III- *Dynamics*

1- Introduction

1-1-Definition:

Dynamics (kinetics) is the study of motion by taking into account the causes that generate it

1-2-Inertial Frame of Reference (Galilean)

In the case of relative motion, the reference frames have been defined as "$\mathcal{R}$", and "$\mathcal{R}_1$", one is assumed to be absolute (fixed), the other is mobile. But the question for "$\mathcal{R}$", it is fixed with respect to what? As a result, it is assumed that a frame of reference is fixed according to the problem under study where the laws of physics become simpler.
The frame of reference in which an isolated (free) object maintains its state of motion(constant velocity) is a privileged reference frame called an inertial frame.

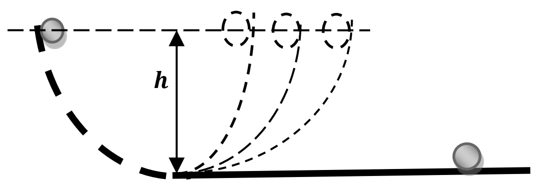
1.3- Observation:

- If a ball is dropped, from a height" h", into a smooth tank (frictionless), it goes down and up again at the same level " h" regardless of the slope.

- If the second side of the bowl is flattened, then it has been lowered, the ball follows a horizontal path and continues its path with a uniform rectilinear movement.

Result:

An isolated ball follows a uniform straight path.



2- Principle of inertia

In an inertial frame of reference (Galilean), a free body (isolated or not subjected to any external forces), continues to move in a straight line at a constant speed (uniform rectilinear motion) if it was already in motion, if it is in rest, it remains at rest.

Note: *The principle of inertia brings us closer to the concept of force.*

3- Mass and momentum

3.1- Mass

The greater the mass of a body, the more difficult to stop or move it.

Mass is the amount of matter in a body that characterizes its ability to resisting the change of motion (velocity), it characterizes its inertia.

3.2- Momentum

- For two bodies with the same velocity, it is easier to stop or move the one with the smaller mass.
- For two bodies with the same mass, it is easier to stop or move the one with the lower velocity.

3-2-1-Definition

The product of a body's mass by its velocity defines the **momentum** *denoted* "$\vec{P}$".

$$\vec{P} = m\vec{v} \quad [kg.m/s]$$

Note: *The principle of inertia can be stated as follows:*

An isolated body of constant mass has a constant momentum.

3-2-2-Momentum of a Particle System

Let be an isolated system consisting "n" of particles of respective velocities "$\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n$". We define the center of mass "G" whose vector position "$\vec{r}_G$" such that:

$$\vec{r}_G = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$$

$\vec{r}_i$: is the position vector for the i^{th} particle of mass "m_i"

Then:

$$\frac{d\vec{r}_G}{dt} = \vec{v}_G = \frac{\sum_{i=1}^{n} m_i \frac{d\vec{r}_i}{dt}}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i \vec{v}_i}{\sum_{i=1}^{n} m_i}$$

$\sum_{i=1}^{n} m_i = M$ the total mass

Then:

$$\vec{v}_G = \frac{\sum_{i=1}^{n} \vec{P}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} \vec{P}_i}{M} \qquad \Rightarrow \qquad \frac{\sum_{i=1}^{n} m_i \vec{v}_i}{\sum_{i=1}^{n} m_i} = \sum_{i=1}^{n} \vec{P}_i$$

Hence: the momentum (linear momentum) of the system

$$\vec{P} = M\vec{v}_G = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \cdots + \vec{P}_n = \sum_{i=1}^{n} \vec{P}_i$$

The momentum of system of "n" particles is the same as if all its mass were concentrated at its center of mass that whose velocity is $\vec{v}_G$.

3-2-3-Conservation of Momentum

a – Conservation of momentum

Let be a system consisting of two particles $[(m_1, \vec{v}_1);(m_2, \vec{v}_2)]$ in interaction. Due to the change in their velocities, each of the particles follows a curvilinear path.

