $2, 4, 8, 16, \dots$ — the subsequence (a_{2^n})

The following are not subsequences of (a_n).

- $6, 4, 8, \dots$ — the terms are not in the right order
- $2, 4, 0, \dots$ — not all the terms are in (a_n)
- $1, 2, 3, \dots, 2020$ — finite so not a sequence.

Now let's give a formal definition of a subsequence.

Definition. Let $(a_n)_{n\geqslant 1}$ be a sequence. A *subsequence* $(b_r)_{r\geqslant 1}$ of $(a_n)_{n\geqslant 1}$ is defined by a function $f : \mathbb{N} \to \mathbb{N}$ such that f is strictly increasing (if $p < q$ then $f(p) < f(q)$), and $b_r = a_{f(r)}$ for $r \geqslant 1$.

We often write $f(r)$ as n_r. Then $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence of natural numbers, and $b_r = a_{n_r}$ so the sequence (b_r) has terms $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$.

Remark.

- Formally, (a_n) corresponds to a function $\alpha : \mathbb{N} \to \mathbb{R}$ or $\alpha : \mathbb{N} \to \mathbb{C}$. Then a subsequence of (a_n) corresponds to a function $\alpha \circ f$, where $f : \mathbb{N} \to \mathbb{N}$ is strictly increasing.

- Subscripts are 'dummy variables'. We can write (a_n) as (a_r) or (a_m) or (a_α) or (a_x). It is conventional to use a letter close to n in the alphabet, to help us remember that it is a natural number. We can use any letter for the subscripts in the subsequence (b_r), except that if we write our original sequence as (a_n) then we should avoid using n for the subsequence too.

- It's sometimes useful to know that $n_r \geqslant r$ for $r \geqslant 1$. (Exercise: prove this inequality, using induction.)

Proposition 34 (Subsequences of a convergent sequence)**.** *Let (a_n) be a sequence. If (a_n) converges, then every subsequence (a_{n_r}) of (a_n) converges. Moreover, if $a_n \to L$ as $n \to \infty$ then every subsequence also converges to L.*

Remark. So if (a_n) is a sequence, and it has two subsequences that tend to different limits, then (a_n) does not converge. This follows from Proposition 34, and can be a useful strategy for showing that a sequence does not converge.

Proof. Assume that (a_n) converges to L.

Let (a_{n_r}) be a subsequence of (a_n).

Take $\varepsilon > 0$.

Since $a_n \to L$, there is N such that if $n \geqslant N$ then $|a_n - L| < \varepsilon$.

If $r \geqslant N$, then $n_r \geqslant r \geqslant N$ (see remark before this result),

so $|a_{n_r} - L| < \varepsilon$.

So $a_{n_r} \to L$ as $r \to \infty$. □

Example. Let $a_n = \begin{cases} 0 & \text{if } n \text{ is prime} \\ 1 + \frac{1}{n} & \text{otherwise} \end{cases}$.

Claim. (a_n) does not converge.

Proof. Idea: the subsequence of terms with prime subscripts tends to 0, and the subsequence of terms with non-prime subscripts tends to 1, so (a_n) doesn't converge.

Let the primes be $p_1 < p_2 < p_3 < \cdots$. Let $P = \{p_1, p_2, p_3, \dots\}$.

Note that there are infinitely many primes, so $(a_{p_r})_{r\geqslant 1}$ is a subsequence.

We have $a_{p_r} = 0$ for all $r \geqslant 1$, so $a_{p_r} \to 0$ as $r \to \infty$.

Let the elements of $\mathbb{N} \setminus P$ be $n_1 < n_2 < n_3 < \cdots$.

Note that there are infinitely many non-primes, so $(a_{n_r})_{r\geqslant 1}$ is a subsequence.

We have $a_{n_r} = 1 + \frac{1}{n_r}$ for $r \geqslant 1$, and so we see that $a_{n_r} \to 1$ as $r \to \infty$.

So (a_n) has subsequences that converge to different limits, so, by Proposition 34, (a_n) does not converge. □

22 Algebra of Limits — part one

Example. This is an unofficial example. We'll return to it once we've proved some results.

Let $a_n = \dfrac{7n^5 - n\sin(n^2+5n)+3}{4n^5-3n^2+n+2}$.
What can we say about (a_n)?

Intuitively...

