4 Real functions

4.1 Preliminaries

Definition 4.1. A function is a relation f between two sets E and F such that, every element $x \in E$ has at most a relation with an element $y \in F$ denoted by $f(x)$ and we write

$$\begin{array}{rcl} f : E & \longrightarrow & F \\ x & \longmapsto & y := f(x) \end{array}$$

The domain of definition of f is the set defined by

$$D_f := \{x \in E : f(x) \text{ exists}\}.$$

4.2 Limits

Definition 4.2. Let I be an open interval, $x_0 \in I$, $\ell \in \mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function.

1. We say the function f has a left limit ℓ at x_0 and we write $\lim_{x \to x_0^-} f(x) = \ell$, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I : -\delta < x - x_0 < 0 \Longrightarrow |f(x) - \ell| \leq \varepsilon.$$

2. We say the function f has a right limit ℓ at x_0 and we write $\lim_{x \to x_0^+} f(x) = \ell$, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I : 0 < x - x_0 < \delta \Longrightarrow |f(x) - \ell| \leq \varepsilon.$$

3. We say the function f has a limit ℓ at x_0 and we write $\lim_{x \to x_0} f(x) = \ell$, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I : 0 < |x - x_0| < \delta \Longrightarrow |f(x) - \ell| \leq \varepsilon.$$

Or equivalently (prove it), if $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = \ell.$

☞ **Remark 4.1.** • We also denote the limit by "arrow" notation $f(x) \to \ell$ as $x \to x_0$ and say $f(x)$ goes to ℓ as x goes to x_0.

• It follows directly from the above definition that

$$\lim_{x \to x_0} f(x) = \ell \Longleftrightarrow \lim_{x \to x_0} |f(x) - \ell| = 0$$

☞ **Example 4.1.** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function.

1. If $f(x) = 2x$. Show that $\lim_{x\to 1} f(x) = 2$.

2. If $f(x) = x \sin \frac{1}{x}$. Show that $\lim_{x\to 0} f(x) = 0$

3. If $f(x) = \mathbf{sgn}\ x := \begin{cases} +1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$, **(the sign function)**. Show that

$$\lim_{x\to 0^-} f(x) = -1, \quad \lim_{x\to 0^+} f(x) = +1.$$

although the corresponding limit does not exist.

4. If $f(x) = \sin \frac{1}{x}$. Show that $\lim_{x\to 0} f(x)$ does not exist.

Definition 4.3 (Limits as $x \to \pm\infty$). Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function well defined for all $x < -M$ and $x > M$ for certain $M > 0$ and $\ell \in \mathbb{R}$. We say the limit of f equal to ℓ at $+\infty$ (resp. $-\infty$) and we write $\lim_{x\to +\infty} f(x) = \ell$ (resp. $\lim_{x\to -\infty} f(x) = \ell$ if

$$\forall \varepsilon > 0, \exists A > 0, \forall x \in, I : x > A \Longrightarrow |f(x) - \ell| \leq \varepsilon$$

$$(\text{resp. } \forall \varepsilon > 0, \exists B < 0, \forall x \in, I : x < B \Longrightarrow |f(x) - \ell| \leq \varepsilon$$

Proposition 4.1 (Algebraic properties). Let $f, g, h : I \longrightarrow \mathbb{R}$ be functions and $x_0 \in I$. Suppose that

$$\lim_{x\to x_0} f(x) = L, \quad \lim_{x\to x_0} g(x) = M.$$

Then

- $\lim_{x\to x_0} \lambda f(x) = \lambda L$ for every $\lambda \in \mathbb{R}$.
- $\lim_{x\to x_0} (f(x) + g(x)) = L + M$.
- $\lim_{x\to x_0} f(x)g(x) = LM$.
- $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$ and $g(x) \neq 0, \forall x \in I$.

Proof. We will prove each part separately using the epsilon-delta definition of limits.

