1) Kinematic features of a particle

The kinetic properties of a moving particle can be studied by focusing attention on its kinetic features during its movement. This includes tracking its change in position relative to a frame of reference, examining its path (trajectory) of movement, measuring its velocity, and estimating its acceleration.

1.1) Position and vector position

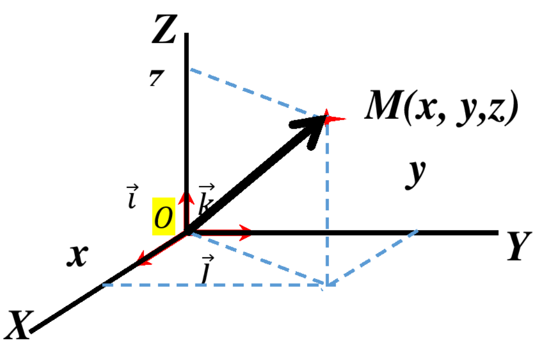
The position of a moving particle at a specific moment (t) is determined with respect to a specified reference frame (origin) through its coordinates. If the motion occurs in one dimension, the position of the moving entity is expressed with a single coordinate. In the case of motion in a plane, it is represented by two coordinates. Meanwhile, three coordinates are employed to express motion in three-dimensional space.

The position vector *for the moving particle 'M' at the moment 't' is a vector extending from the origin to the location of the point. It is symbolized by '$\overrightarrow{OM}$' and is analytically formulated as follows:* $\overrightarrow{OM} = \overrightarrow{r_M} = x\,\vec{\imath} + y\vec{\jmath} + z\,\vec{k}$

The position vector *is characterized by:*

- ***Starting point (the tail) of the vector:*** *The origin point 'O'*
- ***Ending point (Head or Tip) of the vector:*** *The location of the moving particle 'M'.*
- ***Direction of the vector:*** *From point 'O' to position 'M'.*
- ***Support of the vector:*** *The straight line connecting the origin 'O' to the position 'M'.*
- ***The magnitude of the vector:*** $\|\overrightarrow{OM}\| = \sqrt{x^2 + y^2 + z^2}$

The position vector $\overrightarrow{OM}$ is represented in the Cartesian coordinate system as shown in the figure



1.2) The Displacement Vector

The displacement vector for a moving particle between two moments, $\boldsymbol{t_i}$ and $\boldsymbol{t_f}$, is a vector that represents the particle's transition from position M_i to position M_f. It is symbolized by $\overrightarrow{M_\imath M_f}$ and extends from the initial position to the final position. It can be expressed as the difference between the position vectors, as follows:

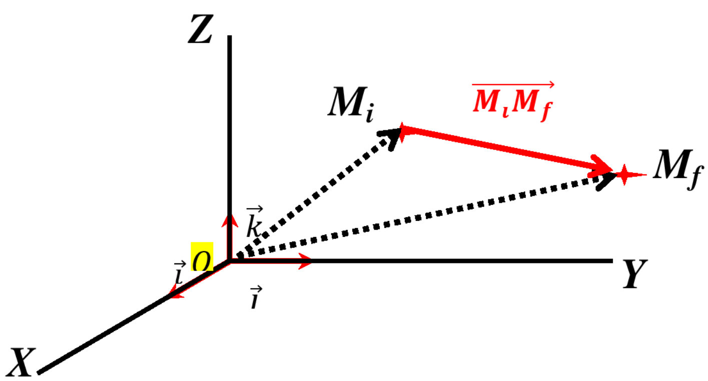
$$\overrightarrow{\boldsymbol{M_\imath M_f}} = \overrightarrow{\boldsymbol{OM_f}} - \overrightarrow{\boldsymbol{OM_\imath}}$$

$$\overrightarrow{\boldsymbol{M_\imath M_f}} = (\boldsymbol{x_f} - \boldsymbol{x_i})\,\vec{\boldsymbol{\imath}} + (\boldsymbol{y_f} - \boldsymbol{y_i})\,\vec{\boldsymbol{\jmath}} + (\boldsymbol{z_f} - \boldsymbol{z_i})\,\vec{\boldsymbol{k}}$$

