5 Usual function

5.1 Definition of $\arcsin$ and a$\arccos$ Functions

Definition 5.1. 1. The arcsine function, denoted as $\arcsin(x)$ or $\sin^{-1}(x)$, is the inverse of the sine function $\boxed{\sin : [-\pi/2, \pi/2] \longrightarrow [-1, 1]}$. In other words, $\arcsin : [-1, 1] \longrightarrow [-\pi/2, \pi/2]$ such that $\forall x \in [-1, 1]$, we have

$$\arcsin(x) = \theta \quad \text{where} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad \text{and} \quad \sin(\theta) = x.$$

2. The arccos function, denoted as $\arccos(x)$ or $\cos^{-1}(x)$, is the inverse of the cosine function $\boxed{\cos : [0, \pi] \longrightarrow [-1, 1]}$. In other words, $\arccos : [-1, 1] \longrightarrow [0, \pi]$ such that $\forall x \in [-1, 1]$, we have

$$\arccos(x) = \theta \quad \text{where} \quad 0 \leq \theta \leq \pi \quad \text{and} \quad \cos\theta = x.$$

Proposition 5.1. (**Properties of the Arcsine and Arccos Function**).

1. $\boxed{\arcsin(-x) = -\arcsin(x),\ \arccos(-x) = \pi - \arccos x}$ $\forall x \in [-1, 1]$.

2. Derivative:

$$\boxed{\forall x \in]-1, 1[: \arcsin' x = \frac{1}{\sqrt{1-x^2}}, \quad \arccos' x = -\frac{1}{\sqrt{1-x^2}}}$$

3. Inverse of sine and cosine:

$$\boxed{\arcsin(\sin\theta) = \theta, \quad \text{for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}}$$

$$\boxed{\arccos(\cos\theta) = \theta, \quad \text{for } 0 \leq \theta \leq \pi}$$

5.2 Definition of hyperbolic functions ch and ash

The hyperbolic sine and hyperbolic cosine functions are defined on $\mathbb{R}$ as follows:

$$\text{sh}(x) = \frac{e^x - e^{-x}}{2}$$
$$\text{ch}(x) = \frac{e^x + e^{-x}}{2}$$

Let us denote that those functions are differentiable and we have

$$\text{sh}'(x) = \frac{e^x - +e^{-x}}{2} = \text{ch}x$$
$$\text{ch}'(x) = \frac{e^x - e^{-x}}{2} = \text{sh}x$$

Definition 5.2. 1. The arcsine function, denoted as $\text{argsh}(x)$ or $\text{sh}^{-1}(x)$, is the inverse of the sine function $\boxed{\text{sh} : \mathbb{R} \longrightarrow \mathbb{R}}$. In other words, $\text{argsh} : [-1, 1] \longrightarrow \mathbb{R}$ such that $\forall x \in [-1, 1]$, we have

$$\text{argsh}x = y \quad \text{where} \quad \text{sh}y = x.$$

2. The arccos function, denoted as $\text{argch}(x)$ or $\text{ch}^{-1}(x)$, is the inverse of the cosine function $\boxed{\text{ch} : [0, +\infty] \longrightarrow [1, +\infty[}$. In other words, $\text{argch} : [1, +\infty[\longrightarrow [0, +\infty]$ such that $\forall x \in [-1, 1]$, we have

$$\text{argch}(x) = y \quad \text{where} \quad \text{ch}y = x.$$

Exercise 52. Calculate $\text{argch}0,\ \text{argsh}0,\ \text{argch}1,\ args1$

Proposition 5.2 (Derivative). The functions argch and argsh are differentiable and we have

$$\boxed{\forall x \in \mathbb{R} : \text{argsh}'x = \frac{1}{\sqrt{x^2+1}}}$$

$$\boxed{\forall x \in]1, +\infty[: \text{argch}'x = \frac{1}{\sqrt{x^2-1}}}$$

5.3 Exercises

Exercise 53. Show that for all $x \in [-1, 1]$, we have

$$\boxed{\sin(\arccos x) = \sqrt{1 - x^2} = \cos(\arcsin x)}$$

Exercise 54. Let $f : D \to [-1, 1]$ be the function defined by $f(x) = \sin x$ where $D = [\frac{\pi}{2}, \frac{3\pi}{2}]$.

1. Verify that f is bijective and determine its inverse f^{-1} in terms of arcsin.
2. Same question for $f(x) = \cos x$ and $D = [2022\pi,\ 2023\pi]$.

Exercise 55. 1. Calculate $\arcsin(\sin\frac{\pi}{3}),\ \arccos\cos(\frac{\pi}{3}),\ \arccos(\sin\frac{\pi}{3})$.

2. Calculate $\arccos(\cos\frac{4\pi}{3}),\ \arccos(\cos\frac{7\pi}{3}),\ \arcsin(\sin\frac{2\pi}{3}),\ \arcsin(\sin\frac{7\pi}{3})$.