Part 1: Let $\lambda \in \mathbb{R}^*$ and $\varepsilon > 0$. Since $\lim_{x \to x_0} f(x) = L$, there exists $\delta > 0$ such that for all $x \in I$ with $0 < |x - x_0| < \delta$, we have $|f(x) - L| < \frac{\varepsilon}{|\lambda|}$. Now, for such x, we have

$$|\lambda f(x) - \lambda L| = |\lambda| \cdot |f(x) - L| < |\lambda| \cdot \frac{\varepsilon}{|\lambda|} = \varepsilon.$$

This shows that $\lim_{x \to x_0} \lambda f(x) = \lambda L$.

Part 2: Let $\varepsilon > 0$. Since $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $x \in I$ we have

$$0 < |x - x_0| < \delta_1 \implies |f(x) - L| < \frac{\varepsilon}{2}$$

$$0 < |x - x_0| < \delta_2 \implies |g(x) - M| < \frac{\varepsilon}{2}$$

.

Let $\delta = \min(\delta_1, \delta_2)$. For all $x \in I$ with $0 < |x - x_0| < \delta$, we have

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\lim_{x \to x_0}(f(x)+g(x)) = L+M$. **Part 3:** Let $\varepsilon > 0$. Since $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $x \in I$ we have

$$0 < |x - x_0| < \delta_1 \implies |f(x) - L| < \epsilon$$

$$0 < |x - x_0| < \delta_2 \implies |g(x) - M| < \epsilon$$

where $\epsilon > 0$ will be chosen later. Let $\delta = \min(\delta_1, \delta_2)$. For all $x \in I$ with $0 < |x - x_0| < \delta$, we have

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &\leq |f(x)||g(x) - M| + |M||f(x) - L| \\ &\leq (|L| + \epsilon)\epsilon + |M|\epsilon = \epsilon^2 + (|L| + |M|)\epsilon. \end{aligned}$$

We can make the expression smaller than ε by appropriately choosing ϵ. **Part 4:** Let $\varepsilon > 0$. Since $M \neq 0$, there exists $\delta_1 > 0$ such that for all $x \in I$ with $0 < |x - x_0| < \delta_1$, we have $|g(x) - M| < \frac{|M|}{2}$.

Additionally, since $\lim_{x \to x_0} f(x) = L$, there exists $\delta_2 > 0$ such that for all $x \in I$ with $0 < |x - x_0| < \delta_2$, we have $|f(x) - L| < \frac{\varepsilon |M|}{2}$.

Let $\delta = \min(\delta_1, \delta_2)$. For all $x \in I$ with $0 < |x - x_0| < \delta$, we have

$$\left| \frac{f(x)}{g(x)} - \frac{L}{M} \right| = \frac{|f(x)M - g(x)L|}{|g(x)M|} \leq \frac{|f(x) - L| \cdot |M| + |g(x) - M| \cdot |L|}{|g(x)| \cdot |M|} < \frac{\frac{\varepsilon |M|}{2} \cdot |M| + \frac{|M|}{2} \cdot |L|}{\frac{|M|}{2} \cdot |M|}.$$

Since $|M|$ is not zero, we can choose δ small enough such that the expression becomes smaller than ε. $\square$

4.3 Continuity

In this paragraph, I is an open interval, $x_0 \in I$, $f : I \longrightarrow \mathbb{R}$ is a function well defined for all $x \in I$.

Definition 4.4 (Continuity). 1. We say that f is continuous at x_0 if $\boxed{\lim_{x \to x_0} f(x) = f(x_0)}$, that is,

$$\boxed{\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I : |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon.}$$

2. We say f is continuous (on I) if it is continuous at every point $x_0 \in I$.

3. We say that f is continuous from the left at x_0 if $\boxed{\lim_{x \to x_0^-} f(x) = f(x_0)}$, that is

$$\boxed{\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I : -\delta < x - x_0 \leq 0 \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon.}$$

4. We say that f is continuous from the left at x_0 if $\boxed{\lim_{x \to x_0^+} f(x) = f(x_0)}$, that is

$$\boxed{\forall \varepsilon > 0, \exists \delta > 0, \forall x \in I : 0 \leq x - x_0 < +\delta \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon.}$$

☞ **Remark 4.2.** • It follows from the above definition that f is continuous at x_0 if and only if

$$\boxed{\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0).}$$

☞ **Example 4.2.** 1. The function $f(x) = x^2$ is continuous at all points in $\mathbb{R}$. Indeed

2. The function $f(x) = \begin{cases} x \ln x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$ is continuous at 0^+

3. The sign function $f(x) = \textbf{sgn}\ x := \begin{cases} +1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$ is not continuous at 0 since $\lim_{x \to x_0} f(x)$ does not exist.

