Chapter N°3 : Limits, Continuous Functions and Derivatives

0.1 Concept of function

To specify a function f you must

1. give a rule which tells you how to compute the value $f(x)$ of the function for a given real number x, and,
2. say for which real numbers x the rule may be applied.

The set of numbers for which a function is defined is called its **domain**. The set of all possible numbers $f(x)$ as x runs over the domain is called the **range or image** of the function.

Definition 1 *A real function f is function from a subset D of $\mathbb{R}$ to $\mathbb{R}$*

For example : definition domain of the function $x \mapsto \dfrac{2}{\sqrt{2-x}}$, is $D =]-\infty, 2[$.

Definition 2 *(Arithmetic and function) Let f and g two functions defined on $D \subset \mathbb{R}$*

1. ***the sum*** *: $f+g : D \to \mathbb{R},\ (f+g)(x) = f(x)+g(x)$,*
2. ***the product*** *: $f.g : D \to \mathbb{R},\ (f.g)(x) = f(x).g(x)$,*
3. ***the multiplies*** *: $\lambda.g : D \to \mathbb{R},\ (\lambda.f)(x) = \lambda.f(x)$,*

Definition 3 *Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ tow functions. Then :*
— $f \geq g$ *si* $\forall x \in D \quad f(x) \geq g(x)$;
— $f \geq 0$ *si* $\forall x \in D \quad f(x) \geq 0$;
— $f > 0$ *si* $\forall x \in D \quad f(x) > 0$;
— f *constant on* D *if* $\exists\, a \in \mathbb{R}\, \forall x \in D \quad f(x) = a$;
— f *is zero on* D *if* $\forall\, x \in D \quad f(x) = 0$;

Definition 4 *Let $f : D \to \mathbb{R}$ is function. We say that :*
— f *is bounded from above over* D *if* $\exists\, M \in \mathbb{R}\, \forall x \in D \quad f(x) \leq M$;
— f *is bounded from below* D *if* $\exists\, m \in \mathbb{R}\, \forall x \in D \quad f(x) \geq m$;
— f *is bounded over* D *lf* $\exists\, M \in \mathbb{R}\, \forall x \in D \quad |f(x)| \leq M$.

Definition 5 *Let $f : D \to \mathbb{R}$ is function. We say that :*
— f *is increasing on* D *if* $\forall x, y \in D,\ x \leq y \Rightarrow f(x) \leq f(y)$;
— f *is strictly increasing on* D *if* $\forall x, y \in D,\ x < y \Rightarrow f(x) < f(y)$;
— f *is decreasing on* D *si* $\forall x, y \in D,\ x \leq y \Rightarrow f(x) \geq f(y)$;
— f *is strictly decreasing on* D *if* $\forall x, y \in D,\ x < y \Rightarrow f(x) > f(y)$;
— f *is monotone (resp. strictly monotone) on* D *if* f *is increasing or decreasing (resp. strictly increasing or strictly decreasing) on* D.

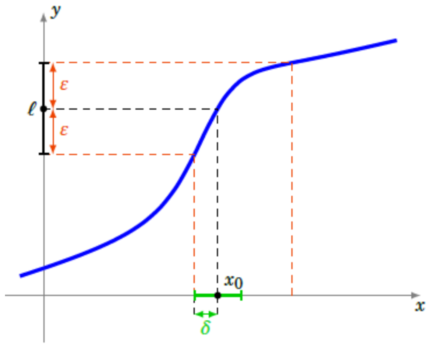
Definition 6 *Let $f : \mathbb{R} \to \mathbb{R}$ is a function, and T is a real number, $T > 0$. The function f is periodic with period T if $\forall x \mathbb{R}, \quad f(x+T) = f(x)$.*

0.2 Limit and Continuity of Real Functions

Definition 7 *(limit) Let $f : I \to \mathbb{R}$ is a real function. Let $x_0 \in \mathbb{R}$ is point of I*
Let $l \in \mathbb{R}$. We sa that f for a limit l at x_0 if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in I\ |x - x_0| < \delta \Rightarrow |f(x) - l| < \epsilon$$

it is also said that $f(x)$ tends to l as x tends x_0. And $\lim\limits_{x \to x_0} f(x) = l$.