- the numerator grows like $7n^5$ — the other terms are much smaller for large n, which is all we care about;

- the denominator grows like $4n^5$

so we might conjecture that $a_n \to \frac{7}{4}$ as $n \to \infty$.

To prove this (and lots more!), we'll prove a bunch of results that are extremely useful in practice. Collectively, these are known as the 'Algebra of Limits', and we'll quote "by AOL" in arguments.

Theorem 35 (Algebra of Limits, part 1)**.** *Let (a_n) and (b_n) be sequences with $a_n \to L$ and $b_n \to M$ as $n \to \infty$. Let c be a constant.*

(i) (constant) If $a_n = c$, so (a_n) is a constant sequence, then $a_n \to c$ as $n \to \infty$.

(ii) (scalar multiplication) The sequence (ca_n) converges, and $ca_n \to cL$ as $n \to \infty$.

(iii) (addition) The sequence $(a_n + b_n)$ converges, and $a_n + b_n \to L + M$ as $n \to \infty$.

(iv) (subtraction) The sequence $(a_n - b_n)$ converges, and $a_n - b_n \to L - M$ as $n \to \infty$.

(v) (modulus) The sequence $(|a_n|)$ converges, and $|a_n| \to |L|$ as $n \to \infty$.

Proof. (i) This is immediate from the definition.

(ii) If $c = 0$, then we're done by (i). So assume that $c \neq 0$.

Take $\varepsilon > 0$.

Since $a_n \to L$, there is N such that if $n \geqslant N$ then $|a_n - L| < \varepsilon$.

Now if $n \geqslant N$ then $|ca_n - cL| = |c||a_n - L| < |c|\varepsilon$.

So (ca_n) converges to cL.

OR...

Take $\varepsilon > 0$.

Since $a_n \to L$, there is N such that if $n \geqslant N$ then $|a_n - L| < \frac{\varepsilon}{|c|}$.

Now if $n \geqslant N$ then $|ca_n - cL| = |c||a_n - L| < \varepsilon$.

So (ca_n) converges to cL.

(iii) Take $\varepsilon > 0$.

Since $a_n \to L$ as $n \to \infty$ there is N_1 such that if $n \geqslant N_1$ then $|a_n - L| < \varepsilon$.

Since $b_n \to M$ as $n \to \infty$ there is N_2 such that if $n \geqslant N_2$ then $|b_n - M| < \varepsilon$.

Let $N = \max\{N_1, N_2\}$. If $n \geqslant N$, then $|a_n - L| < \varepsilon$ and $|b_n - M| < \varepsilon$, so

$$\begin{aligned} |(a_n + b_n) - (L + M)| &\leqslant |a_n - L| + |b_n - M| \text{ (by triangle inequality)} \\ &< 2\varepsilon. \end{aligned}$$

So $(a_n + b_n)$ converges to $L + M$.

(iv) This follows from (ii) and (iii).

(v) This was Proposition 27.

$\square$

Remark. In (iii), I ended up showing that we can make $|(a_n+b_n)-(L+M)|$ less than 2ε by going far enough along the sequence. But the definition says ε, not 2ε, so isn't this a problem?

Well, no, it's not a problem. We need to show that we can make $|(a_n + b_n) - (L + M)|$ less than any positive real number — and that's what we've done. The important thing is that 2 was a (positive) constant: it didn't depend on n.

We could instead have chosen N_1 and N_2 corresponding to $\frac{\varepsilon}{2}$ (so if $n \geqslant N_1$ then $|a_n - L| < \frac{\varepsilon}{2}$ and similarly for b_n), and then we'd have got ε at the end. But if I'd done that then it might have seemed more mysterious: you might have wondered "how would I have known to choose $\frac{\varepsilon}{2}$?"

In practice, sometimes I doodle on scrap paper and consequently know what to choose at the start, and sometimes I just work through and see what happens, and if I get 2ε or 1000ε at the end then it doesn't matter. I illustrated these two alternative approaches in (ii) — but really they're the same, and both are fine.

Example.

Claim. Let $a_n = \dfrac{1}{2^n} + \left(1 + (-1)^n \dfrac{1}{\sqrt{n}}\right) + \dfrac{n\cos(n^3+1)}{5n^2+1}$. Then $a_n \to 1$ as $n \to \infty$.