- at the moment "$t = t_0$" the two particles are in position A_1 and A_2
- at the moment "$t = t_1$" the two particles are in position B_1 and B_2

The position vector of the center of mass of the system is:

$$\vec{r}_G = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

The momentum is:

- At "$t = t_0$": $\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2$
- At "$t = t_1$": $\vec{P'} = m_1\vec{v'}_1 + m_2\vec{v'}_2$

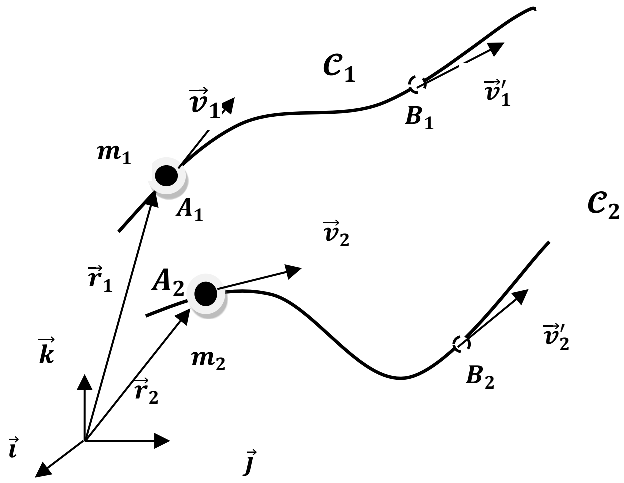
The velocity of the center of mass of the system is:

- At "$t = t_0$" :

$$\vec{v}_G = \frac{d\vec{r}_G}{dt} = \frac{\sum_{i=1}^{n} m_i \frac{d\vec{r}_i}{dt}}{\sum_{i=1}^{n} m_i} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

- At "$t = t_1$" :

$$\overrightarrow{v'}_G = \frac{d\vec{r}_G}{dt} = \frac{\sum_{i=1}^{n} m_i \frac{d\overrightarrow{r'}_i}{dt}}{\sum_{i=1}^{n} m_i} = \frac{m_1\overrightarrow{v'}_1 + m_2\overrightarrow{v'}_2}{m_1 + m_2}$$



Since the system is isolated, the center of mass moves at a constant speed.

$$\vec{v}_G = \overrightarrow{v'}_G$$

- At "$t = t_0$" : $\vec{P} = M\vec{v}_G$

- At "$t = t_1$" : $\overrightarrow{P'} = M\overrightarrow{v'}_G$

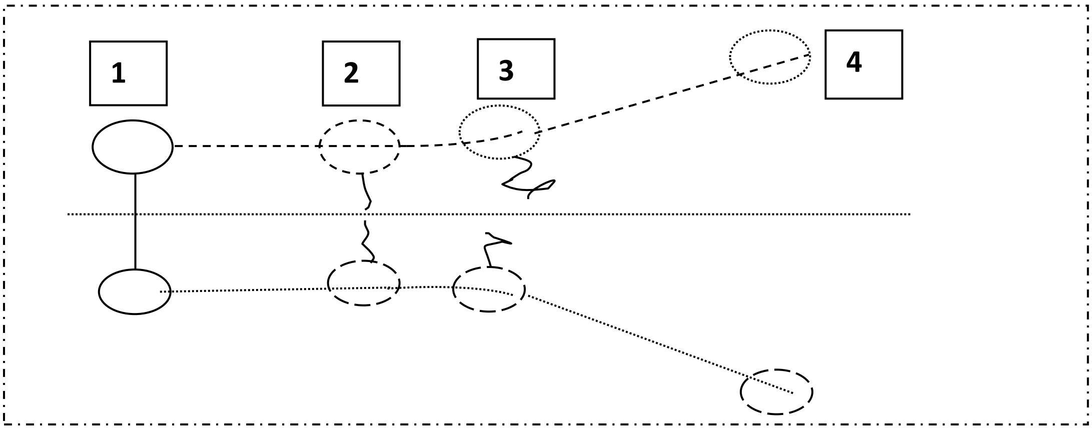
$$\vec{v}_G = \overrightarrow{v'}_G \qquad \Rightarrow \qquad \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \frac{m_1\overrightarrow{v'}_1 + m_2\overrightarrow{v'}_2}{m_1 + m_2}$$

$$m_1\vec{v}_1 + m_2\vec{v}_2 = M\vec{v}_G = m_1\overrightarrow{v'}_1 + m_2\overrightarrow{v'}_2 = M\overrightarrow{v'}_G$$

$$\Rightarrow \qquad \vec{P} = \overrightarrow{P'}$$

b - Equality of changes in momentum

Let be two magnetic disks linked by a string and thrown on a blower table which constitutes an isolated system.