0.2.1 Limits and inequalities.

- If there is some neighborhood of a such that $f(x) \leq g(x)$, and the limits of f and g exist at a, then
$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$
- If the limits of f and g exist at a, and
$$\lim_{x \to a} f(x) < \lim_{x \to a} g(x),$$
then in some neighborhood of a $f(x) < g(x)$.
- **Squeeze theorem :** If in some neighborhood of a $f(x) < g(x) < h(x)$, and the limits of f and h exist at a, and
$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x),$$
then the limit of g also exists at a, and
$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x)$$
- 0 **times bounded is** 0. If $\lim_{x \to a} f(x) = 0$, and $g(x)$ is bounded, then
$$\lim_{x \to a} f(x)g(x) = 0.$$

Definition 8 *The function f is continuous at a if and only if there exists the limit of the function at a and the limit is $f(a)$. Or*
$$\lim_{x \to a} f(x) = f(a)$$

In logical symbolism,
$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in D_f : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

For example :

$\lim_{x \to 4} \frac{1}{x} = \frac{1}{4}$ by definition we can choose $\delta = 12\epsilon$ so that f continuous at 4.

f is continuous on D_f, if its continuous at every point in D_f.

If f is *not continuous* at a we say that f is *discontinuous* at a or that a is a point of discontinuity of f.

Remark 9
- *Polynomials function is continuous on* $\mathbb{R}$
- *Constant function is continuous on the interval.*
- *Functions* $\cos, \sin$ *are continuous on* $\mathbb{R}$
- $\ln$ *is continuous on* $]0, +\infty[$
- *Exponential function is continuous on* $\mathbb{R}$

However the integer part function is not continuous at every point $a \in \mathbb{Z}$ because does not limit at these points. But is $a \in \mathbb{R}/\mathbb{Z}$ is continuous at a.

Corollary 10 — *A function f is continuous from right at a if and only if there exists its* ***right-hand side limit*** *at a and it is $f(a)$.*
— *A function f is continuous from left at a if and only if there exists its* ***left-hand side limit*** *at a and it is $f(a)$.*

For example : **signum** function is not continuous at 0

$$\lim_{x\to 0^+} sgn(x) = 1, \quad \lim_{x\to 0^+} sgn(x) = -1.$$

Definition 11 *A function f is* ***uniformly continuous*** *on a set D if for every $\epsilon > 0$, there exists $\delta > 0$, such that $|f(x_1) - f(x_2)| < \epsilon$ for all $x_1, x_2 \in D_f$, and $|x_1 - x_2| < \delta$.*

More briefly

$$\forall \epsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in D_f : |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon$$

For example : $f : x \mapsto x^2$ uniformly continuous on $[0, 1]$.

0.2.2 Continuous functions in interval.

Theorem 12 ***Weierstrass theorem*** *: If a function is continuous in a bounded and closed interval, then the function has* ***maximum*** *and* ***minimum*** *value.*

Theorem 13 ***Intermediate value theorem*** *: If the function $f(x)$ is continuous in the bounded and closed $[a, b]$ interval, then every value y between $f(a)$ and $f(b)$ is attained c in $[a, b]$, such that $y = f(c)$*

In logical symbolism this theorem has the following expression :

$$f \in \mathcal{C}([a, b]), \text{ and } f(a).f(b) < 0 \Rightarrow \exists c \in]a, b[, \text{ such that } f(c) = 0.$$

Exercise 1 : Let $f : [0, 1] \to [0, 1]$ continuous function. Prove that f has at least fixed point.