Proof. We showed earlier that $\dfrac{1}{2^n} \to 0$ and $1 + (-1)^n \dfrac{1}{\sqrt{n}} \to 1$ and also $\dfrac{n\cos(n^3+1)}{5n^2+1} \to 0$ as $n \to \infty$ (see Section 16).

So, by AOL, (a_n) converges, and $a_n \to 0 + 1 + 0 = 1$ as $n \to \infty$. □

Example.

Claim. Let $a_n = (-1)^n + \frac{n}{2^n}$ for $n \geqslant 1$. Then (a_n) does not converge.

Proof. Suppose, for a contradiction, that (a_n) converges.

Note that $\left(\frac{n}{2^n}\right)$ converges (this was an earlier example).

So, by AOL, the sequence with n^{th} term $(-1)^n = a_n - \frac{n}{2^n}$ converges.

But we showed earlier that $((-1)^n)$ does not converge (or we could now note that it has subsequences tending to different limits 1 and -1). This is a contradiction. $\square$

23 Algebra of Limits — part two

Theorem 36 (Algebra of Limits, part 2)**.** *Let (a_n) and (b_n) be sequences with $a_n \to L$ and $b_n \to M$ as $n \to \infty$.*

(vi) (product) The sequence $(a_n b_n)$ converges, and $a_n b_n \to LM$ as $n \to \infty$.

(vii) (reciprocal) If $M \neq 0$, then the sequence $\left(\frac{1}{b_n}\right)$ converges, and $\frac{1}{b_n} \to \frac{1}{M}$ as $n \to \infty$.

(viii) (quotient) If $M \neq 0$, then the sequence $\left(\frac{a_n}{b_n}\right)$ converges, and $\frac{a_n}{b_n} \to \frac{L}{M}$ as $n \to \infty$.

Remark. You might wonder whether the sequences $\left(\frac{1}{b_n}\right)$ and $\left(\frac{a_n}{b_n}\right)$ in (vii) and (viii) are defined. This is a good question. The answer is that — as we'll show in the proof — if $M \neq 0$ then a tail of (b_n) has all its terms nonzero, and hence there's a tail of $\left(\frac{1}{b_n}\right)$ that exists, and similarly for $\left(\frac{a_n}{b_n}\right)$. When we talk about convergence of these sequences, it's enough to consider a tail.

Proof. (vi) We're going to want to study

$$\begin{aligned} |a_n b_n - LM| &= |a_n(b_n - M) + M(a_n - L)| \\ &\leqslant |a_n||b_n - M| + |M||a_n - L| \end{aligned}$$

— this use of the triangle inequality can help us to see how to proceed.

Take $\varepsilon > 0$. We may assume that $\varepsilon < 1$.

Since $a_n \to L$, there is N_1 such that if $n \geqslant N_1$ then $|a_n - L| < \varepsilon$.

Since $b_n \to M$, there is N_2 such that if $n \geqslant N_2$ then $|b_n - M| < \varepsilon$.

Let $N = \max\{N_1, N_2\}$.

If $n \geqslant N$, then $|a_n - L| < \varepsilon$ and $|b_n - M| < \varepsilon$ and $|a_n| < |L| + \varepsilon$, so

$$\begin{aligned}
|a_n b_n - LM| &= |a_n(b_n - M) + M(a_n - L)| \\
&\leqslant |a_n||b_n - M| + |M||a_n - L| \\
&< (|L| + \varepsilon) \cdot \varepsilon + |M| \cdot \varepsilon \\
&< \varepsilon(1 + |L| + |M|).
\end{aligned}$$

Since $1 + |L| + |M|$ is constant, this is enough to show that $(a_n b_n)$ converges, and the limit is LM.

(vii) Assume that $M \neq 0$.

Idea: (1) eventually b_n is close to M, so can't be 0. (2) $\left|\frac{1}{b_n} - \frac{1}{M}\right| = \frac{|b_n - M|}{|M||b_n|}$ – eventually the numerator is small, and $|b_n|$ is close to $|M|$.

Take $\varepsilon > 0$.

Since $b_n \to M$ and $|M| > 0$, there is N_1 such that if $n \geqslant N_1$ then $|b_n - M| < \frac{|M|}{2}$, so (by the Reverse Triangle Inequality)

$$|b_n| \geqslant ||b_n + (M - b_n)| - |M - b_n|| > \frac{|M|}{2} > 0.$$

So the tail $(b_n)_{n \geqslant N_1}$ has all terms nonzero, so we can consider the sequence $\left(\frac{1}{b_n}\right)_{n \geqslant N_1}$.

