Chapter N°2 : Numerical Sequences

What is a sequence ?

0.1 Introduction

A sequence is an infinite list, technically an infinite tuple, of numbers, like this :

$$2, 4, 6, 8, 10, ...$$

or like one of these

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, ...$$
$$1, 0, 1, 0, 1, 0, 1, 0, ...$$

An integral part of Analysis I is the study of various properties of real sequences and their relationships. To think flexibly about those relationships, it helps to be aware of some different ways of representing sequences and some advantages and disadvantages of those representations.

For example that we invest a sum S at an annual rate of 10 percent. If S_n represents the sum that we will obtain after n years, we have

$$S_0 = S, \quad S_1 = S.1, 1, ..., \quad S_n = S.(1, 1)^n.$$

At the end of $n = 10$ yers, then we will have $S_{10} = S.(1.1)^{10} \approx S.2, 59$.

Definition 1 *A function $u : \mathbb{N} \to \mathbb{R}$ whose domain of definition the set $\mathbb{N}$ is called a* ***sequence****.*

The values $u(n) = u_n$ of the function u are called the terms of the sequence. In this connection the sequence itself is denoted (u_n), and also written as $u_1, u_2, ..., u_n, ...$. It is called a sequence of elements in $\mathbb{R}$.
The element u_n is called the n-th term of sequence.

0.2 Convergence and divergence of sequence

Definition 2 *If $\lim\limits_{n\to\infty} u_n = l$, we say that the sequence (u_n) convergent to l or tends to l and write $u_n \longrightarrow l \quad as\ n \to +\infty$ A sequence having a limit is said to be convergent. A sequence that does not have a limit is said to be divergent. And*

$$\lim_{n\to\infty} u_n = l \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}(n > N) : |u_n - l| < \epsilon.$$

We call the natural number N the *threshold* for the given ϵ.

Remark 3 *We say that the limit of the sequence (u_n) is in ∞, or (u_n) diverges to ∞, if*

$$\forall A \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}(n > N) : u_n > A.$$

The notations :

$$\lim_{n\to\infty} u_n = \infty, \text{ or } \lim u_n = \infty, \text{ or } u_n \to \infty \text{ if } n \to \infty \text{ or } u_n \to \infty$$

Remark 4 *We say that the limit of the sequence (u_n) is in $-\infty$, or (u_n) diverges to $-\infty$, if*

$$\forall A \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}(n > N) : u_n < A.$$

The notations :

$$\lim_{n\to\infty} u_n = -\infty, \ or\ \lim u_n = -\infty, \ or\ u_n \to -\infty \ if\ n \to \infty \ or\ u_n \to -\infty$$

Exercise 1 : Using the definition of limit, verify that

1. $\lim_{n\to\infty} \frac{1}{n} = 0,$ since $|\frac{1}{n} - 0| = \frac{1}{n} < \epsilon,$ when $n > N = [\frac{1}{\epsilon}]$ (We recall that $[\frac{1}{\epsilon}]$ is the integer part of the number $[\frac{1}{\epsilon}]$.)
2. $\lim_{n\to\infty} \frac{1+n}{n} = 1,$ since $|\frac{1+n}{n} - 1| = \frac{1}{n} < \epsilon,$ when $n > N = [\frac{1}{\epsilon}]$
3. $\lim_{n\to\infty} 1 + \frac{(-1)^n}{n} = 1,$ since $|1 + \frac{(-1)^n}{n} - 1| = \frac{1}{n} < \epsilon,$ when $n > N = [\frac{1}{\epsilon}]$
4. $\lim_{n\to\infty} \frac{\sin n}{n} = 0,$ since $|\frac{\sin n}{n} - 0| = \frac{1}{n} < \epsilon,$ when $n > N = [\frac{1}{\epsilon}]$
5. $\lim_{n\to\infty} \frac{1}{q^n} = 0,$ if $|q| > 1$

Example 1 :

— The sequence $1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, ...$ whose n-th term is $u_n = n^{(-1)^n},\ n \in \mathbb{N}$ is divergent.