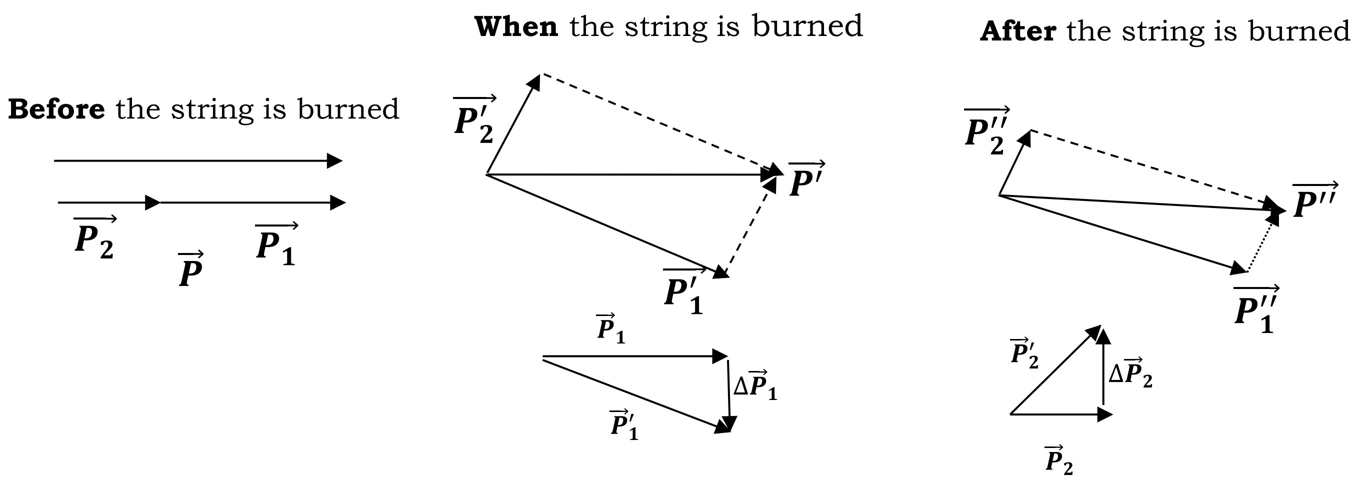
- **1** Position before burning the string

  The system is isolated, and the disks are still linked.
- **2** Position where the string is burned:

  The system is isolated and the disks begin to repel each other.
- **3** Position after the string is burned:

  The system is still isolated, but the disks become non-isolated and repel each other (interact) and change their velocities.
- **4** Position after a moment of disk separation:

  The system is still isolated, but the disks become free again and continue in a straight path.



Since the system is isolated, momentum is conserved, $\vec{P} = \vec{P'} = \vec{P''}$

but for the disks constituting this system are interacting, which changes their momentum $\overrightarrow{P_1}$ and $\vec{P}_2$.

Since: $\quad \vec{P} = \vec{P'} \quad \Rightarrow \quad \vec{P}_1 + \vec{P}_2 = \vec{P'}_1 + \vec{P'}_2$

$$\Rightarrow \quad \vec{P'}_1 - \vec{P}_1 = \vec{P}_2 - \vec{P'}_2$$

The change in momentum is:

$$\Delta\vec{P} = \vec{P'} - \vec{P}$$

$$\Rightarrow \quad \Delta\vec{P}_1 = \vec{P'}_1 - \vec{P}_1 \quad \text{and} \quad \Delta\vec{P}_2 = \vec{P'}_2 - \vec{P}_2$$

$$\Rightarrow \quad \Delta\vec{P}_1 = -\Delta\vec{P}_2$$

i.e., the variations in momentum are equal and opposite

4- Newton's Laws

4.1: 1st Law: Law of Inertia

In an inertial frame of reference, the momentum of a free body is conserved, i.e., the body (system) is in uniform rectilinear motion or at rest depending on its initial state

4.2: 2nd Law: Fundamental Principle of Dynamics

This law is already mentioned, i.e., any change in velocity (or change in momentum) of an isolated (free) system is the result of an interaction that results in a ***force***.

