Contents

1 Sequences

1.1 Definitions

Definition 1.1. A real sequence (or sequence) is a mapping

$$\boxed{\begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{R} \\ n & \longmapsto & u_n \end{array}}$$

It is denoted by $(u_n)_{n\in\mathbb{N}}$ and u_n is called general term of the sequence.

Definition 1.2.
- A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded below if

$$\boxed{\exists a \in \mathbb{R}, \forall n \in \mathbb{N} : u_n \geq a}$$

- A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded above if

$$\boxed{\exists b \in \mathbb{R}, \forall n \in \mathbb{N} : u_n \leq b}$$

- A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded if it is bounded below and bounded above. In other words if

$$\boxed{\exists C > 0, \forall n \in \mathbb{N} : |u_n| \leq C}$$

Definition 1.3. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence.

1. We say $(u_n)_{n\in\mathbb{N}}$ is increasing (resp. decreasing) if

$$\boxed{\forall n \in \mathbb{N} : u_n \leq u_{n+1} \text{ (resp. } u_{n+1} \leq u_n)}$$

2. We say $(u_n)_{n\in\mathbb{N}}$ is constant if $\boxed{\forall n \in \mathbb{N} : u_n = u_{n+1}}$
3. We say $(u_n)_{n\in\mathbb{N}}$ is monotone if it increasing or decreasing.

☞ **Example 1.1.**
- The sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n = (-1)^n$ is not monotone.
- The sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n = \frac{n+1}{2n+1}$ is not monotone. Indeed, for all $n \in \mathbb{N}$:

$$\begin{aligned} u_{n+1} - u_n = \frac{n+2}{2n+3} - \frac{n+1}{2n+1} &= \frac{(n+2)(2n+1) - (2n+3)(n+1)}{(2n+3)(2n+1)} \\ &= \frac{-1}{(2n+3)(2n+1)} < 0 \end{aligned}$$

- Consider the geometric sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n = a^n$ is monotone if and only if $a \geq 0$. Indeed, if $a > 0$, we have for all $n \in \mathbb{N}$:

$$\boxed{\frac{u_{n+1}}{u_n} = \frac{a^{n+1}}{a^n} = a.}$$

  Thus, if $a < 1$, the sequence is decreasing, if $a > 1$ it is increasing and if $a = 1$ or $a = 0$, it is constant. Now if $a < 0$, then $u_{n+1} - u_n = a^n(a-1)$ which is positive if n is even and negative if n is odd.

1.2 Convergence

Definition 1.4. Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence and $\ell \in \mathbb{R}$.

1. We say ℓ is a limit of the sequence $(u_n)_{n\in\mathbb{N}}$ and we write $\lim_{n\to+\infty} = \ell$ if

$$\boxed{\forall\varepsilon > 0, \exists N \in \mathbb{N},\ \forall n \geq N : |u_n - \ell| \leq \varepsilon}$$

$$\boxed{(\text{ or } \forall\varepsilon > 0, \exists N \in \mathbb{N} : n \geq N \Longrightarrow |u_n - \ell| \leq \varepsilon)}$$

Definition 1.5.

1. We say the sequence $(u_n)_{n\in\mathbb{N}}$ has $+\infty$ as a limit and we write $\lim_{n\to+\infty} = +\infty$, if

$$\boxed{\forall A > 0,\ \exists n \in \mathbb{N},\ \forall n \geq N : u_n \geq A.}$$

2. We say the sequence $(u_n)_{n\in\mathbb{N}}$ has $-\infty$ as a limit and we write $\lim_{n\to+\infty} = -\infty$, if

$$\boxed{\forall B < 0,\ \exists n \in \mathbb{N},\ \forall n \geq N : u_n \leq B.}$$

Definition 1.6. Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence and $\ell \in \mathbb{R}$. We say the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent (or converges to ℓ) if it has a limit $\ell \in \mathbb{R}$. Otherwise, we say it is divergent.