The displacement vector is characterized by:

- ***Starting point (Tail) of the vector:*** *The position of the particle 'M_i' at moment 't_i.'*
- ***Head (Tip) of the vector:*** *The position of the moving particle 'M_f' at moment 't_f.'*
- ***Direction of the vector:*** *From position M_i to position M_f*
- ***Support of the vector:*** *The straight line connecting position M_i to position M_f*
- ***The magnitude of the vector:*** $\left\|\overrightarrow{\boldsymbol{M_\imath M_f}}\right\| = \sqrt{(\boldsymbol{x_f} - \boldsymbol{x_i})^2 + (\boldsymbol{y_f} - \boldsymbol{y_i})^2 + (\boldsymbol{z_f} - \boldsymbol{z_i})^2}$

The position vector $\overrightarrow{M_\imath M_f}$ is represented in the Cartesian coordinate system as shown in the figure



1.3) Velocity vector

Generally, velocity represents the change in distance travelled by a moving object over a specific time duration. By means of calculating velocity, we can classify it into two forms: average velocity and instantaneous velocity.

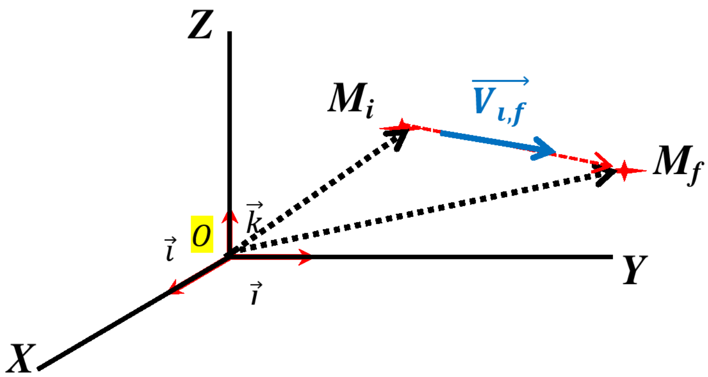
1.3.1) Average velocity

Average velocity, denoted as $\overrightarrow{V_a}$ *or* $\overrightarrow{V_{\iota,f}}$*, serves as a vector quantity to describe the velocity over an extended time interval between two distinct moments* t_i *and* t_f*. Mathematically, it is articulated as the ratio of the displacement vector* $\overrightarrow{M_\iota M_f}$ *to the time interval* $\Delta t = t_f - t_i$ *during this displacement.*

The vector of the average velocity ($\overrightarrow{V_{\iota,f}}$) *shares the same direction and support as the displacement vector* ($\overrightarrow{M_\iota M_f}$)*, while its magnitude is determined by the ratio of the displacement vector's length to the time interval.*

$$\overrightarrow{V_{\iota,f}} = \frac{\overrightarrow{M_\iota M_f}}{\Delta t} = \frac{\overrightarrow{OM_f} - \overrightarrow{OM_\iota}}{t_f - t_i} = \frac{(x_f - x_i)}{t_f - t_i}\vec{\imath} + \frac{(y_f - y_i)}{t_f - t_i}\vec{\jmath} + \frac{(z_f - z_i)}{t_f - t_i}\vec{k}$$

$$\left\|\overrightarrow{V_{\iota,f}}\right\| = \sqrt{\left(\frac{x_f - x_i}{t_f - t_i}\right)^2 + \left(\frac{y_f - y_i}{t_f - t_i}\right)^2 + \left(\frac{z_f - z_i}{t_f - t_i}\right)^2}$$



1.3.2) Instantaneous velocity

The instantaneous velocity, represented by $\vec{V}$*, is a vector describing the velocity of an particle at a specific moment "*t*" or within an Infinitesimally small time interval (*Δt*) approaching (tending to) zero. Mathematically, the vector of instantaneous velocity is defined as the limiting value of the ratio of the displacement vector to the time interval as* Δt *approaches (tends to) zero.*

$$\vec{V}(t) = \lim_{\Delta t \to 0} \frac{\overrightarrow{M_\iota M_f}}{\Delta t}$$

In mathematical calculations, the limit process is expressed by the derivative of the position vector with respect to time.