Theorem 14 ***Inverse of a continuous function*** *: If a function is continuous and invertible in a bounded and closed interval, then the range of the function is a closed interval, and in this interval the inverse function is continuous.*

0.3 Differential Calculus

Definition 15 *Let f be a function which is defined on some interval I and a be some number in the interval. The derivative of the function f at a is the value of the limit*

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

— f is said to be differentiable at a if this limit exists.
— f is called differentiable on the interval I, if it is differentiable at ever point a in I

Other notation : one can substitute $x - a = h$ then

$$\lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = f'(a)$$

The function f has a tangent line at point a if and only if f is differentiable at a. The equation of the tangent line is

$$y = f'(a)(x - a) + f(a)$$

If $f(x)$ is differentiable at a, then the function is continuous at a. The converse of the theorem is not true :
for example, $f(x) = |x|$ is continuous at 0, but not differentiable at 0!

0.3.1 Derivative rules.

If f and g are differentiable at a, then

— for any $c \in \mathbb{R}$ f is differentiable at a, and

$$(c.f)'(a) = cf'(a)$$

— $f + g$ is differentiable at a, and

$$(f + g)'(a) = f'(a) + g'(a).$$

— $f.g$ is differentiable at a, and

$$(f.g)'(a) = f'(a).g(a) + f(a).g'(a)$$

— If $g(a) \neq 0$, then $\frac{f}{g}$ differentiable at a and

$$(\frac{f}{g})' = \frac{f'(a).g(a) + f(a).g'(a)}{g^2(x)}$$

Definition 16 *Chain rule. If g is differentiable at a, and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a, and*

$$(f \circ g)'(a) = g'(a)f'(g(a)).$$

Definition 17 *Derivative of the inverse function. If f is continuous and has an inverse in a neighbourhood of the point a, and it is differentiable at a, and $f'(a) \neq 0$, then f^{-1} is differentiable at $f(a)$, and*

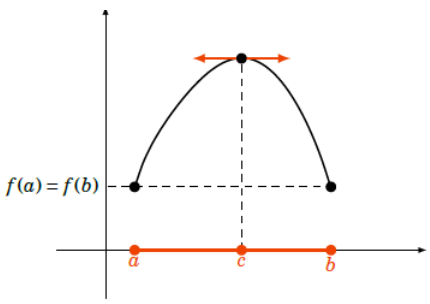
$$(f^{-1})'(f(a)) = \frac{1}{f(a)}.$$

0.3.1.1 Derivatives of usual functions

The table on the left is a summary of the main formulas to know, x is a variable. The table in the right is that of the compositions, $u : x \mapsto u(x)$

Function	Derivative	Function	Derivative
x^n	$nx^{n-1} \quad n \in \mathbb{Z}$	u^n	$nu'u^{n-1} \quad n \in \mathbb{Z}$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\frac{1}{u}$	$-\frac{u'}{u^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\sqrt{u}$	$\frac{u'}{2\sqrt{u}}$
x^α	$\alpha x^{\alpha-1} \quad \alpha \in \mathbb{R}$	u^α	$\alpha u'u^{\alpha-1} \quad \alpha \in \mathbb{R}$
e^x	e^x	e^u	$u'e^u$
$\ln x$	$\frac{1}{x}$	$\ln u$	$-\frac{u'}{u}$
$\cos x$	$-\sin x$	$\cos u$	$-u' \sin u$
$\sin x$	$\cos x$	$\sin u$	$u' \cos u$
$\tan x$	$\frac{1}{\cos^2 x} = 1 + \tan^2 x$	$\tan u$	$\frac{u'}{\cos^2 u} = u'(1 + \tan^2 u)$

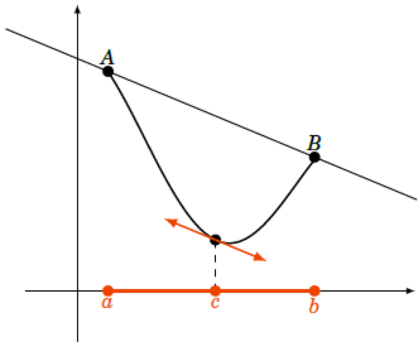
Theorem 18 ***Rolle's theorem****. If f is continuous on a closed interval $[a, b]$, and differentiable on the open interval $]a, b[$, and $f(a) = f(b)$, then there exists a $c \in]a, b[$ such that $f'(c) = 0$.*