— One can verify similarly that the sequence $1, -1, 1, -1, ...$, for which $u_n = (-1)^n$, has not limit.

0.2.1 Properties of the Limit of a Sequence

1. A sequence assuming only one value will be called a constant sequence.
2. If there exists a number a and an index p such that $u_n = a$ for all $n > p$, the sequence (u_n) will be called **ultimately constant.** For example the sequence of n-th term $u_n = \max\{\frac{11}{10}, \frac{n+1}{n}\}$, for $n \geq 10,\ (u_n)$ is ultimately constant.
3. sequence (u_n) is bounded if there exists M such that $|u_n| < M$ for all $n \in \mathbb{N}$.

Theorem 5 *a. An ultimately constant sequence converges.*
b. A convergent sequence cannot have two different limits.
c. A convergent sequence is bounded.

Proof 6 *b This is the most important part of the theorem.*
Let l_1, l_2 tow limits of the sequence (u_n) such that $l_1 > l_2$. put $\epsilon < \frac{(l_1-l_2)}{2}$, so, since the sequence u_n converge respectively to l_1 and l_2 :

$$\exists N_1 \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, (n > N) \Rightarrow |u_n - l_1| < \epsilon$$

$$\exists N_1 \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, (n > N) \Rightarrow |u_n - l_2| < \epsilon$$

But that for $n > max(N_1, N_2)$:

$$l_1 - l_2 = |l_1 - l_2| = |l_1 - u_n + u_n - l_2| \leq |l_1 - u_n| + |u_n - l_2| < 2\epsilon$$

but this is impossible.

c Let $\lim_{n\to\infty} u_n = a$. Setting $\epsilon = 1$ in the definition of limit, we find N such that $|u_n - a| < 1$ for all $n > N$. Then for $n > N$ we have $|u_n| < |a| + 1$. If we now take $M > \max\{|u_1|, ..., |u_n|, |a| + 1\}$

Theorem 7 *(Arithmetic proprieties of limits) Let (a_n), and (b_n) two convergent real sequences, with $a_n \longrightarrow a$ and $b_n \longrightarrow b$. Then we have*

1. *The sequence $(a_n + b_n)$ convergent with*

$$\lim_{n\to\infty}(a_n + b_n) = a + b \quad (\textit{sum rule}).$$

2. *The sequence $(a_n.b_n)$ convergent with*

$$\lim_{n\to\infty}(a_n.b_n) = a.b \quad (\textit{product rule}).$$

3. *For $c \in \mathbb{R}$,the sequence $(c.a_n)$ convergent with*

$$\lim_{n\to\infty}(c.a_n) = c.a \quad (\textit{constant multiple rule}).$$

4. If $b_n \neq 0$, for all $n \in \mathbb{N}$ and $b \neq 0$ Then the sequence $(\frac{a_n}{b_n})$ convergent with

$$\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{a}{b} \quad (\textit{quotient rule}).$$

Exercise 2 : Find the limits of the following sequences

$$\lim_{n\to\infty}\frac{\sin(n^2+1)}{n^2+1}, \quad \lim_{n\to\infty}(\frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n+1}-\sqrt{n}}), \quad \lim_{n\to\infty}\frac{\sin n}{n}(\sqrt{n+1}-\sqrt{n}), \quad \lim_{n\to\infty}(\sqrt{n^4+n^2}-n^2),$$

Theorem 8 *(Passage to the Limit and Inequalities)*

— *Let (a_n), and (b_n) two convergent real sequences, with $a_n \longrightarrow a$ and $b_n \longrightarrow b$. If $a < b$, then there exists an index $N \in \mathbb{N}$ such that $a_n < b_n$, for all $n > N$.*

— *Suppose the sequences $(a_n), (b_n)$, and (c_n) ave such that $a_n < b_n < c_n$ for all $n > N \in \mathbb{N}$. the sequences (a_n) and (c_n) both converge to the same limit, then the sequence (a_n) also converges to that limit.*

For example $-1 \leq \sin(n^2+1) \leq 1$, then $\frac{-1}{n^2+1} \leq \frac{\sin(n^2+1)}{n^2+1} \leq \frac{1}{n^2+1}$ by passage to the limit $\lim_{n\to\infty}\frac{\sin(n^2+1)}{n^2+1} = 0$.