4. The function $f(x) = \begin{cases} x^2 & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$ is not continuous at 0, since $\lim_{x \to x_0} f(x) = 0 \neq 1 := f(0)$.

5. Study the continuity of the following function

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 1 \\ x^1 + 1 & \text{if} \quad x \geq 1 \end{cases}$$

Theorem 4.2. If f and g are continuous functions at x_0, then so are λf, $f+g$ and fg. If in addition $g(x_0) \neq 0$, then f/g is continuous at x_0.

Proof. Exercise □

Theorem 4.3. Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function such that $f(a)f(b) \leq 0$. Then, there exists $c \in [a,b]$ such that $f(c) = 0$.

Proof. Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that $f(a)f(b) \leq 0$. We aim to show that there exists $c \in [a,b]$ such that $f(c) = 0$. Without loss of generality, assume $f(a) \leq f(b)$. If $f(a) = 0$ or $f(b) = 0$, we are done, so let's consider the case where $f(a) < 0$ and $f(b) > 0$. Define the set

$$S = \{x \in [a,b] \mid f(x) \leq 0\}.$$

Notice that $a \in S$ since $f(a) \leq 0$, and $b \notin S$ since $f(b) > 0$. Therefore, S is nonempty and bounded above by b, so $\sup S$ exists.

Let $c = \sup S$. We will show that $f(c) = 0$.

Since c is the supremum of S, for any $\varepsilon > 0$, there exists $x \in S$ such that $c - \varepsilon < x \leq c$. This implies $f(x) \leq 0$.

Because f is continuous, as ε approaches 0, $f(x)$ approaches $f(c)$. Since $f(x) \leq 0$ for all $x \in S$, we have $f(c) \leq 0$.

Suppose, for the sake of contradiction, that $f(c) < 0$. Then by continuity of f, there exists $\delta > 0$ such that for all x with $|x - c| < \delta$, we have $f(x) < 0$. This contradicts the fact that $c = \sup S$.

Hence, we must have $f(c) \geq 0$.

Since we've shown both $f(c) \leq 0$ and $f(c) \geq 0$, it follows that $f(c) = 0$.

Thus, in all cases, there exists $c \in [a,b]$ such that $f(c) = 0$, completing the proof of the Intermediate Value Theorem.

□

Theorem 4.4 ((Weierstrass extreme value)). If $f : [a,b] \longrightarrow \mathbb{R}$ is continuous on the closed and bounded interval $[a,b]$. Then f is bounded on $[a,b]$ and attains its maximum and minimum values on $[a,b]$. That is

$$\exists c_1, c_2 \in [a,b] : f(c_1) = \min_{x \in [a,b]} f(x), \;\; f(c_2) = \max_{x \in [a,b]} f(x)$$

4.4 Uniform continuity

Definition 4.5. Let $f : I \longrightarrow \mathbb{R}$ be a function. We say f is uniformly continuous if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in I : |x - y| \leq \delta \Longrightarrow |f(x) - f(y)| \leq \varepsilon.$$

☞ **Remark** **4.3.** In other words, f is uniformly continuous if $f(x) - f(y) \to 0$ as $x - y \to 0$.