Also, there is N_2 such that if $n \geqslant N_2$ then $|b_n - M| < \varepsilon$.

Let $N = \max\{N_1, N_2\}$. If $n \geqslant N$, then

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \frac{|M - b_n|}{|M||b_n|} < \frac{\varepsilon}{|M|} \cdot \frac{2}{|M|}.$$

Since $\frac{2}{|M|^2}$ is a positive constant, this shows that $\left(\frac{1}{b_n}\right)_{n \geqslant N_1}$ converges, and the limit is $\frac{1}{M}$.

(viii) This follows from (vi) and (vii).

□

Example. Let $a_n = \dfrac{7n^5 - n\sin(n^2+5n) + 3}{4n^5 - 3n^2 + n + 2}$ (we saw this example at the start of Section 22).

Claim. $a_n \to \frac{7}{4}$ as $n \to \infty$.

Proof. Idea: the important terms (for large n) are $7n^5$ and $4n^5$.

We have

$$a_n = \frac{7 - \frac{1}{n^4}\sin(n^2+5n) + \frac{3}{n^5}}{4 - \frac{3}{n^3} + \frac{1}{n^4} + \frac{2}{n^5}}.$$

Now

$$0 \leqslant \left|\frac{1}{n^4}\sin(n^2+5n)\right| \leqslant \frac{1}{n^4} \leqslant \frac{1}{n}$$

and $\frac{1}{n} \to 0$, so by Sandwiching $\frac{1}{n^4}\sin(n^2+5n) \to 0$, and several other terms also tend to 0 (eg by Sandwiching),

so, by AOL, (a_n) converges, and

$$a_n \to \frac{7 - 0 + 0}{4 - 0 + 0 + 0} = \frac{7}{4}$$

as $n \to \infty$. □

Proposition 37 (Reciprocals and infinite/zero limits)**.** *Let (a_n) be a sequence of positive real numbers. Then $a_n \to \infty$ as $n \to \infty$ if and only if $\frac{1}{a_n} \to 0$ as $n \to \infty$.*

Proof. Exercise (using the definitions). □

24 Orders of magnitude

When we're studying a sequence (a_n), it can be really useful to develop some intuition about the behaviour of a_n for large n, in order to make a conjecture about the convergence (or otherwise) of the sequence, and to select a proof strategy. (This is what we did in the example at the end of the last section, for example.)

Example.

- Let $a_n = \dfrac{8n^2 + 1000000n + 1000000}{14n^6 + n^3 + n}$.

  Intuitively, the key term in the numerator is $8n^2$, and the key term in the denominator is $14n^6$. Even with the amusingly large coefficients in the numerator, when n is large these terms will be much smaller than $8n^2$.

  So it feels like the sequence grows roughly like $\frac{8}{14n^4}$, so should tend to 0.

  We can formalise this using AOL. Dividing through top and bottom by n^6 (since this is the key term), we get

  $$a_n = \frac{\frac{8}{n^4} + \frac{1000000}{n^5} + \frac{1000000}{n^6}}{14 + \frac{1}{n^3} + \frac{1}{n^5}} \to \frac{0+0+0}{14+0+0} = 0$$

  as $n \to \infty$.

- We showed in Lemma 25 that $\frac{n}{2^n} \to 0$ as $n \to \infty$.

  This is an example of the idea that 'exponentials beat polynomials'. But while 'exponentials beat polynomials' is a useful slogan for intuition, it is not suitable for rigorous proofs!

- We've seen a couple of examples where we used that $|\cos x| \leqslant 1$ and $|\sin x| \leqslant 1$ for all x — this can be useful.

- We'll show in the next section that $\frac{\log n}{n} \to 0$ as $n \to \infty$. Intuitively, polynomials grow faster than logarithms.

Definition. Let (a_n) and (b_n) be sequences. We write $a_n = O(b_n)$ as $n \to \infty$ if there is a constant $C \in \mathbb{R}^{>0}$ and there is N such that if $n \geqslant N$ then $|a_n| \leqslant C|b_n|$. This is 'big O' notation.

If $b_n \neq 0$ for all n (or all sufficiently large n), then we write $a_n = o(b_n)$ as $n \to \infty$ if $\frac{a_n}{b_n} \to 0$ as $n \to \infty$. This is 'little o' notation.