☞ **Example 1.2.**

1. If $u_n = c$, $\forall n \in \mathbb{N}$, then $(u_n)_{n\in\mathbb{N}}$ is convergent to the limit c. Indeed, we have

$$\boxed{\forall\varepsilon > 0, \exists N = 0,\ \forall n \geq N : |u_n - c| = |c - c| = 0 < \varepsilon.}$$

2. If $u_n = \frac{1}{n}$ $\forall n \in \mathbb{N}^*$, then $(u_n)_{n\in\mathbb{N}^*}$ converges to 0. Indeed. Let $\varepsilon > 0$ and $n \in \mathbb{N}^*$. We have

$$\boxed{|u_n - 0| \leq \varepsilon \Longleftrightarrow \frac{1}{n} \leq \varepsilon \Longleftrightarrow n \geq \frac{1}{\varepsilon}.}$$

   Hence it suffices to choose $N \geq \frac{1}{\varepsilon}$, that is for example $N = [1/\varepsilon] + 1$. Then $n \geq N \Longrightarrow |u_n - 0| \leq \varepsilon$.

3. If $u_n = \frac{n+1}{2n+1}$ $\forall n \in \mathbb{N}^*$, then $(u_n)_{n\in\mathbb{N}^*}$ converges to $1/2$. Indeed. Let $\varepsilon > 0$ and $n \in \mathbb{N}^*$. We have

$$\boxed{|u_n - \tfrac{1}{2}| \leq \varepsilon \Longleftrightarrow \tfrac{1}{4n+2} \leq \varepsilon \Longleftrightarrow n \geq \tfrac{1}{4\varepsilon} - \tfrac{1}{2}.}$$

Hence it suffices to choose $N \geq \frac{1}{4\varepsilon} - \frac{1}{2}$, that is for example $N = [1/4\varepsilon] + 1$. Then $n \geq N \Longrightarrow |u_n - 1/2| \leq \varepsilon$.

4. $u_n = (-1)^n$. Then the sequence $(u_n)_{n\in\mathbb{N}}$ is not convergent. Indeed, if not there exists $\ell \in \mathbb{R}$ such that $\lim_{n\to\infty} u_n = \ell$. Hence taking $\varepsilon = 1/2$. according to the the definition of the limit,

$$\boxed{\exists N \in \mathbb{N},\ \forall n \geq N : |u_n - \ell| \leq \tfrac{1}{2}.}$$

but we have for $n = 2N$

$$\boxed{2 = |u_{n+1} - u_n| \leq |u_{n+1} - \ell| + |u_n - \ell| \leq 1/2 + 1/2 = 1}$$

which is a contradiction.

5. If $u_n = n^2$, then $\lim_{\to+\infty} u_n = +\infty$. Indeed, given any $A > 0$. Then

$$\boxed{u_n \geq A \Longleftrightarrow n \geq \sqrt{A}.}$$

Therefore, $\forall n \geq [\sqrt{A}] + 1$, we have $n \geq A$ which implies that $u_n = n^2 \geq A$.

☞ **Example 1.3.** Calculate the limit of the sequence $(u_n)_{n\in\mathbb{N}}$ in the following cases

1. $u_n = a^n$, $a \in \mathbb{R}_+$
2. $u_n = n(e^{1/n} - 1)$.
3. $u_n = \sum\limits_{k=0}^{n} \frac{1}{2^k}$.
4. $u_n =$

1. For $u_n = a^n$, where $a \in \mathbb{R}_+$:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} a^n = \begin{cases} +\infty, & \text{if } a > 1 \\ 0, & \text{if } 0 < a < 1 \\ 1, & \text{if } a = 1 \end{cases}$$

2. For $u_n = n(e^{1/n} - 1)$:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} n(e^{1/n} - 1) = 0$$

3. For $u_n = \sum_{k=0}^{n} \frac{1}{2^k}$:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{2^k} = 2$$

Proposition 1.1 (Uniqueness). A convergent sequence has a unique limit.