☞ **Example** **4.3.** 1. $f : [0,1] \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous. Indeed, given $\varepsilon > 0$. We have

$$|f(x) - f(y)| = |x^2 - y^2| = |x+y||x-y| \leq 2|x-y|.$$

Taking $\delta = \varepsilon/2$, so

$$|x-y| \leq \delta \Longrightarrow 2|x-y| \leq \varepsilon \Longrightarrow |f(x) - f(y)| \leq \varepsilon.$$

2. $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous. Indeed, for $\varepsilon = 2$, taking $x_n = n + 1/n$, $y_n = n$. Then $\forall \delta > 0$, there exists $n \in \mathbb{N}^*$ such that $|x_n - y_n| = 1/n \leq \delta$ and

$$|f(x_n) - f(y_n)| = |(n + 1/n)^2 - n^2| = 2 + 1/n^2 \geq 2 = \varepsilon.$$

3. $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ defined by $f(x) = 1/x$ is not uniformly continuous. Indeed, for $\varepsilon = 1$, taking $x_n = 1/n$, $y_n = \frac{1}{n+1}$. Then $\forall \delta > 0$, there exists $n \in \mathbb{N}^*$ such that $|x_n - y_n| \leq 1/n \leq \delta$ and

$$|f(x_n) - f(y_n)| = |(n+1) - n| = 1 \geq 1 = \varepsilon.$$

Proposition **4.5.** Every uniformly continuous function is continuous

Proof. Let $f : I \longrightarrow \mathbb{R}$ be uniformly continuous function. Given any $x_0 \in I$, then □

Theorem **4.6.** Let $f :]a, b[$ be a continuous function such that $\lim_{x \to a+} f(x)$, $\lim_{x \to b-} f(x)$ exist and finite. Then f is uniformly continuous.

Proof. Let $\varepsilon > 0$ be given. We need to show that there exists a $\delta > 0$ such that for all $x, y \in]a, b[$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

Since $\lim_{x \to a+} f(x)$ exists, there exists a $\delta_1 > 0$ such that if $a < x < x + \delta_1 < b$, then $|f(x + \delta_1) - f(x)| < \varepsilon/2$. Similarly, since $\lim_{x \to b-} f(x)$ exists, there exists a $\delta_2 > 0$ such that if $a < x - \delta_2 < x < b$, then $|f(x) - f(x - \delta_2)| < \varepsilon/2$.

Now, choose $\delta = \min(\delta_1, \delta_2)$. Let $x, y \in]a, b[$ such that $|x - y| < \delta$. Without loss of generality, assume $x < y$. Then, we have $|x - (x - \delta_2)| = \delta_2$, and $|(x + \delta_1) - y| = \delta_1$. Therefore, by the triangle inequality, we get

$$|f(x) - f(y)| \leq |f(x) - f(x - \delta_2)| + |f(x + \delta_1) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, for any $x, y \in]a, b[$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$, which shows that f is uniformly continuous. □

4.5 Differentiable functions

I is an open interval, $x_0 \in I$, $f : I \longrightarrow \mathbb{R}$ is a function well defined at all points of I

Definition 4.6. 1. We say that f is differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists and finite.}$$

This limit is denoted by $f'(x_0)$ and called derivative of f at x_0. Thus

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}.$$

If f is differentiable at all point of I, we say f is differentiable.

2. We say that f is left-differentiable at x_0 if the left limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists and finite.}$$

This limit is denoted by $f'(x_0^-)$ and called left-derivative of f at x_0.

3. We say that f is right-differentiable at x_0 if the right limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists and finite.}$$

This limit is denoted by $f'(x_0^+)$ and called right-derivative of f at x_0.

☞ **Remark 4.4.** • It is sometimes convenient to let $x = x_0 + h$ and the above limit becomes

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

• It is easy to see that f is differentiable at x_0 if and only if it is left and right differentiable at x_0 and $f'(x_0^+) = f'(x_0^-)$.

☞ **Example 4.4.** Study the differentiability of the following functions

1. $f(x) = C$, $C \in \mathbb{R}$. Given $x_0 \in \mathbb{R}$. We have

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{C-C}{x-x_0} = 0$$

Thus f is differentiable and $f'(x_0) = 0$.

2. $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$. Given $x_0 \in \mathbb{R}$. We have

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{x^2-x_0^2}{x-x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0$$

Thus f is differentiable and $f'(x_0) = 2x_0$.