Let us investigate what happens if we consider a sequence (a_n) with $(an) \longrightarrow +\infty$and a sequences (b_n) which is controlled by (a_n) by

$$\forall n \in \mathbb{N} : a_n \leq b_n$$

Theorem 9 *(Comparison theorem for sequences tending to $+\infty$). Let (a_n) and (b_n) be real sequences. Suppose that $(an) \longrightarrow +\infty$ and that there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have $b_n > a_n$. Then, $(b_n) \longrightarrow +\infty$*

0.3 Monotonicity, boundedness, and convergence

- A numerical sequence (u_n) is **strictly increasing** if $u_n < u_{n+1}, \quad \forall n \in \mathbb{N}$
- A numerical sequence (u_n) is **increasing** if $u_n \leq u_{n+1}, \quad \forall n \in \mathbb{N}$
- A numerical sequence (u_n) is **strictly decreasing** if $u_n > u_{n+1}, \quad \forall n \in \mathbb{N}$
- A numerical sequence (u_n) is **decreasing** if $u_n \geq u_{n+1}, \quad \forall n \in \mathbb{N}$

For example the sequence (u_n) such that the n-th term $u_n = n^2$ est strictly increasing.

Remark 10 *A numerical sequence is said to be* ***monotonic*** *if it is increasing or decreasing.*

Remark 11 *Let (u_n) a numerical sequence*

- *If $\frac{u_{n+1}}{u_n} \geq 1$ is* ***increasing*** *and* ***strictly increasing*** *if $\frac{u_{n+1}}{u_n} > 1, \quad \forall n \in \mathbb{N}$*
- *If $\frac{u_{n+1}}{u_n} \leq 1$ is* ***decreasing*** *and* ***strictly decreasing*** *if $\frac{u_{n+1}}{u_n} < 1, \quad \forall n \in \mathbb{N}$*

Theorem 12 *Every increasing sequence (u_n) that is bounded above is convergent.*

From that follows in a straight forward way

Corollary 13 *Every decreasing sequence that is bounded below is convergent.*

From Theorem 10 and Corollary 13 follows immediately

Corollary 14 *Every monotonic bounded sequence is convergent.*

0.4 Adjacent sequences

In general. Two sequences are adjacent if the first is increasing, the second is decreasing, and their difference converges to 0.

Definition 15 *Two real sequences* (u_n) *and* (v_n) *are called adjacent if* (u_n) *is increasing,* (v_n) *is decreasing, and* $\lim_{n\to\infty}(an - bn) = 0$.

Example 2 : The two sequences (u_n) and (v_n) defined by

$$u_n = 1 + \frac{1}{n+1}, \quad v_n = 1 - \frac{1}{n+1}, \quad \forall n \in \mathbb{N}$$

are adjacent.
Exercise 2 : Let $\forall n \in \mathbb{N}$

$$u_n = \sum_{k=1}^{k=n} \frac{1}{k^2}, \text{ and } v_n = u_n + \frac{2}{n+1}$$

Prove that (u_n) and (v_n) are adjacent.
Exercise 3 : (U_n) and (V_n) be two sequences such that :

$$\begin{cases} U_0 \leq V_0, \quad 0 < \beta < \alpha \\ U_n = \dfrac{\alpha U_{n-1} + \beta V_{n-1}}{\alpha + \beta} \\ V_n = \dfrac{\alpha V_{n-1} + \beta U_{n-1}}{\alpha + \beta} \end{cases}$$

1. Let $W_n = U_n - V_n$. Prove that (W_n) is geometric sequence. Identify q and W_0.
2. Prove that (U_n) is an increasing and that (V_n) is decreasing.
3. Deduce that (U_n) and (V_n) are adjacent sequences.
4. Find the limit l in terms of U_0 and V_0.