Remark.

- Sandwiching tells us that if $a_n = O(b_n)$ and $b_n \to 0$ as $n \to \infty$ then $a_n \to 0$ as $n \to \infty$.
- Big O and little o notation give us precise ways to make precise statements about comparative rates of growth of sequences. Please use them precisely!

Example. This example is in a Moodle quiz. Before you read on to the next section, please go to the Moodle course page for Analysis I, and try the quiz for section 24 (it's a short multiple choice quiz).

25 Monotonic sequences

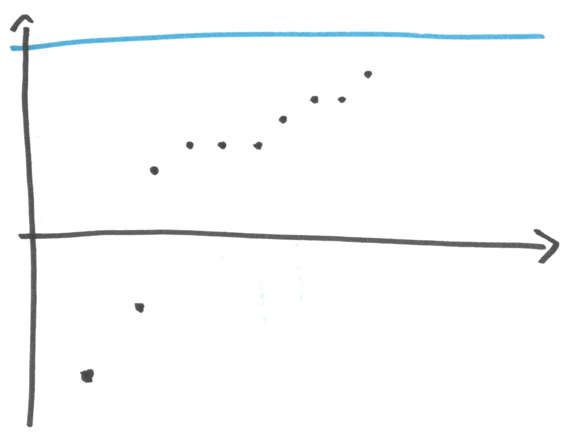
Definition. Let (a_n) be a real sequence.

- We say that (a_n) is *monotonic increasing*, or *monotone increasing*, or *increasing*, if $a_n \leqslant a_{n+1}$ for all n.
- We say that (a_n) is *strictly increasing* if $a_n < a_{n+1}$ for all n.
- We say that (a_n) is *monotonic decreasing*, or *monotone decreasing*, or *decreasing*, if $a_n \geqslant a_{n+1}$ for all n.

- We say that (a_n) is *strictly decreasing* if $a_n > a_{n+1}$ for all n.
- We say that (a_n) is *monotonic*, or *monotone*, if it is increasing or decreasing.

Example. Notice that a constant sequence is both increasing and decreasing. This might seem counterintuitive!

We know that a convergent sequence is bounded. What can we say about a bounded monotone sequence?



Theorem 38 (Monotone Sequences Theorem). *Let (a_n) be a real sequence.*

(i) If (a_n) is increasing and bounded above, then (a_n) converges.

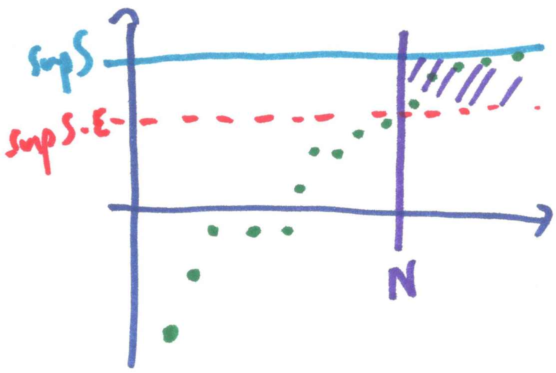
(ii) If (a_n) is decreasing and bounded below, then (a_n) converges.

Remark.
- So 'a bounded monotone sequence converges'.
- The result applies to tails of sequences too: if (a_n) has a tail that is monotone and bounded, then it converges.

Proof. (i) Assume that (a_n) is increasing and bounded above.

Idea: $\{a_n : n \geqslant 1\}$ has a supremum, and (a_n) converges to this.

The set $S = \{a_n : n \geqslant 1\}$ is non-empty and bounded above, so, by Completeness, it has a supremum.



Take $\varepsilon > 0$.

By the Approximation Property, there is N such that $\sup S - \varepsilon < a_N \leqslant \sup S$.

If $n \geqslant N$, then $\sup S - \varepsilon < a_N \leqslant a_n \leqslant \sup S$,

so $|a_n - \sup S| < \varepsilon$.

So (a_n) converges, and $a_n \to \sup S$ as $n \to \infty$.

(ii) If (a_n) is decreasing and bounded below, then $(-a_n)$ is increasing and bounded above, so (ii) follows from (i).

$\square$

Lemma 39. *Let (a_n) be a real sequence that is increasing and not bounded above. Then $a_n \to \infty$ as $n \to \infty$.*

Proof. Take $M \in \mathbb{R}$.