Theorem 19 ***Mean value theorem.*** *If f is continuous on the closed interval $[a,b]$, and differentiable on the open interval $]a,b[$, then there exists a $c \in]a,b[$ such that*

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Therefore, for any function that is continuous on $[a,b]$, and differentiable on $]a;b[$, there exists $c \in]a,b[$ such that the secant joining the endpoints of the interval $[a,b]$ is parallel to the tangent at c.



0.3.2 Relationship between monotonicity and derivative.

Let $f(x)$ be continuous on $[a;b]$, and differentiable on $]a,b[$,.

— $f(x)$ is monotonically increasing on $[a,b]$ if and only if for all $x \in]a,b[$, $f'(x) \geq 0$.
— If for all $x \in]a,b[$, $f'(x) > 0$ then $f(x)$ is strictly monotonically increasing on $[a,b]$.
The converse of the statement is not true, for example $f(x) = x^3$ is strictly monotonically increasing, but $f'(0) = 0$.
— $f(x)$ is monotonically decreasing on $[a,b]$ if and only if for all $x \in]a,b[$, $f'(x) \leq 0$.
— If for all $x \in]a,b[$, $f'(x) < 0$ then $f(x)$ is strictly monotonically decreasing on $[a,b]$.

Theorem 20 ***L'Hospital's rule.*** *Let's assume that f and g are differentiable in a punctured neighbourhood of a, f and g have limits at a, and either both limits are 0 or both limits are ∞, that is, the limit of the quotient of the two function is critical. In this case if there exists the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$, then also exists the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$, and*

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

This theorem is also valid for one-sided limits or limits at infinity or minus infinity.
Exercise 2 : Calculate the limits (Applying L'Hospital's theorem)

$$\lim_{x\to 0} \frac{\ln(1+x)-x}{x^2}, \quad \lim_{x\to 0} \frac{e^{x^2}-\cos x}{x^2}$$

Exercise 3 : Let f numerical function defined as follow

$$f(x) = \begin{cases} \dfrac{\ln(2-e^x)}{x}, & x < 0 \\ -1-x, & x \geq 0. \end{cases}$$

— Study the continuity of f on $\mathbb{R}$.
— Study the differentiability of f on $\mathbb{R}$.
— f' is continuous on $\mathbb{R}$? (i.e $f \in \mathcal{C}^1(\mathbb{R})$)

0.4 Higher derivatives

Definition 21 *Let f be a differentiable function, and let f' be its derivative. The derivative of f' (if it has one) is written f'' and is called* ***the second derivative*** *of f. Similarly, the derivative of the second derivative, if it exists, is written f''' and is called* ***the third derivative*** *of f. Continuing this process, one can define, if it exists, the n-th derivative as the derivative of the (n-1)-th derivative. These repeated derivatives are called* ***higher-order derivatives.*** *The n-th derivative is also called the* ***derivative of order*** *n (or n-th-order derivative : first, second, third-order derivative, ...) and denoted $f^{(n)}$.*

For example : The higher derivatives of the function $f : x \mapsto f(x) = \ln(1+x)$ are :

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2}, ..., f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$$

The question : By induction show that $f^{(n)(x)}$ is true.

Generalizes the product rule (which is also known as **Leibniz's rule**). It f and g are n-times differentiable functions, then the product $f.g$ is also n-times differentiable and its n-th derivative given by

$$(f.g)^n = f^{(n)}.g + C_n^1 f^{(n-1)} g^{(1)} + ... + C_n^k f^{(n-k)} g^{(k)} + ... + f.g^{(n)} = \sum_{k=0}^{k=n} C_n^k f^{(n-k)} g^{(k)}$$

Exercise 4 : Using Leibnitz formula ; calculate the derivative $7-th$ of the function $h(x) = x^3 . \ln x$