Proof. Suppose that $(u_n)_{n\in\mathbb{N}}$ has two limits $\ell_1,\ \ell_2$ such that $\ell_1 \neq \ell_2$. Take $\varepsilon = |\ell_1 - \ell_2| > 0$. From the definition of the limit, there are $N_1,\ N_2 \in \mathbb{N}$ such that

$$\boxed{\forall n \geq N_1 : |u_n - \ell_1| < \varepsilon/2, \quad \forall n \geq N_2 : |u_n - \ell_2| < \varepsilon/2.}$$

Hence for $n \geq \max\{N_1, N_2\}$, we have

$$\boxed{\varepsilon = |\ell_1 - \ell_2| \leq |u_n - \ell_1| + |u_n - \ell_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon}$$

contradiction. □

Proposition 1.2. A convergent sequence is bounded.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a convergent sequence to the limit ℓ. Hence taking $\varepsilon = 1$. According to the the definition of the limit,

$$\boxed{\exists N \in \mathbb{N},\ \forall n \geq N : |u_n - \ell| \leq 1.}$$

Then

$$\forall n \geq N : |u_n| \leq |u_n - \ell| + |\ell| \leq 1 + \ell := M_0 \tag{1.1}$$

and we have

$$\forall n \leq N : |u_n| \leq M_1 := \max\{|u_0|, |u_1|, ..., |u_N|\} \tag{1.2}$$

From (1.1) and (1.2), we deduce that $\forall n \in \mathbb{N} : |u_n| \leq M := \max\{M_0, M_1\}$. Hence the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded. □

Proposition 1.3. Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two sequences converging respectively to u and v. Then

1. $\boxed{\lim_{n\to+\infty} (\lambda u_n) = \lambda u,\ \forall \lambda \in \mathbb{R}}$.
2. $\boxed{\lim_{n\to+\infty} (u_n + v_n) = u + v}$.
3. $\boxed{\lim_{n\to+\infty} (u_n v_n) = uv}$.
4. $\boxed{\lim_{n\to+\infty} \frac{1}{u_n} = \frac{1}{u},\ \text{if } u \neq 0 \text{ and } u_n \neq 0,\ \forall n \in \mathbb{N}.}$

Proof. 1. If $\lambda = 0$, then $\lambda u_n = 0 \to 0$, as $n \to +\infty$. If not, let $\varepsilon > 0$. According to the the definition of the limit, $\exists N \in \mathbb{N}$:

$$\boxed{\forall n \geq N : |u_n - u| \leq \tfrac{\varepsilon}{|\lambda|}.}$$

Hence

$$\boxed{\boxed{\forall n \geq N : |\lambda u_n - \lambda u| = |\lambda||u_n - u| \leq \varepsilon.}}$$

2. Let $\varepsilon > 0$. According to the the definition of the limit, there are $N_1, N_2 \in \mathbb{N}$:

$$\boxed{\forall n \geq N_1 : |u_n - u| \leq \tfrac{\varepsilon}{2}, \quad \forall n \geq N_2 : |v_n - v| \leq \tfrac{\varepsilon}{2}}$$

Hence, taking $N := \max\{N_1, N_2\}$

$$\boxed{\boxed{\forall n \geq N : |(u_n + v_n) - (u + v)| \leq |u_n - u| + |v_n - v| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.}}$$

3. Let $\varepsilon > 0$.

$$\boxed{|u_n v_n - uv| = |u_n(v_n - v) + v(u_n - u)| \leq |u_n||v_n - v| + |v||u_n - u|}$$

Since $(u_n)_{n\in\mathbb{N}}$ converges, it is bounded (see Proposition 1.2). Therefore, there is $M > 0$ such that

$$\boxed{\forall n \in \mathbb{N} : |u_n| \leq M.}$$

$$\boxed{|u_n v_n - uv| \leq M|v_n - v| + v|u_n - u| \leq M'(|v_n - v| + |u_n - u|), \ M' = \max\{M, v\}.}$$

Otherwise, according to the the definition of the limit, there are $N_1, N_2 \in \mathbb{N}$:

$$\boxed{\forall n \geq N_1 : |u_n - u| \leq \tfrac{\varepsilon}{2M'}, \quad \forall n \geq N_2 : |v_n - v| \leq \tfrac{\varepsilon}{2M'}.}$$

Hence, $\forall n \geq N := \max\{N_1, N_2\}$, we have

$$\boxed{\boxed{|u_n v_n - uv| \leq M'(\frac{\varepsilon}{2M'} + \frac{\varepsilon}{2M'}) = \varepsilon}.}$$

4. The proof is left as an exercise.

□

Proposition 1.4. Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.