Since (a_n) is not bounded above, there is N such that $a_N > M$.

Then, since (a_n) is increasing, if $n \geqslant N$ then $a_n \geqslant a_N > M$. $\square$

Example. Let $a_n = \left(1 + \frac{1}{n}\right)^n$.

On Sheet 1, you proved that (a_n) is increasing and that (a_n) is bounded above (by 3). So, by the Monotone Sequences Theorem, (a_n) converges. Say $a_n \to L$ as $n \to \infty$. Then, since limits preserve weak inequalities, we see that $2 \leqslant L \leqslant 3$.

(Secretly, we know more about L, but that's strictly unofficial for now.)

Example. Let $c \geqslant 0$. In this example, we'll show that $\sqrt{c}$ exists. (This generalises earlier work on $\sqrt{2}$, and uses a different strategy.)

Define (a_n) by $a_1 = 1$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$ for $n \geqslant 1$.

This is a legitimate definition, since (by induction) $a_n \neq 0$ for $n \geqslant 1$.

Claim. (a_n) converges, and if $a_n \to L$ then $L^2 = c$.

Proof.

- (a_n) bounded below:

  by a straightforward induction argument, we have $a_n > 0$ for all n.

- study $a_n^2 - c$:

  for $n \geqslant 1$, we have

$$\begin{aligned}
a_{n+1}^2 - c &= \frac{1}{4}\left(a_n + \frac{c}{a_n}\right)^2 - c \\
&= \frac{1}{4}\left(a_n^2 + 2c + \frac{c^2}{a_n^2}\right) - c \\
&= \frac{1}{4}\left(a_n^2 - 2c + \frac{c^2}{a_n^2}\right) \\
&= \frac{1}{4}\left(a_n - \frac{c}{a_n}\right)^2 \\
&\geqslant 0,
\end{aligned}$$

  so $a_{n+1}^2 \geqslant c$ for $n \geqslant 1$.

- $(a_n)_{n\geqslant 2}$ decreasing:

  for $n \geqslant 2$, we have

$$a_{n+1} - a_n = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right) - a_n = \frac{1}{2}\left(\frac{c}{a_n} - a_n\right) = \frac{1}{2a_n}(c - a_n^2) \leqslant 0,$$

  so $a_{n+1} \leqslant a_n$ for $n \geqslant 2$.

So, by the Monotone Sequences Theorem, (a_n) converges.

Say $a_n \to L$ as $n \to \infty$.

Then also $a_{n+1} \to L$ as $n \to \infty$ (it's a tail of the sequence).

But if $L \neq 0$ then

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right) \to \frac{1}{2}\left(L + \frac{c}{L}\right)$$

by AOL.

Since limits are unique, we have $L = \frac{1}{2}\left(L + \frac{c}{L}\right)$,

so, rearranging, $L^2 = c$.

Also, we have $a_n > 0$ for all n, and limits preserve weak inequalities, so $L \geqslant 0$.

So $\sqrt{c}$ exists ($L = \sqrt{c}$).

In the case that $L = 0$, since limits preserve weak inequalities and $a_n^2 \geqslant c$ for $n \geqslant 2$ we have $c \leqslant 0$, so $c = 0$ and $L^2 = c$. □

Lemma 40. *We have* $\frac{\log n}{n} \to 0$ *as* $n \to \infty$.

Proof. Let $a_n = \frac{\log n}{n}$.

Then $a_n \geqslant 0$ for all n, so (a_n) is bounded below.

Also, by properties of log we see that $(a_n)_{n\geqslant 100}$ is decreasing.

So, by the Monotone Sequences Theorem, (a_n) converges. Say $\frac{\log n}{n} \to L$ as $n \to \infty$.

Since limits preserve weak inequalities, we have $L \geqslant 0$.

Now

$$a_{2n} = \frac{\log(2n)}{2n} = \frac{\log 2 + \log n}{2n} \to 0 + \frac{L}{2}$$

by AOL,

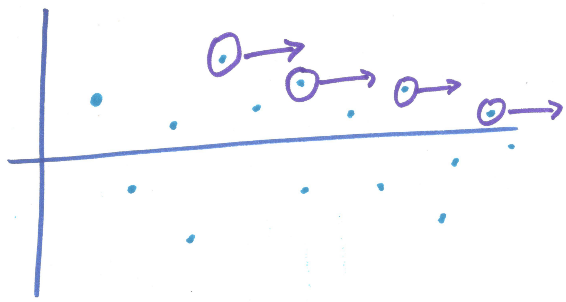
but also (a_{2n}) is a subsequence of (a_n) so $a_{2n} \to L$ as $n \to \infty$.