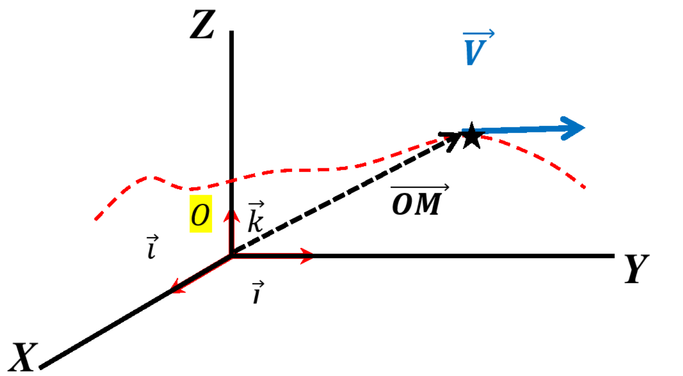
$$\vec{V}(t) = \frac{d(\overrightarrow{OM})}{dt} = \frac{d(x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k})}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$$

$$\|\vec{V}(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

The vector of instantaneous velocity represents a tangential vector to the path at the location where the instantaneous velocity is calculated.

Note: The vector of average velocity between two moments t_i and t_f equals the vector of instantaneous velocity at the midpoint of this interval $\frac{t_i+t_f}{2}$

$$\vec{V}\left(\frac{t_i + t_f}{2}\right) = \overrightarrow{V_{i,f}}$$



1.4)Acceleration vector

Acceleration is defined as the rate of change in velocity with respect to time. Based on how acceleration is calculated, we can distinguish between two types: average acceleration and instantaneous acceleration.

1.4.1) Average acceleration

Average acceleration, denoted as $\overrightarrow{a_a}$ or $\overrightarrow{a_{i,f}}$, describes the change in velocity over a specific time interval, typically between two distinct moments t_i and t_f. Mathematically, the Average acceleration is expressed as the ratio of the change in velocity ($\overrightarrow{\Delta V_{i,f}} = \overrightarrow{V_f} - \overrightarrow{V_i}$) to the *change in time ($\Delta t = t_f - t_i$). The vector of average acceleration ($\overrightarrow{a_{\iota,f}}$) shares the same direction and support as the vector of velocity change ($\overrightarrow{\Delta V_{\iota,f}}$), While its magnitude is the ratio of the magnitude of the velocity change vector ($\left\|\overrightarrow{\Delta V_{\iota,f}}\right\|$) to the duration of the time interval $\Delta t = t_f - t_i$) .*

$$\overrightarrow{a_{\iota,f}} = \frac{\overrightarrow{\Delta V_{\iota,f}}}{\Delta t} = \frac{\overrightarrow{V_f} - \overrightarrow{V_\iota}}{t_f - t_i} = \frac{(V_{xf} - V_{xi})}{t_f - t_i}\vec{\imath} + \frac{(V_{yf} - V_{yi})}{t_f - t_i}\vec{\jmath} + \frac{(V_{zf} - V_{zi})}{t_f - t_i}\vec{k}$$

$$\left\|\overrightarrow{a_{\iota,f}}\right\| = \frac{\left\|\overrightarrow{\Delta V_{\iota,f}}\right\|}{t_f - t_i}$$

1.4.2) Instantaneous acceleration

The instantaneous acceleration vector, denoted by $\vec{a}$, signifies the acceleration at a precise moment t, specifically within an infinitesimally small time interval (approaching or tending to zero).

The acceleration vector is mathematically defined as the limit of the rate of change in velocity over an infinitesimally small period of time, approaching or tending to zero.