3. $f(x) = x^n$, $n \in \mathbb{N}^*$. Given $x_0 \in \mathbb{R}$. We have

$$\begin{aligned}
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &:= \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} \\
&= \lim_{x \to x_0} \frac{(x - x_0) \sum_{k=0}^{n-1} x^{n-1-k} x_0^k}{x - x_0} \\
&= \lim_{x \to x_0} \left(\sum_{k=0}^{n-1} x^{n-1-k} x_0^k \right) \\
&= n x_0^{n-1}
\end{aligned}$$

Thus, f is differentiable at x_0 and $f'(x_0) = n x_0^{n-1}$. Since this holds for every $x_0 \in \mathbb{R}$, then f is differentiable and $f'(x) = n^{n-1}$.

4. $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x}$.

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{1/(x+h) - 1/x}{h} \\
&= \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.
\end{aligned}$$

Thus f is differentiable and $f'(x) = -\frac{1}{x^2}$.

5. $f :]0, +\infty[$ defined by $f(x) = \sqrt{x}$.

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h}) + \sqrt{x}} = \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Thus f is differentiable and $f'(x) = \frac{1}{2\sqrt{x}}$.

6. $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = |x|$.

   - If $x > 0$ then given h such that $-x < h < x$. Then

   $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h} = 1.$$

   Hence f is differentiable at x and $f'(x) = 1$.

- If $x < 0$ then given h such that $-x < h < x$. Then
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{|x+h|-|x|}{h} = \lim_{h\to 0} \frac{-(x+h)+x}{h} = -1.$$
Hence f is differentiable at x and $f'(x) = -1$.
- If $x = 0$, then, we have
$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = +1.$$
and
$$\lim_{h\to 0^-} \frac{k(0+h)-k(0)}{h} = \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{-h}{h} = -1.$$
Therefore, the limit of difference quotient does not exist. It follows that f is not differentiable at 0.

Proposition 4.7. If f is differentiable at x_0, then it is continuous at x_0.

Proof. We have
$$|f(x)-f(x_0)| = \left|\frac{f(x)-f(x_0)}{x-x_0}\right| |x-x_0|$$
passing to the limit as $x \to x_0$, taking into account that f is differentiable at x_0, we obtain $\lim_{x\to x_0} |f(x)-f(x_0)| = 0$ which means that f is continuous at x_0. □

Theorem 4.8. Let f, g be a differentiable functions at x_0 then so are λf, $f+g$, fg and f/g if $g(x_0) \neq 0$.

1. $(\lambda f)'(x_0) = \lambda f'(x_0)$
2. $(f+g)'(x_0) = f'(x_0) + g'(x_0)$
3. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
4. If $g(x_0) \neq 0$ then $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g(x_0)^2}$. In particular, we have
$$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}$$

Proof. We will prove each part separately.

1. Let λ be a constant. By the definition of the derivative, we have

$$(\lambda f)'(x_0) = \lim_{h\to 0} \frac{\lambda f(x_0+h) - \lambda f(x_0)}{h}.$$

Using the linearity of the limit, we can factor out λ and obtain

$$(\lambda f)'(x_0) = \lambda \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lambda f'(x_0).$$

2. The derivative of the sum of two functions is the sum of their derivatives:

$$(f+g)'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) + g(x_0+h) - f(x_0) - g(x_0)}{h}.$$

Using the linearity of the limit, we can separate the limit into two parts and apply the definition of the derivatives of f and g:

$$(f+g)'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} + \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} = f'(x_0) + g'(x_0).$$

3. For the product rule, we consider

$$(fg)'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h}.$$

We can rewrite the above expression as

$$(fg)'(x_0) = \lim_{h\to 0} \left(f(x_0+h)\frac{g(x_0+h) - g(x_0)}{h} + g(x_0)\frac{f(x_0+h) - f(x_0)}{h} \right).$$