The rate of change in momentum in an interval time produces the applied force.

$$\vec{F} = \sum_i \vec{F}_i^{ex} = \frac{\Delta \vec{P}}{\Delta t}$$

Where: $\begin{cases} \vec{F}: Net\ force \\ \vec{P}: \text{system momentun} \end{cases}$

In the limit case with an infinitesimal change:

$$\vec{F} = \lim_{\Delta t \to 0} \left(\frac{\Delta \vec{P}}{\Delta t} \right) = \frac{d\vec{P}}{dt}$$

Note: In the case where the mass of the system is constant, the 2nd law becomes

$$\vec{F} = \sum_i \vec{F}_i^{ex} = \frac{d\vec{P}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt}$$

$$\Longrightarrow \qquad \vec{F} = m\vec{a}$$

4.3: 3rd Law: Law of Reciprocity (Action and Reaction Law)

As already pointed out, the momentum exchanging during the interaction between two particles in the system are the same but opposite.

$$\vec{P} = \vec{P'} \Rightarrow \vec{P}_1 + \vec{P}_2 = \vec{P'}_1 + \vec{P'}_2$$

$$\Rightarrow \vec{P'}_1 - \vec{P}_1 = \vec{P}_2 - \vec{P'}_2$$

$$\Rightarrow \Delta\vec{P}_1 = -\Delta\vec{P}_2$$

If:

$\vec{F}_{12}$**:** is the action of particle (**1**) on particle (**2**)

$\vec{F}_{21}$ **:** is the action of particle **(2)** on particle **(1)**

So: $\vec{F}_{12} = \frac{\Delta\vec{P}_2}{\Delta t}$ and $\vec{F}_{21} = \frac{\Delta\vec{P}_1}{\Delta t}$ Since $\Delta\vec{P}_1 = -\Delta\vec{P}_2 \Rightarrow$

at the limit: $\Delta\vec{P}_1 \to d\vec{P}_1$ and $\Delta\vec{P}_2 \to d\vec{P}_2$

$$\Rightarrow \vec{F}_{12} = \frac{d\vec{P}_2}{dt} \quad \text{and} \quad \vec{F}_{21} = \frac{d\vec{P}_1}{dt}$$

$$\Delta\vec{P}_1 = -\Delta\vec{P}_2 \Rightarrow d\vec{P}_1 = -d\vec{P}_2$$

$$\Rightarrow \vec{F}_{21} = -\vec{F}_{12}$$

Result:

If one body exerts an action on another, the latter reacts with an equal and opposite force

5- Some laws of force

According to the fundamental law of dynamics, we have:

$$\vec{F} = m\vec{a} = m\ddot{\vec{r}}$$

Where: $\vec{F} = \vec{F}\,(\dot{\vec{r}}, \vec{r}, t)$

5.1- Constant force

In this case, the net force is:

$$\vec{F} = \vec{F}\left(\dot{\vec{r}}, \vec{r}, t\right) = \vec{F}_0 = constante$$

$$\vec{F} = \vec{F}\left(\dot{\vec{r}}, \vec{r}, t\right) = \vec{F}_0 = m\ddot{\vec{r}} \qquad \Longrightarrow \qquad \ddot{\vec{r}} = \frac{\vec{F}_0}{m} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$$

$$\Longrightarrow \qquad d\left(\frac{d\vec{r}}{dt}\right) = \frac{\vec{F}_0}{m}dt \qquad \Longrightarrow \qquad \int_{\dot{\vec{r}}_0}^{\dot{\vec{r}}} d\left(\dot{\vec{r}}\right) = \frac{\vec{F}_0}{m}\int_{t_0}^{t} dt$$

$$\Longrightarrow \qquad \dot{\vec{r}} - \dot{\vec{r}}_0 = \frac{\vec{F}_0}{m}(t - t_0) \qquad \Longrightarrow \qquad \dot{\vec{r}} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}_0 + \frac{\vec{F}_0}{m}(t - t_0)$$