0.5 Recurrence Sequence (or Recursively-defined sequences)

Definition 16 *Let* $f : \mathbb{R} \to \mathbb{R}$ *a real function. A recursively-defined sequence is an relation that expresses each element of a sequence as a function of the preceding one.*

$$u_0 \in \mathbb{R}, \quad u_{n+1} = f(u_n)$$

Then a recurrence sequence is defined by two data u_0 the first term and recurrence relation

$$u_0, \quad u_1 = f(u_0), \quad u_2 = f(u_1) = fof(u_0), \ldots \quad , u_n = fof\ldots of(u_0)$$

Example 3 : Let

$$u_0 = 2, \text{ and for } n \geq 0, \quad u_{n+1} = f(u_n)$$

$$u_1 = 1 + \sqrt{2}, \quad u_2 == fof(2) = 1 + \sqrt{1 + \sqrt{2}}, \quad , u_3 = fofof(2) = 1 + \sqrt{1 + \sqrt{1 + \sqrt{2}}}, \ldots$$

Example 4 :(Fubunacci sequence) is defined as follow : The succeeding terms are dependent on the last two preceding terms

$$F_0 = 0, \ F_1 = 1, \text{ and for } n \geq 2, \quad F_n = F_{n-1} + F_{n-2}$$

In fact,it is easier to list these out in a list by just adding the previous two terms to get the next term.

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots$$

The Fibonacci sequence has a long history in mathematics and you can find out more about it on line at any number of websites. The Fibonacci sequence is named after the 13th-century Italian mathematician known as Fibonacci,who used it to solve a problem concerning the breeding of rabbits.This sequence also occurs in numerous applications in plant biology.

Remark 17 *Two simple examples of recursive definitions are for* ***arithmetic sequences and geometric sequences.*** *An arithmetic sequence has a common difference, or a constant difference between each term.*

$$u_n - u_{n-1} = r \quad or \quad u_n = u_{n-1} + r$$

A geometric sequence has a common ratio

$$u_n = q.u_{n-1}, \quad or \quad \frac{u_n}{u_{n-1}} = q$$

Again, in this case it is relatively easy to find a formula for the n-th term : $u_n = u_0.q^n$

Finding an explicit expression for u_n as in the above example is often not possible, because solving recursions can be very difficult or even impossible. How, then, can we say anything about the limiting behavior of are cursively defined sequence ? The following procedure will allow us to identify candidates for limits : A **fixed point** of a function is a point x so that $f(x) = x$. For recursive sequences this translates as if the sequence u_n is can be given as $u_{n+1} = f(u_n)$ and if l is a fixed point for $f(x)$, then if $u_n = l$ is equal to the fixed point for some k, then all successive values of u_n are also equal to l for $k > n$

Proposition 18 *If f is associate function of a convergent recursive sequence to l, then l is solution of the equation $f(l) = l$.*

Example 3 : Assume that $\lim_{n\to\infty} u_n$ exists for

$$u_{n+1} = \sqrt{3u_n}, \quad \text{with } u_0 = 2$$

Find $\lim_{n\to\infty} u_n$
Since the problem tells us that the limit exists, we don't have to worry about existence. We may assume that $\lim_{n\to\infty} u_n = l$ The problem that remains is to identify the limit. To do this we need to note that if $\lim_{n\to\infty} u_n = l$ then it is true that $\lim_{n\to\infty} u_{n+1} = l$, ,since these are exactly the same sequence. Now, we compute the fixed points.We solve

$$\lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} = \sqrt{3u_n} \Leftrightarrow l = \sqrt{3l},$$

This has two solutions, namely,$l = 0$ and $l = 3$. When $u_0 = 2$, we have $u_n > 2$ for all $n = n_1, n_2, n_3, ...$ so we can exclude $l = 0$ as the limiting value. This leaves only one possibility,and we conclude that

$$\lim_{n\to\infty} u_n = 3.$$

0.6 Cauchy Criterion

We now have a test that allows us to establish that a monotonic sequence converges without knowing its limit.