Applying the definition of derivatives and continuity, we get

$$(fg)'(x_0) = f(x_0)g'(x_0) + g(x_0)f'(x_0).$$

4. Finally, for the quotient rule, we have

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x_0) &= \lim_{h\to 0} \frac{\frac{f(x_0+h)}{g(x_0+h)} - \frac{f(x_0)}{g(x_0)}}{h} = \lim_{h\to 0} \frac{f(x_0+h)g(x_0) - f(x_0)g(x_0+h)}{hg(x_0+h)g(x_0)} \\
&= \lim_{h\to 0} \frac{\frac{f(x_0+h)-f(x_0)}{h} g(x_0) - f(x_0+h)\frac{g(x_0+h)-g(x_0)}{h}}{g(x_0+h)g(x_0)} \\
&= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}
\end{aligned}$$

This completes the proof. □

Theorem 4.9. Let I, J be two open intervals, $x_0 \in I$ and $f : I \longrightarrow J$, $g : J \longrightarrow \mathbb{R}$ be two functions such that $f(x_0) \in J$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$ then $g \circ f$ is differentiable at x_0 and we have

$$\boxed{(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)}.$$

Proof. Since f is differentiable at x_0, by definition, there exists a derivative $f'(x_0)$ given by

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Similarly, since g is differentiable at $y_0 = f(x_0)$, there exists a derivative $g'(f(x_0))$ given by

$$g'(f(x_0)) = \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0}.$$

Now consider the composition of the two functions $g \circ f : I \to \mathbb{R}$. The derivative of this composition at x_0 is given by

$$(g \circ f)'(x_0) = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}.$$

We set $y = f(x)$ which go to $y_0 = f(x_0)$ as $x \to x_0$ since f is continuous. Then, we have

$$\begin{aligned}
(g \circ f)'(x_0) &= \lim_{x \to x_0} \frac{g(y) - g(y_0)}{y - y_0} \frac{y - y_0}{x - x_0} \\
&= \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
&= g'(y_0)f'(x_0) = g'(f(x_0))f'(x_0),
\end{aligned}$$

which completes the proof. □

☞ **Example 4.5.** 1. $f(x) = \sqrt{x^2 + 1}$, calculate $f'(x)$.

$$\boxed{f'(x) = 2x\frac{1}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}}$$

2. $g(t) = f(x)$, $x = e^t$. Calculate $g'(t)$. We have

$$\boxed{g'(t) = (f(e^t))' = e^t f'(e^t) = xf'(x)}$$

4.6 Mean value theorem

$[a, b]$ is a closed bounded interval with $a < b$.

Lemme 4.10. Let $f :]a, b[\longrightarrow \mathbb{R}$ be a differentiable function. Suppose that f has an extreme value at a $c \in]a, b[$. Then $f'(c) = 0$

Proof. Let $f :]a, b[\to \mathbb{R}$ be a differentiable function, and suppose that f has an extreme value at $c \in]a, b[$. We aim to show that $f'(c) = 0$. Since f has an extreme value at c, it means that either $f(c)$ is a maximum or a minimum value. Without loss of generality, let's consider the case where $f(c)$ is a maximum. By the definition of a maximum, for any $x \in]a, b[$, we have $f(x) \leq f(c)$. This implies that the difference quotient

$$\boxed{\frac{f(x) - f(c)}{x - c} \geq 0, \ \forall x < c}, \text{ and } \boxed{\frac{f(x) - f(c)}{x - c} \leq 0, \ \forall x > c}.$$

Then, taking the limit as x approaches c, we have

$$\boxed{\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0}, \quad \text{and } \boxed{\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0}.$$

By the differentiability of f at c, those limits can be expressed as the derivative of f at c:

$$f'(c) \leq 0 \text{ and } f'(c) \geq 0$$

which implies $f'(c) = 0$.

□

Theorem 4.11 (Rolle's theorem). Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and differentiable on $]a, b[$ such that $f(a) = f(b)$. Then

$$\boxed{\exists c \in]a, b[: f'(c) = 0}$$

☞ **Remark 4.5.** It is absolutely necessary to suppose f differentiable at all points of $]a, b[$. Consider the function $f(x) = |x|$ on $[-1, 1]$. Clearly $f(-1) = f(1)$, but there is no point c where $f'(c) = 0$.