Finally:

$$\int_{\vec{r}_0}^{\vec{r}} d\vec{r} = \int_{t_0}^{t}\left[\dot{\vec{r}}_0 + \frac{\vec{F}_0}{m}(t - t_0)\right]dt$$

$$\Longrightarrow \qquad \vec{r} = \frac{1}{2}\frac{\vec{F}_0}{m}(t - t_0)^2 + \dot{\vec{r}}_0(t - t_0) + \vec{r}_0$$

It is the law of uniformly varied motion

Example: Free Fall $\qquad \vec{F}_0 = m\vec{g} \Longrightarrow \vec{a} = \vec{g} = \ddot{\vec{r}}$

$$\Longrightarrow \qquad \vec{v} = \frac{d\vec{r}}{dt} = \vec{v}_0 + \vec{a}(t - t_0)$$

$$\Longrightarrow \qquad \vec{r} = \frac{1}{2}\vec{a}(t - t_0)^2 + \vec{v}_0(t - t_0) + \vec{r}_0$$

Since the motion is done in a straight line

$$\Longrightarrow \qquad r = \frac{1}{2}a(t - t_0)^2 + v_0(t - t_0) + h_0$$

5.2- Time-dependent force

$$\vec{F}\left(\dot{\vec{r}},\vec{r},t\right) = \vec{F}(t)$$

$$\ddot{\vec{r}} = \frac{\vec{F}(t)}{m} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) \qquad \Rightarrow \qquad \int_{\dot{\vec{r}}_0}^{\dot{\vec{r}}} d\left(\dot{\vec{r}}\right) = \frac{1}{m}\int_{t_0}^{t}\vec{F}(t)dt$$

$$\Rightarrow \quad \dot{\vec{r}} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}_0 + \frac{1}{m}\int_{t_0}^{t}\vec{F}(t)dt \qquad \Rightarrow \quad \int_{\vec{r}_0}^{\vec{r}} d\vec{r} = \int_{t_0}^{t}\left[\dot{\vec{r}}_0 + \frac{1}{m}\int_{t_0}^{t}\vec{F}(t)dt\right]dt$$

Finally: $\quad \vec{r} = \int_{t_0}^{t}\left[\dot{\vec{r}}_0 + \frac{1}{m}\int_{t_0}^{t}\vec{F}(t)dt\right]dt + \vec{r}_0$

Example: Point Charge **Q** in a Variable Electric Field $E(t) = E_0 sin(\omega t)$.

We know the force of an electric charge is: $\quad F = QE$

$$F = QE_0 sin(\omega t) \Rightarrow F = ma = QE_0 sin(\omega t)$$

$$\Rightarrow \qquad a = \frac{QE_0 sin(\omega t)}{m}$$

$$r = \int_0^t\left[\dot{r}_0 + \int_0^t \frac{QE_0 sin(\omega t)}{m}\right]dt + r_0 = r_0 + v_0 t + \frac{QE_0}{m\omega^2}(\omega t - sin\omega t)$$

If we take the following initial conditions: $t_0 = 0$; $r_0 = 0$; $v_0 = 0$

$$r = \frac{QE_0}{m\omega^2}(\omega t - sin\omega t)$$

5.3- Velocity-dependent force

$$\vec{F}\left(\dot{\vec{r}},\vec{r},t\right) = \vec{F}\left(\dot{\vec{r}}\right) = \vec{F}(\vec{v})$$

$$\Rightarrow \quad \ddot{\vec{r}} = \frac{\vec{F}(\vec{v})}{m} = \frac{d}{dt}(\vec{v})$$

$$\Rightarrow \quad dt = m\frac{dv}{F(v)} \qquad \Rightarrow \quad t - t_0 = \int_{v_0}^{v} m\frac{dv}{F(v)}$$

$$\Rightarrow \quad t = t_0 + f(v; v_0)$$

But:

$$ma = \frac{mdv}{dt} = m\frac{dv}{dr}.\frac{dr}{dt} = mv\frac{dv}{dr} = F(v)$$

$$\Longrightarrow \quad dr = m\frac{vdv}{F(v)}$$

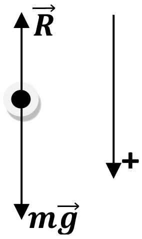
$$\Longrightarrow \quad \int_{r_0}^{r} dr = m\int_{v_0}^{v}\frac{vdv}{F(v)} \qquad \Longrightarrow \qquad r = r_0 + m\int_{v_0}^{v}\frac{vdv}{F(v)}$$