So, by uniqueness of limits, $\frac{L}{2} = L$, so $L = 0$. $\square$

26 Convergent subsequences

Theorem 41 (Scenic Viewpoints Theorem). *Let (a_n) be a real sequence. Then (a_n) has a monotone subsequence.*

Proof. Idea: consider the 'peaks' of the sequence.



Let $V = \{k \in \mathbb{N} : \text{ if } m > k \text{ then } a_m < a_k\}$. (The elements of V are 'peaks' or 'scenic viewpoints': if $k \in V$ then a_k is higher than all subsequent terms.)

Case 1: V is infinite.

Say the elements of V are $k_1 < k_2 < \cdots$.

Then $(a_{k_r})_r$ is a subsequence of (a_n)

and it is monotone decreasing (if $r < s$ then $k_r < k_s$ so $a_{k_r} > a_{k_s}$).

Case 2: V is finite.

Then there is N such that if $k \in V$ then $k < N$.

Let $m_1 = N$. Then $m_1 \notin V$ so there is $m_2 > m_1$ with $a_{m_2} \geqslant a_{m_1}$.

Also, $m_2 \notin V$ so there is $m_3 > m_2$ with $a_{m_3} \geqslant a_{m_2}$.

Continuing inductively, we construct $m_1 < m_2 < m_3 < \cdots$ such that $a_{m_1} \leqslant a_{m_2} \leqslant a_{m_3} \leqslant \cdots$.

Then $(a_{m_r})_r$ is an increasing subsequence of (a_n). $\square$

Theorem 42 (Bolzano-Weierstrass Theorem)**.** *Let (a_n) be a bounded real sequence. Then (a_n) has a convergent subsequence.*

Proof. By the Scenic Viewpoints Theorem, (a_n) has a monotone subsequence.

This monotone subsequence is bounded (because the whole sequence is), so by the Monotone Sequences Theorem (Theorem 38) it converges. $\square$

Remark.
- This proof of the Bolzano-Weierstrass Theorem was very short, because we did all the work in the Monotone Sequences Theorem and Scenic Viewpoints Theorem! I have another favourite proof of Bolzano-Weierstrass. I've turned it into a quiz 'proof sorter' activity on Moodle.

- The Monotone Sequences Theorem and Scenic Viewpoints Theorem don't make sense for complex sequences. But Bolzano-Weierstrass potentially could ...

Corollary 43 (Bolzano-Weierstrass Theorem for complex sequences)**.** *Let (z_n) be a bounded complex sequence. Then (z_n) has a convergent subsequence.*

Proof. Study real and imaginary parts, and repeatedly pass to subsequences
Write $z_n = x_n + \mathrm{i}y_n$ where x_n, $y_n \in \mathbb{R}$.

Say (z_n) is bounded by M, so $|z_n| \leqslant M$ for all n.

Then (x_n) and (y_n) are also bounded by M, and they are real sequences.

By Bolzano-Weierstrass, (x_n) has a convergent subsequence, say $(x_{n_r})_r$.

Now $(y_{n_r})_r$ is a bounded real sequence, so by Bolzano-Weierstrass it has a convergent subsequence, say $(y_{n_{r_s}})_s$.

Note that $(x_{n_{r_s}})_s$ is a subsequence of the convergent sequence $(x_{n_r})_r$ and hence converges.

So, by Theorem 33, $(z_{n_{r_s}})_s$ converges (since its real and imaginary parts converge). $\square$

27 Cauchy sequences

Example. Let (a_n) be a convergent sequence.

Then $a_{n+1} - a_n \to 0$ as $n \to \infty$.

We can prove this directly from the definition (with the triangle inequality), or using tails and the Algebra of Limits.

But it is not the case that if $a_{n+1} - a_n \to 0$ as $n \to \infty$ then (a_n) converges.

For example, consider $a_n = \sqrt{n}$. Certainly (a_n) does not converge. But

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0$$

as $n \to \infty$.