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\overrightarrow{V_{\iota,f}}}{\Delta t}$$

In mathematical terms, when the time interval becomes extremely small, the limit of a function represents its derivative. Consequently, we can express the formula for instantaneous acceleration as follow

$$\vec{a}(t) = \frac{d(\vec{V})}{dt} = \frac{d(V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k})}{dt} = \frac{dV_x}{dt}\vec{\imath} + \frac{dV_y}{dt}\vec{\jmath} + \frac{dV_z}{dt}\vec{k}$$

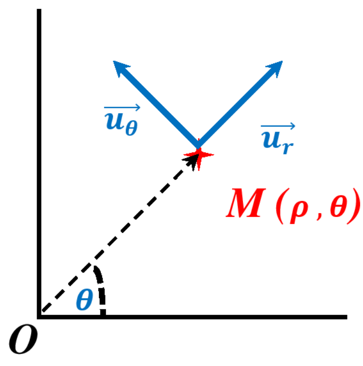
$$\|\vec{a}(t)\| = \sqrt{\left(\frac{dV_x}{dt}\right)^2 + \left(\frac{dV_y}{dt}\right)^2 + \left(\frac{dV_z}{dt}\right)^2}$$

Note: The vector of average acceleration between two moments t_i and t_f equals the vector of instantaneous acceleration at the midpoint of this interval $\frac{t_i+t_f}{2}$

$$\vec{a}\left(\frac{t_i + t_f}{2}\right) = \overrightarrow{a_{\iota,f}}$$

1.5) Velocity and acceleration in polar coordinate system

*The polar coordinates of point M represented by both Radial **(ρ)** and Angular **(θ)** coordinates, and given as **M (ρ, θ)**, and the position vector for the point M can be written in the polar coordinate system as follows:* $\overrightarrow{OM} = \rho\overrightarrow{u_r}$ *where* $\rho = \|\overrightarrow{OM}\|$ *and* $\overrightarrow{u_r} = \frac{\overrightarrow{r_M}}{\rho} = \frac{\overrightarrow{OM}}{\|\overrightarrow{OM}\|}$



*The expression for the instantaneous **velocity** vector in **polar** coordinates is given as follows:*

$$\vec{V}(t) = \frac{d(\overrightarrow{OM})}{dt} = \frac{d(\rho\overrightarrow{u_r})}{dt} = \frac{d\rho}{dt}\overrightarrow{u_r} + \rho\frac{d\overrightarrow{u_r}}{dt}$$

$$\vec{u}_r = \cos\theta \times \vec{\imath} + \sin\theta \times \vec{\jmath}$$

$$\vec{u}_\vartheta = -\sin\theta \times \vec{\imath} + \cos\theta \times \vec{\jmath}$$

$$\frac{d\,(\vec{u}_r)}{dt} = \frac{d(\cos\theta \times \vec{\imath} + \sin\theta \times \vec{\jmath})}{dt} = -\frac{d\theta}{dt}\sin\theta \times \vec{\imath} + \frac{d\theta}{dt}\cos\theta \times \vec{\jmath}$$

where $\frac{d\,\vec{\imath}}{dt} = \vec{0}$ $\frac{d\,\vec{\jmath}}{dt} = \vec{0}$ $\frac{d\vartheta}{dt} = \dot{\theta}$

$$\frac{d\,(\vec{u}_r)}{dt} = \frac{d\theta}{dt}(-\sin\theta \times \vec{\imath} + \cos\theta \times \vec{\jmath}) = \dot{\theta}\,\vec{u}_\theta$$

$$\frac{d\,(\vec{u}_\theta)}{dt} = \frac{d(-\sin\theta \times \vec{\imath} + \cos\theta \times \vec{\jmath})}{dt} = -\frac{d\theta}{dt}\cos\theta \times \vec{\imath} - \frac{d\theta}{dt}\sin\theta \times \vec{\jmath}$$

where $\frac{d\,\vec{\imath}}{dt} = \vec{0}$ $\frac{d\,\vec{\jmath}}{dt} = \vec{0}$ $\frac{d\theta}{dt} = \dot{\theta}$

$$\frac{d\,(\vec{u}_\theta)}{dt} = -\frac{d\theta}{dt}(\cos\theta \times \vec{\imath} + \sin\theta \times \vec{\jmath}) = -\dot{\theta}\,\vec{u}_r$$