Definition 19 *A sequences (u_n) is called fundamental or Cauchy sequence if for any $\epsilon > 0$ there exists an index $N \in \mathbb{N}$ such that $|u_m - u_n| < \epsilon$ whenever $n > N$, and $m > N$ i.e*

$$\forall\epsilon > 0, \exists N \in \mathbb{N}, \forall m > n > N : |u_m - u_n| < \epsilon.$$

For example, the sequence $(\frac{1}{n})_{n\mathbb{N}}$ is a Cauchy sequence.

Theorem 20 *(Cauchy's converges criterion) A numerical sequence converges if and only if is a Cauchy sequence.*

For example the sequence $(-1)^n$ is not Cauchy sequence, because has not limit. Even though this fact is obvious we shall give a formal verification. The negation of the statement that (u_n) is cauch sequence is the following :

$$\exists\epsilon > 0, \forall N \in \mathbb{N}, \exists m > n > N : |u_m - u_n| \geq \epsilon.$$

That is, there exists $\exists\epsilon > 0$, such that for any $N \in \mathbb{N}$, two numbers n, m larger than N for which $|u_m - u_n| \geq \epsilon$. In our cases it suffices to set $\epsilon = 1$ then for any $N \in \mathbb{N}$, we shall have

$$|u_{N+1} - u_{N+2}| = |1 - (-1)| = 2 > 1 = \epsilon$$

0.7 Cauchy sequences and convergence

It probably will not surprise you to learn that every convergent sequence is Cauchy, and vice versa.

Theorem 21 *Let (u_n) be a real sequence. Then, (u_n) is* ***convergent*** *iff (u_n) is a* ***Cauchy sequence****.*

To prove Theorem 15, we first prove a lemma 16 :

Lemma 22 *(Cauchy sequences are bounded). Let (u_n) be a Cauchy sequence. Then, there exists a non-negative constant C such that*

$$|u_n| \leq C, \quad \forall n \in \mathbb{N}$$

Proof of Theorem 15 :

— [⇒] Let $\epsilon > 0$ be arbitrary. Since $(u_n) \longrightarrow l$ there exists $N \in \mathbb{N}$ such that

$$\forall m > n > N, |u_m - l| < \frac{\epsilon}{2} \text{ and } |u_n - l| < \frac{\epsilon}{2}$$

So for all $\forall m > n > N$ we have

$$|u_m - u_n| = |u_m - l + l + u_n| < |u_n - l| + |u_m - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

So (u_n) is a Cauchy sequence.

— [⇒] Since (u_n) is a Cauchy sequence, it is bounded by Lemma 16. Hence, by the Bolzano-Weierstrass theorem, (u_n) has a convergent subsequence, say (u_{n_i}). Suppose that (u_{n_i}) convergent to l. We will show that (u_n) convergent to l also.
To this end, let $\epsilon > 0$ be arbitrary. Then, since (u_n) is Cauchy, there exists $N_1 \in \mathbb{N}$ such that

$$\forall n, n_i \geq N_1, |u_n - u_{n_i}| < \frac{\epsilon}{2}$$

Also, since (u_{n_i}) convergent to l, there exists $N_2 in \mathbb{N}$ such that

$$\forall i \geq N_2, |u_{n_i} - l| < \frac{\epsilon}{2}$$

Let $N = \max\{N_1, N_2\}$. Then, since $n_i > i$, we have

$$\forall n > N, |u_n - l| = |u_n - l - u_{n_i} + u_{n_i}| < |u_{n_i} - u_n| + |u_{n_i} - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence (u_n) convergent to l, i.e. (u_n) is convergent.

Exercise 3 :

— Show that $u_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is Cauchy sequence.

— Show that $u_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}, \quad u_n = \sum_{k=1}^{n} \frac{3}{2 \ln k}$ are not Cauchy sequence.