Exercise 56. 1. Show that $\arctan a + \arctan b = \arctan\frac{a+b}{1-ab}$, with $ab < 1$

2. Calculate $\arctan(1/2) + \arctan(1/3)$

Exercise 57. 1. Calculate

$$\boxed{C = \sum_{k=0}^{n} \text{ch}(kx), \quad S = \sum_{k=0}^{n} \text{sh}(kx)}$$

2. Linearize $\text{sh}x.\text{ch}(2x),\ \text{ch}x.\text{ch}^2x$
3. Verify that $\text{sh}(2x) = 2\text{sh}x\text{ch}x$ and then calculate

$$P = \text{ch}x.\text{ch}(\frac{x}{2}).\text{ch}(\frac{x}{2^2}).......\text{ch}(\frac{x}{2^n}).$$

Exercise 58. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \text{argch}\sqrt{1+x^2}$.

1. Determine the domain of definition of f.
2. Calculate $\text{argch}(\text{ch}t)$, for all $t \in \mathbb{R}$
3. Show that $\forall x \in \mathbb{R} : f(x) = \text{argsh}|x|$.
4. Calculate $f'(x)$, for all $x \in \mathbb{R}^*$.
5. Is f differentiable at 0?.

Exercise 59. (**Assignment**)
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function defined by

$$f(x) = \begin{cases} \arctan \frac{1}{x^2} & \text{if} \quad x \neq 0 \\ \ell & \text{if} \quad x = 0 \end{cases}$$

1. Determine ℓ.
2. Show that f is differentiable on $\mathbb{R}^*$ and calculate f'.
3. Show that f is differentiable at 0 and calculate $f'(0)$ (Apply MVT between 0 and x).
4. Deduce that f is $\mathcal{C}^\infty$.
5. Calculate g' where g is the function defined on $\mathbb{R}$ by $g(x) = \arctan x^2$.
6. Calculate $\arctan x^2 + \arctan \frac{1}{x^2}$, $\forall x \in \mathbb{R}^*$ and deduce $\arctan x + \arctan \frac{1}{x}$, $\forall x \in \mathbb{R}^*$.
7. Show that $g : [0, +\infty[\to [0, \pi/2[$ is bijective and calculate g^{-1}.
8. Calculate $(g^{-1})'$ in two ways.

Reminder

$$\cos(a+b) = \cos a \cos b - \sin a \sin b, \quad \cos(a-b) = \cos a \cos b + \sin a \sin b$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b, \quad \sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$\cos(2x) = \cos^2 x - \sin^2 x, \quad \sin(2x) = 2\sin x \cos x$$

$$\arccos : [-1,1] \to [0,\pi], \quad \arcsin : [-1,1] \to [-\pi/2, \pi/2]$$

$$\arccos' x = \frac{-1}{\sqrt{1-x^2}}, \quad \arcsin' x = \frac{1}{\sqrt{1-x^2}}$$

$$\arctan : \mathbb{R} \to]-\pi/2, \pi/2[, \quad \arctan' x = \frac{1}{1+x^2}$$

$$\mathrm{ch}x = \frac{e^x+e^{-x}}{2}, \ \mathrm{sh}x = \frac{e^x-e^{-1}}{2}, \quad \mathrm{ch}x + \mathrm{sh}x = e^x$$

$$\mathrm{ch}'x = \mathrm{sh}x, \quad \mathrm{sh}'x = \mathrm{ch}x, \quad \mathrm{ch}^2x - \mathrm{sh}^2x = 1$$

$$\mathrm{ch}(a+b) = \mathrm{ch}a\mathrm{ch}b + \mathrm{sh}a\mathrm{sh}b, \quad \mathrm{ch}(a-b) = \mathrm{ch}a\mathrm{ch}b - \mathrm{sh}a\mathrm{sh}b$$

$$\mathrm{sh}(a+b) = \mathrm{sh}a\mathrm{ch}b + \mathrm{ch}a\mathrm{sh}b, \quad \mathrm{sh}(a-b) = \mathrm{sh}a\mathrm{ch}b - \mathrm{ch}a\mathrm{sh}b$$

$$\mathrm{ch}(2x) = \mathrm{ch}^2x + \mathrm{sh}^2x, \ \ \mathrm{sh}(2x) = 2\mathrm{sh}x\mathrm{ch}x$$

$$\mathrm{argch} : [1, +\infty[\to [0, +\infty], \quad \mathrm{argsh} : \mathbb{R} \to \mathbb{R}$$

$$\mathrm{argch}'x = \frac{1}{\sqrt{x^2-1}}, \quad \mathrm{argsh}'x = \frac{1}{\sqrt{1+x^2}}$$

$$\mathrm{argth} : [-1, +1] \to \mathbb{R}, \quad \mathrm{argth}'x = \frac{1}{x^2-1}$$