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1 Sequences

1.1 Cauchy sequence

Definition 1.1. The sequence $(u_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence, if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall p, q \geq N : |u_p - u_q| \leq \tfrac{\varepsilon}{2}$$

Proposition 1.1. *A convergent sequence is Cauchy.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a convergent sequence with limit ℓ. This means that for any $\varepsilon > 0$, there exists an N such that

$$\forall n \geq N : |u_n - \ell| < \tfrac{\varepsilon}{2}$$

Now, let's choose two arbitrary indices p and q such that $p, q \geq N$. Then, by the triangle inequality,

$$|u_p - u_q| \leq |u_p - \ell| + |\ell - u_q| \leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

This shows that for any $\varepsilon > 0$, there exists an N such that for all $p, q \geq N$, $|u_p - u_q| \leq \varepsilon$, which is the definition of a Cauchy sequence. Hence, a convergent sequence is a Cauchy sequence. □

Proposition 1.2. *Every Cauchy sequence is convergent*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Let $\varepsilon > 0$. Then, there exists an $N_1 \in \mathbb{N}$ such that

$$\forall p, q \geq N_1 : |u_p - u_q| < \frac{\varepsilon}{2}. \tag{1.1}$$

Since $(u_n)_{n\in\mathbb{N}}$ is Cauchy, it is also bounded. By the Bolzano-Weierstrass theorem **??**, there exists a convergent subsequence $(u_{k_n})_{n\in\mathbb{N}}$ of (u_n). Let ℓ be the limit of this subsequence. Then there exists an $N_2 \in \mathbb{N}$ such that

$$\forall n \geq N_2 : |u_{k_n} - \ell| < \frac{\varepsilon}{2} \tag{1.2}$$

Now, we will show that the entire sequence (u_n) converges to ℓ. By the definition of the subsequence, we have $k_n \to +\infty$ as $n \to +\infty$, Hence, there exists $N_3 \in \mathbb{N}$ such that

$$\forall n \geq N_3 : k_n \geq N_1. \tag{1.3}$$

Choose $N = \max\{N_1, N_2, N_3\}$. Then, for all $n \geq N$, we have from (1.1),(1.2) and (1.3) :

$$|u_n - \ell| \leq |u_n - u_{k_n}| + |u_{k_n} - \ell| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which proves that $\lim_{n\to\infty} u_n = \ell$ and the proposition is proved. □