Example: frictional force (air resistance) acting on a body in free fall: $\vec{R} = -k\vec{v}$

$$\sum \vec{F}^{ex} = m\vec{g} + \vec{R} \quad \Longrightarrow \quad mg - kv = \frac{mdv}{dt}$$

$$\Longrightarrow \quad \frac{dv}{\left(g-\frac{k}{m}v\right)} = dt \quad \Longrightarrow \quad \int\frac{dv}{\left(g-\frac{k}{m}v\right)} = \int dt$$

If we take



$$g - \frac{k}{m}v = u \Longrightarrow -\frac{k}{m}dv = du$$

So

$$\int\frac{du}{u} = -\frac{m}{k}\int dt \quad \Longrightarrow \quad Ln(u) = -\frac{m}{k}t$$

If at $t_0 = 0, \quad v_0 = 0 \quad \Longrightarrow \quad v = \alpha(1 - e^{-\beta t})\beta = \frac{m}{k}$

Where $\alpha = \frac{mg}{k}$

5.4- Position-dependent force

$$\vec{F}\left(\dot{\vec{r}}, \vec{r}, t\right) = \vec{F}(\vec{r})$$

Generally, these types of forces are conservative, so they derive from a potential.

$$F = -\frac{dV}{dr}$$

Where V **:** is a potential function (potential energy)

$$F = -\frac{dV}{dr} = ma = m\ddot{r} \qquad \Longrightarrow \qquad \vec{F}\circ\dot{\vec{r}} = m\ddot{\vec{r}}\circ\dot{\vec{r}}$$

$$\vec{F}\circ\frac{d\vec{r}}{dt} = \frac{1}{2}\frac{d(m\dot{r}^2)}{dt}$$

$$\Longrightarrow \quad \int_{r_0}^{r} \vec{F} \circ d\vec{r} = \int_{\dot{r}_0}^{\dot{r}} d(\tfrac{1}{2} m\dot{r}^2) = -\int_{V_0}^{V} dV$$

$$\Longrightarrow \quad \tfrac{1}{2}\left(m\dot{r}^2 - m{\dot{r}_0}^2\right) = V(r_0) - V(r)$$

$$\Longrightarrow \qquad \tfrac{1}{2} m\dot{r}^2 + V(r) = \tfrac{1}{2} m{\dot{r}_0}^2 + V(r_0) = Constant = E$$

E: total energy (mechanical Energy)

We have:

$$\frac{dr}{dt} = \dot{r} = \mp\sqrt{\frac{2}{m}}\sqrt{E - V(r)} \qquad \Longrightarrow \qquad dt = \mp\sqrt{\frac{m}{2}}.\frac{dr}{\sqrt{E - V(r)}}$$

$$\Longrightarrow \qquad t - t_0 = \mp\sqrt{\frac{m}{2}}.\int_{r_0}^{r} \frac{dr}{\sqrt{E-V(r)}}$$

$$\Longrightarrow \qquad t = t_0 \mp \sqrt{\frac{m}{2}}.\int_{r_0}^{r} \frac{dr}{\sqrt{E-V(r)}} = T(r)$$

Time is a function of "r"**,** conversely, we can determine the function that describes the position of the mobile "$r = R(t)$"

6- Angular momentum

A particle of mass "m" and velocity "$\vec{v}$"**,** has momentum "$\vec{P}$" and is subject to forces given by Newton's second law.

$$\vec{F} = \sum_i \vec{F}_i^{ex} = \frac{d\vec{P}}{dt}$$

$$\Longrightarrow \quad \vec{r} \wedge \vec{F} = \sum_i \vec{r} \wedge \vec{F}_i^{ex} = \sum_i \overrightarrow{\mathcal{M}}_i(\vec{F}_i^{ex})_{/o} = \vec{r} \wedge \frac{d\vec{P}}{dt}$$