Proof. By the Weierstrass extreme value theorem 4.4 f attains its global maximum and minimum values on $[a, b]$. If these are both attained at the endpoints, then f is constant, and $f'((c) = 0$ for all points $c \in]a, b[$. Otherwise, f attains at least one of its global maximum or minimum values at an interior point $c \in]a, b[$. Lemma 4.10 implies that $f'(c) = 0$. □

We extend Rolle's theorem to functions that attain different values at the endpoints.

Theorem 4.12 (Mean value theorem). Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function differentiable on $]a, b[$. Then there exists a point $c \in]a, b[$ such that

$$\boxed{f(b) - f(a) = (b - a)f'(c).}$$

☞ **Remark 4.6.** Graphically, this result says that there is $c \in]a, b[$ such that the slope of the tangent line at the point $(c, f(c))$ is equal to the slope of the chord between the endpoints $(a, f(a))$ and $(b, f(b))$.

Proof. Apply Rolle's theorem 4.11 to the function

$$g(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}\right](x-a).$$

□

Theorem **4.13.** Let $f :]a, b[\longrightarrow \mathbb{R}$ be a differentiable function such that $f'(x) = 0$ for all $x \in]a, b[$. Then f is constant.

Proof. f is constant if $f(x) = f(y)$, $\forall x, y \in]a, b[$. Take arbitrary $x, y \in]a, b[$ with $x < y$. As $]a, b[$ is an interval, $[x, y] \subset]a, b[$. Then f restricted to $[x, y]$ satisfies the hypotheses of the mean value theorem 4.12 . Therefore, there is a $c \in]x, y[$ such that

$$f(x) - f(y) = (x - y)f'(c).$$

Since $f'(c) = 0$, we have $f(x) = f(y)$. Hence, f is constant. □

Proposition **4.14.** Let $f :]a, b[\longrightarrow \mathbb{R}$ be a differentiable function. Then

- f is increasing if and only if $f'x) \geq 0$ for all $x \in]a, b[$.
- f is decreasing if and only if $f'x) \leq 0$ for all $x \in]a, b[$.

Proof. Let us denote that f is increasing (resp. decreasing) if and only if $\frac{f(x)-f(y)}{x-y} \geq 0$, (resp. ≤ 0), $\forall x \neq y$.

Let us prove the first item. Suppose f is increasing. For all $x, c \in]a, b[$with $x \neq c$,

$$\frac{f(x)-f(c)}{x-c} \geq 0$$

Taking a limit as x goes to c, we see that $f'(c) \geq 0$. For the other direction, suppose $f'(c) \geq 0$ for all $c \in]a, b[$. Take any $x, y \in]a, b[$ with $x < y$, and note that $[x, y] \subset]a, b[$. By the mean value theorem 4.12, there is some $c \in]x, y[$ such that

$$f(x) - f(y) = (x - y)f'(c).$$

Hence

$$\frac{f(x)-f(y)}{x-y} = f'(c) \geq 0$$

and so f is increasing. We leave the second item to the reader as exercise. □

4.7 Exercises

Exercise 37. Find the domain of definition of the following functions

$$f(x)=\sqrt{x^2+3x-4},\quad g(x)=\ln(x^2+3x-4),\quad h(x)=\frac{\ln(x+1)}{\sqrt{1-x^2}},\quad k(x)=\frac{1}{[x]-2022}.$$

Exercise 38. Calculate the following limits

$$\lim_{x\to+\infty}\left(\sqrt[3]{x^3+1}-x\right),\quad \lim_{x\to1}\frac{\sqrt{x}-1}{\sqrt{x+8}-3},\quad \lim_{x\to1}\frac{\sqrt[4]{x}-1}{\sqrt{x}-1},\quad \lim_{x\to1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$$

$$\lim_{x\to+\infty}\frac{e^x}{x^6},\ \lim_{x\to+\infty}\frac{\ln(1+x^2)}{x},\quad \lim_{x\to0}\frac{[|x|]}{x^{10}},\ \lim_{x\to+\infty}\frac{1-\cos x}{x^2},\quad \lim_{x\to\pi}\frac{\sin x}{x-\pi}.$$