Nonetheless, intuitively it seems that if eventually all the terms of a sequence are bunched up close together then the sequence might converge.

Definition. Let (a_n) be a sequence. We say that (a_n) is a *Cauchy sequence* if

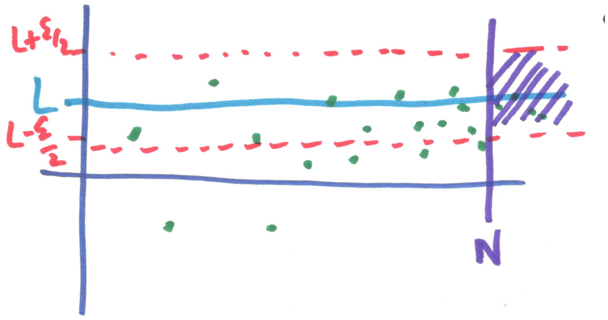
$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall m, n \geqslant N \ |a_n - a_m| < \varepsilon.$$

Remark. Note that this definition makes sense for complex sequences as well as for real sequences.

Proposition 44. *Let (a_n) be a convergent sequence. Then (a_n) is Cauchy.*

Proof. Say $a_n \to L$ as $n \to \infty$.

Take $\varepsilon > 0$.



Since $a_n \to L$, there is N such that if $n \geqslant N$ then $|a_n - L| < \frac{\varepsilon}{2}$.

Take $m, n \geqslant N$. Then $|a_m - L| < \frac{\varepsilon}{2}$ and $|a_n - L| < \frac{\varepsilon}{2}$,

so, by the triangle inequality,

$$\begin{aligned} |a_m - a_n| &= |(a_m - L) + (L - a_n)| \\ &\leqslant |a_m - L| + |a_n - L| < \varepsilon. \end{aligned}$$

So (a_n) is Cauchy. $\square$

Proposition 45. *Let (a_n) be a Cauchy sequence. Then (a_n) is bounded.*

Proof. Idea: use a similar strategy to Proposition 30, where we showed that a convergent sequence is bounded.

Since (a_n) is Cauchy, there is (applying the definition with $\varepsilon = 1$) N such that if $m, n \geqslant N$ then $|a_m - a_n| < 1$.

Now for $n \geqslant N$ we have $|a_n - a_N| < 1$,

so $|a_n| = |(a_n - a_N) + a_N| \leqslant 1 + |a_N|$.

Let $K = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |a_N|\}$.

Then $|a_n| \leqslant K$ for all $n \geqslant 1$.

So (a_n) is bounded. $\square$

Proposition 46. *Let (a_n) be a Cauchy sequence. Suppose that the subsequence $(a_{n_r})_r$ converges. Then (a_n) converges.*

Proof. Idea: eventually all the terms of (a_{n_r}) are really close to the limit L, and eventually all the terms of (a_n) are really close to terms in the subsequence and hence also really close to L.

Say that $a_{n_r} \to L$ as $r \to \infty$.

Take $\varepsilon > 0$.

Then there is N_1 such that if $r \geqslant N_1$ then $|a_{n_r} - L| < \frac{\varepsilon}{2}$.

Also, since (a_n) is Cauchy there is N_2 such that if $m, n \geqslant N_2$ then $|a_m - a_n| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$.

Let $r = N$. Then $n_r \geqslant r \geqslant N_1$ so $|a_{n_r} - L| < \frac{\varepsilon}{2}$

and if $n \geqslant N$ then n, $n_r \geqslant N_2$ so $|a_{n_r} - a_n| < \frac{\varepsilon}{2}$,

so

$$\begin{aligned} |a_n - L| &= |(a_n - a_{n_r}) + (a_{n_r} - L)| \\ &\leqslant |a_n - a_{n_r}| + |a_{n_r} - L| < \varepsilon. \end{aligned}$$

So $a_n \to L$ as $n \to \infty$. □

The following result is really useful! We'll use it in later sections.

Theorem 47 (Cauchy Convergence Criterion)**.** *Let (a_n) be a sequence. Then (a_n) converges if and only if (a_n) is Cauchy.*

Proof. ($\Rightarrow$) This was Proposition 44.

($\Leftarrow$) Assume that (a_n) is Cauchy.

Then (a_n) is bounded, by Proposition 45,

so by the Bolzano-Weierstrass Theorem (Theorem 42), (a_n) has a convergent subsequence, say (a_{n_r}).