$$\vec{V}(t) = \frac{d\rho}{dt}\overrightarrow{u_r} + \rho\frac{d\theta}{dt}\vec{u}_\theta = \dot{\rho}\overrightarrow{u_r} + \rho\dot{\theta}\,\vec{u}_\theta$$

$$\|\vec{V}(t)\| = \sqrt{(\dot{\rho})^2 + (\rho\dot{\theta})^2}$$

The expression for the instantaneous ***acceleration*** *vector in* ***polar*** *coordinates is given as follows:*

$$\vec{a}(t) = \frac{d(\vec{V})}{dt} = \frac{d\,(\dot{\rho}\vec{u_r} + \rho\dot{\theta}\,\vec{u}_\theta)}{dt}$$

$$= \frac{d\,\dot{\rho}}{dt}\vec{u_r} + \dot{\rho}\frac{d\vec{u_r}}{dt} + \frac{d\rho}{dt}\dot{\theta}\,\vec{u}_\theta + \rho\frac{d\dot{\theta}}{dt}\vec{u}_\theta + \rho\dot{\theta}\frac{d\vec{u}_\theta}{dt}$$

$$= \ddot{\rho}\vec{u_r} + \dot{\rho}\dot{\theta}\,\vec{u}_\theta + \dot{\rho}\dot{\theta}\,\vec{u}_\theta + \rho\ddot{\theta}\vec{u}_\theta - \rho\dot{\theta}\dot{\theta}\,\vec{u}_r$$

$$\vec{a}(t) = (\ddot{\rho} - \rho\dot{\theta}^2)\,\vec{u}_r + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\vec{u}_\theta$$

$$\|\vec{a}(t)\| = \sqrt{= (\ddot{\rho} - \rho\dot{\theta}^2)^2 + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})^2}$$

1.6) Velocity and acceleration in cylindrical coordinate system

$$\overrightarrow{OM} = \rho\vec{u_r} + z\,\vec{k}$$

$$\vec{V}(t) = \frac{d\,\rho}{dt}\vec{u_r} + \rho\frac{d\theta}{dt}\,\vec{u}_\theta + \frac{dz}{dt}\vec{k} = \dot{\rho}\vec{u_r} + \rho\dot{\theta}\,\vec{u}_\theta + \dot{z}\,\vec{k}$$

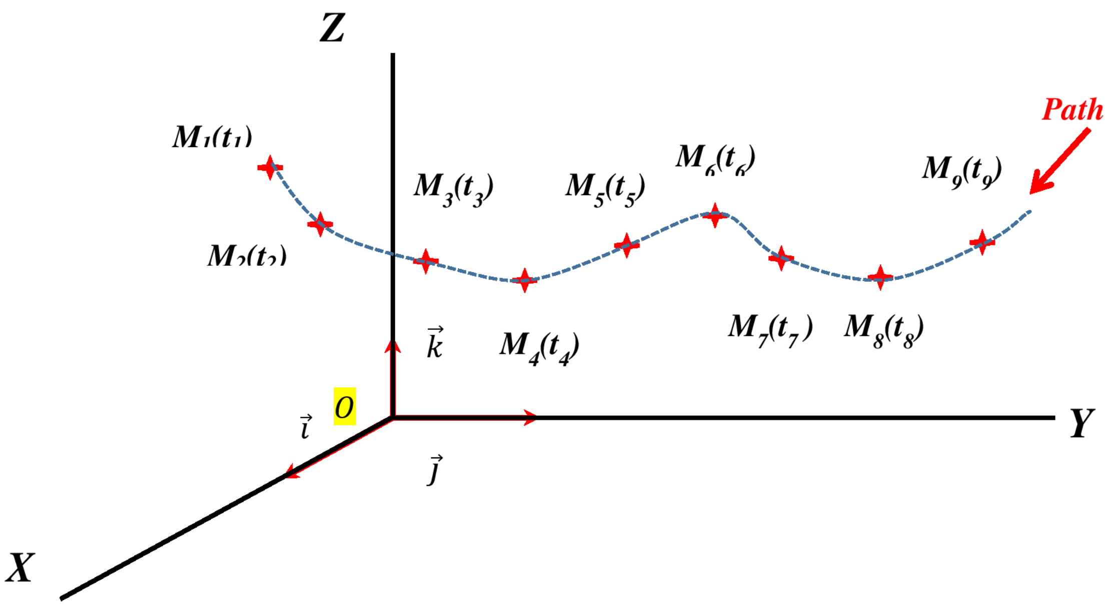
$$\|\vec{V}(t)\| = \sqrt{(\dot{\rho})^2 + (\rho\dot{\theta})^2 + \dot{z}^2}$$