Exercise 39. 1. Using the definition of the derivative, calculate the following limits

$$\lim_{x\to0}\frac{\ln(1+x)}{x},\quad \lim_{x\to0}\frac{e^x-1}{x}$$

2. Deduce the following limits

$$\lim_{x\to+\infty}\left(1+\frac{k}{x}\right)^x,\ k\in\mathbb{R},\quad \lim_{x\to0}\frac{a^x-b^x}{x},\ a,b>0.$$

Exercise 40. 1. Show that

$$\forall x,y\geq0:|\sqrt{x}-\sqrt{y}|\leq\sqrt{|x-y|}$$

2. Deduce that the function $x\mapsto\sqrt{x}$ is uniformly continuous on $\mathbb{R}_+$.

3. Show that the function $x\mapsto\frac{1}{x}$ is not uniformly continuous on $(0,\infty)$ (Choose $x=\frac{1}{n},\ y=\frac{1}{2n}$).

Exercise 41. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be the function defined by

$$f(x)=\begin{cases}x^3+\frac{a}{x^2} & \text{if}\quad x\neq0\\ 0 & \text{if}\quad x=0\end{cases}$$

1. Calculate $\lim_{x\to0}f(x)$
2. Deduce the value of a for which f is continuous.

Exercise 42. Study the continuity of the function defined on $\mathbb{R}$ by $f(x)=[x]$ (consider the two cases: $x\in\mathbb{Z}$ and $x\notin\mathbb{Z}$).

Exercise 43. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a continuous function such that $f(x)=0$ for all $x\in\mathbb{Q}$. Show that $f(x)=0$ for all $x\in\mathbb{R}$.

Exercise 44. 1. Let $f:[0,1]\longrightarrow[0,1]$ be a continuous function. Show that f has a fixed point.

2. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a continuous and decreasing function. Show that f has a unique fixed point.

Exercise 45. 1. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a continuous and periodic function such that $\lim_{x\to+\infty} f(x)$ exists. Show that f is constant.

2. Deduce that $x\mapsto\sin x$ and $x\mapsto\cos x$ do not have limits at $+\infty$ and $-\infty$.

Exercise 46. Calculate the derivatives of the following functions: $\sqrt{\frac{1+x^2}{x-1}}$, $\ln(1+\cos(x^2-x+1))$

Exercise 47. 1. Using the definition of the derivative, calculate the following limits

$$\lim_{x\to 0}\frac{\ln(1+x)}{x},\quad \lim_{x\to 0}\frac{e^x-1}{x}$$

2. Deduce the following limits

$$\lim_{x\to+\infty}\left(1+\frac{k}{x}\right)^x,\ k\in\mathbb{R},\quad \lim_{x\to 0}\frac{a^x-b^x}{x},\ a,b>0.$$

Exercise 48. Let f be the function defined on $\mathbb{R}^*$ by $f(x)=x^2\sin\frac{1}{x^2}$.

1. Show that f can be extended to be continuous on $\mathbb{R}$ and give its extension $\tilde{f}$.
2. Study the differentiability of $\tilde{f}$ and calculate its derivative $\tilde{f}'$
3. Is $\tilde{f}$ of class $\mathcal{C}^1(\mathbb{R})$?

Exercise 49. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a function such that

$$\forall x,y\in\mathbb{R}: |f(x)-f(y)|\leq |x-y|^2.$$

1. Show that f is differentiable and calculate its derivative.
2. Deduce the value of f.

Exercise 50. Show the following inequalities

$$\forall x>-1: \frac{x}{1+x}\leq\ln(1+x)\leq x,$$

$$\forall x\in]0,1[: 1+x\leq e^x\leq\frac{1}{1-x}$$

(Apply the Mean Value Theorem to the functions: $e^x-x-1,\ (1-x)e^x-1$).

Exercise 51. Calculate the nth-order derivatives for $n\in\mathbb{N}$ of the following functions

$$(x^2+x+1)e^x,\quad \frac{e^x}{1-x},\quad \frac{e^{-x}}{1+x}.$$