If we add the quantity "$\frac{d\vec{r}}{dt} \wedge \vec{P} = \mathbf{0}$" that does not modify the previous expression in any way, we will have:

$$\sum_i \overrightarrow{\mathcal{M}}_{i_{/o}} = \vec{r} \wedge \frac{d\vec{P}}{dt} + \frac{d\vec{r}}{dt} \wedge \vec{P} = \frac{d\left(\vec{r} \wedge \vec{P}\right)}{dt}$$

Quantity "$\vec{r} \wedge \vec{P}$" plays an important role in rotational motion than momentum in translation. This amount is called ***angular momentum***.

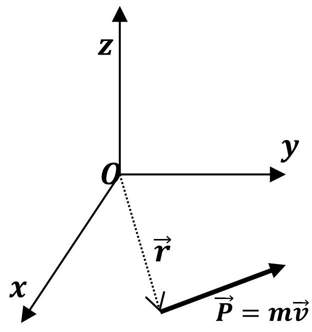
6.1- Definition

The angular momentum with respect to a point "O", denoted "$\vec{L}_O$", of a particle of mass "m" and velocity "$\vec{v}$", is the rotation that results from the effect of its momentum.

$$\vec{L}_O = \vec{\mathcal{M}}(\vec{P})_{/o} = \overrightarrow{OM} \wedge \vec{P} = \vec{r} \wedge \vec{P}.$$

6.2- Relation between angular momentum and resultant forces (Newton's 2nd Law)

Newton's second law for a rotational motion of a body can be written as follows:



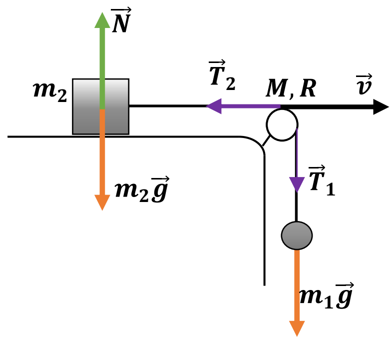
$$\vec{\mathcal{M}}(\vec{F})_{/o} = \sum_i \vec{\mathcal{M}}_{i/o} = \frac{d(\vec{r} \wedge \vec{P})}{dt} = \frac{d\vec{L}_O}{dt}$$

Example:

The mass m_2, slides on frictionless table, driven by the sphere m_1, with the help of a non stretched wire passing through the groove of a pulley of radius R and mass M distributed on its rim.

Calculate

1. The angular momentum with respect to an axis passing through the center of the pulley.
2. The acceleration of the masses m_1 and m_2



- The angular momentum of m_2 :

  $L_2 = |\vec{r}_2 \wedge m_2\vec{v}_2| = m_2 vR$

- The angular momentum of m_1 :

  $L_1 = |\vec{r}_1 \wedge m_1\vec{v}_1| = m_1 vR$

- The angular momentum of M :

  $L_3 = |\vec{R} \wedge M\vec{v}| = MvR$

Pulley mass distributed over the rim (periphery), so the angular momentum is:

$$L_{/\Delta} = L_1 + L_2 + L_3$$

$$\sum_i \mathcal{M}(\vec{F})_{/\Delta} = \frac{dL_{/\Delta}}{dt} = \frac{d(L_1 + L_2 + L_3)}{dt} = \frac{d(m_1 vR + m_2 vR + MvR)}{dt}$$

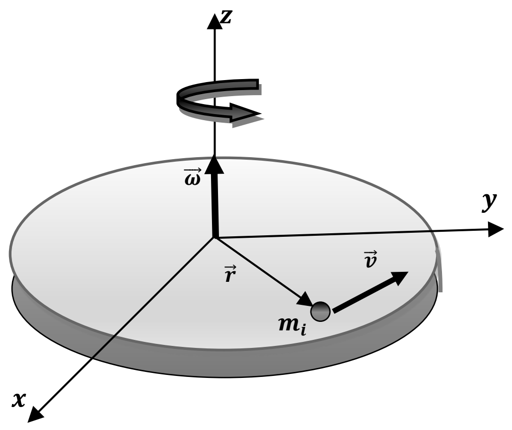
$$\sum_i \mathcal{M}(\vec{F})_{/\Delta} = \mathcal{M}\left(m_1\vec{\vec{g}}\right)_{/\Delta} = m_1 gR = (m_1 + m_2 + M)Ra$$

$$\Rightarrow \qquad a = \frac{m_1 g}{(m_1 + m_2 + M)}$$

6.3- Angular momentum of a rigid (non-deformable) body

6.3.1-Expression of angular momentum

The rotation is about the axis $\overline{oz}$, the point has the velocity $m_i\vec{v}_i$