1. If $(u_n)_{n\in\mathbb{N}}$ is increasing and bounded above, it is convergent.
2. If $(u_n)_{n\in\mathbb{N}}$ is decreasing and bounded below, it is convergent.

Proof.

1. Let $(u_n)_{n\in\mathbb{N}}$ be an increasing sequence that is bounded above. Consider the set
$$A = \{u_n : n \in \mathbb{N}\}.$$
Since A is a set of real numbers and is bounded above, it has a least upper bound (supremum) denoted by α. We claim that $\lim_{n\to\infty} u_n = \alpha$.
Given $\varepsilon > 0$, since $\alpha = \sup A$, there exists an element u_N in A such that
$$\boxed{\alpha - \varepsilon < u_N.}$$
Since (u_n) is increasing,
$$\forall n \geq N : \alpha - \varepsilon < a_N \leq u_n.$$
This implies
$$\forall n \geq N : |u_n - \alpha| = \alpha - u_n \leq \varepsilon.$$
which satisfies the definition of the limit. This proves that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to α.
2. The proof is left as an exercise.

□

Proposition 1.5. Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}}$ be three sequences such that
$$\boxed{\forall n \in \mathbb{N} : u_n \leq v_n \leq w_n.}$$
Then
$$\boxed{\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} w_n = \ell \Longrightarrow \lim_{n\to+\infty} v_n = \ell.}$$

Proof. Assume that $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} w_n = \ell$. We want to show that $\lim_{n\to+\infty} v_n = \ell$.

Given any $\varepsilon > 0$, since $\lim_{n\to+\infty} u_n = \ell$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|u_n - \ell| < \varepsilon$.

Similarly, since $\lim_{n\to+\infty} w_n = \ell$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $|w_n - \ell| < \varepsilon$.

Let $N = \max\{N_1, N_2\}$. For all $n \geq N$, we have $u_n \leq v_n \leq w_n$, which implies
$$\ell - \varepsilon < u_n \leq v_n \leq w_n < \ell + \varepsilon.$$

Therefore, for all $n \geq N$, we have $|v_n - \ell| < \varepsilon$. This shows that $\lim_{n\to+\infty} v_n = \ell$, as desired.

Hence, we have proved the proposition. □

Proposition 1.6. Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ be two sequences such that

$$\boxed{\forall n \in \mathbb{N} : u_n \leq v_n}$$

Then

1. $\lim_{n\to+\infty} u_n = +\infty \Longrightarrow \lim_{n\to+\infty} v_n = +\infty$.
2. $\lim_{n\to+\infty} v_n = -\infty \Longrightarrow \lim_{n\to+\infty} u_n = -\infty$.

Proof.

1. Assume that $\lim_{n\to+\infty} u_n = +\infty$. We want to show that $\lim_{n\to+\infty} v_n = +\infty$.

   Given any $M > 0$, since $\lim_{n\to+\infty} u_n = +\infty$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $u_n > M$.

   Since $u_n \leq v_n$ for all $n \in \mathbb{N}$, it follows that $v_n \geq u_n > M$ for all $n \geq N_1$. This implies that $\lim_{n\to+\infty} v_n = +\infty$.

2. Assume that $\lim_{n\to+\infty} v_n = -\infty$. We want to show that $\lim_{n\to+\infty} u_n = -\infty$.

   Given any $M < 0$, since $\lim_{n\to+\infty} v_n = -\infty$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $v_n < M$.

   Since $u_n \leq v_n$ for all $n \in \mathbb{N}$, it follows that $u_n \leq v_n < M$ for all $n \geq N_2$. This implies that $\lim_{n\to+\infty} u_n = -\infty$.

Hence, both parts of the proposition have been proved. □