$$\vec{a}(t) = (\ddot{\rho} - \rho\dot{\theta}^2)\vec{u}_r + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\vec{u}_\theta + \ddot{z}\,\vec{k}$$

$$\|\vec{a}(t)\| = \sqrt{= (\ddot{\rho} - \rho\dot{\theta}^2)^2 + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})^2 + \ddot{z}^2}$$

1.8) Motion Path (trajectory)

The trajectory is a set of successive positions occupied by a moving particle at consecutive (successive) moments. In other words, the trajectory is the geometric locus of consecutive positions of the moving particle. It represents the connected line passing through all the positions of the moving particle during its motion in chronological order. By examining the shape of the trajectory followed by the moving particle, we can determine the type of its motion, whether it is straight, curved, or circular.



1.8.1) Path (trajectory) Equation

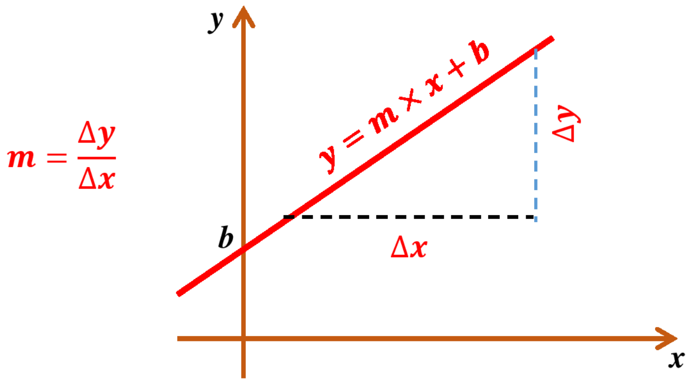
The path equation for a moving particle is a function that defines the relationship between the coordinates of this particle during its movement. Examples of some standard forms of the path equations:

- ***Straight Path Equation:***

  In two-dimensional Cartesian coordinates, a straight path can be represented by the equation:

$$\boldsymbol{y = m \times x + b}$$

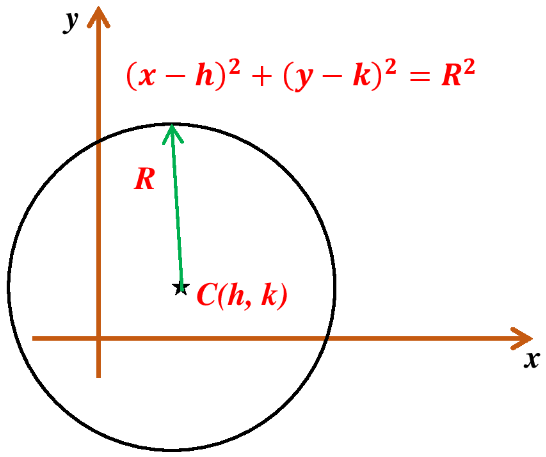
*where **m** is the slope of the line, and **b** is the y-intercept.*



- ***Circular Path Equation:***

The equation for a circle with radius ***R*** *centered at the point C(h, k)*

$$(x-h)^2+(y-k)^2=R^2$$



- ***Parabolic Path Equation:***

The general equation for a parabola in standard form is:

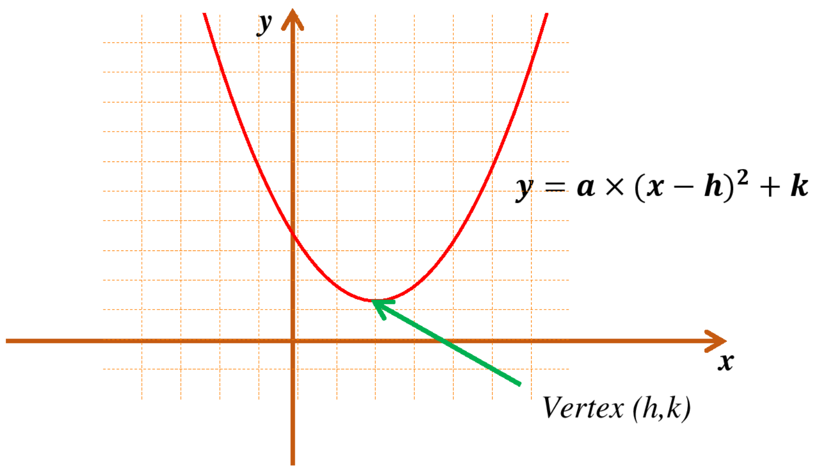
$$y=a\times x^2+b\ \times x+c$$

This is a quadratic equation, where ***a***, ***b***, *and* ***c*** *are constants.*

The vertex form of a parabola is:

$$y=a\times(x-h)^2+k$$

Where (h,k) is the vertex of the parabola.

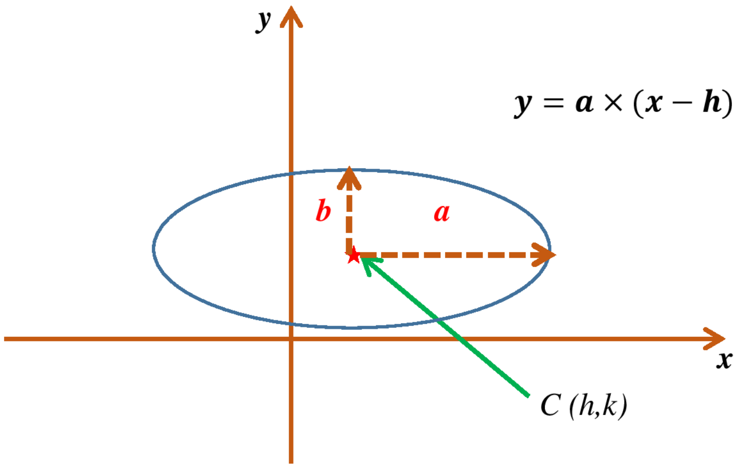


- ***Elliptical Path Equation:***

The general equation for a hyperbola is given by:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

where, ***(h, k)*** *is the center of the hyperbola, and* ***a*** *and* ***b*** *are constants determining the shape of the hyperbola.*



Example01:

The position of a moving particle in the Cartesian coordinate system is described over time by the following equations: $x = 3(1 + cos(2t)), y = 3(2 + sin(2t))$

1- Determine the Path (trajectory) Equation and plot it.

$$\begin{cases} x = 3(1 + cos(2t)) \\ y = 2(2 + sin(2t)) \end{cases} \Rightarrow \begin{cases} x = 3 + 3\,cos(2t) \\ y = 6 + 3\,sin(2t) \end{cases} \Rightarrow \begin{cases} \frac{x-3}{3} = cos(2t) \\ \frac{y-6}{3} = sin(2t) \end{cases}$$

$$\begin{cases} \left(\frac{x-3}{3}\right)^2 = (cos(2t))^2 \\ \left(\frac{y-6}{3}\right)^2 = (sin(2t))^2 \end{cases} \Rightarrow \left(\frac{x-3}{3}\right)^2 + \left(\frac{y-6}{3}\right)^2 = 1$$

$$(x-3)^2 + (y-6)^2 = 3^2$$

The trajectory takes on a circular form as the derived equation represents a circle with a radius of 3, centered at the coordinates (3, 6)