- $\vec{L}_{i/O} = \vec{r}_i \wedge \vec{P}_i$

But:

$$\vec{v}_i = \vec{\omega} \wedge \vec{r}_i \quad \text{and} \quad \vec{\omega} = \omega\vec{k}$$

$$\Rightarrow \qquad \vec{L}_{i/O} = m_i\vec{r}_i \wedge (\vec{\omega} \wedge \vec{r}_i) = m_i\vec{r}_i \wedge \left(\omega\vec{k} \wedge \vec{r}_i\right)$$

$$\Rightarrow \qquad \vec{L}_{i/O} = m_i {r_i}^2 \omega\vec{k}$$

$$L_{i/\overline{oz}} = \vec{L}_{i/O} \circ \vec{k} = [\vec{r}_i \wedge (m_i\vec{v}_i)] \circ \vec{k}$$

$$\Rightarrow \qquad L_{i/\overline{oz}} = m_i {r_i}^2 \omega\vec{k} \circ \vec{k} = m_i {r_i}^2 \omega$$

The angular momentum of a point "m_i" on the solid is "L_i"

The total angular momentum with respect to a point "O" is:

$$\vec{L}_O = \sum_i m_i {r_i}^2 \omega\vec{k}$$

and the total angular momentum with respect to the axis "$\overrightarrow{OZ}$" is:

$$L_{/\overrightarrow{oz}} = L_z = \sum_i L_{i/\overrightarrow{oz}} = \sum_i m_i {r_i}^2 \omega$$

The moment of inertia of a set of points that rotate about an axis is defined as follows:

$$I = \sum_i m_i {r_i}^2$$

For a continuous solid, the moment of inertia is:

$$I = \int r^2 dm$$

The angular momentum with respect to a point "O" is:

$$\vec{L}_O = (\sum_i m_i {r_i}^2).\vec{\omega} = I.\vec{\omega}$$

The angular momentum with respect to the axis $''(\overrightarrow{oz})''$ is:

$$L_{/\overrightarrow{oz}} = L_z = (\sum_i m_i {r_i}^2).\omega = I.\omega$$

Newton's second law becomes:

$$\sum_i \overrightarrow{\mathcal{M}}_{i/o} = \frac{d\vec{L}_O}{dt} = \frac{d(I\vec{\omega})}{dt} = I\vec{\varepsilon}$$

with and: moment of inertia (constant)$\vec{\varepsilon} = \frac{d\vec{\omega}}{dt} I$

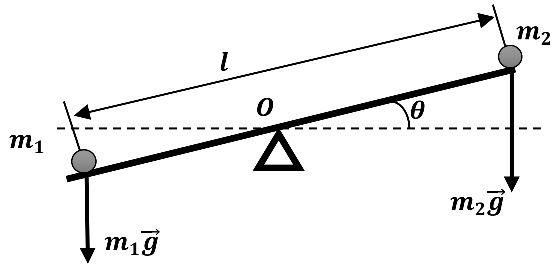
This is Newton's law applied to rotational motion

Example:

On a swing, long of and of $''l''$mass, $''M''$**two boys of masses**, m_1 and, are distracting themselves. Calculate their angular acceleration $m_2\varepsilon$**.**

Knowing that the moments of inertia of the helm and the boys are:

$$I_b = \frac{Ml^2}{12} \qquad I_1 = m_1\left(\frac{l}{2}\right)^2 \qquad I_2 = m_2\left(\frac{l}{2}\right)^2$$



The angular momentum is:

$$\vec{L}_O = I\vec{\omega} = (I_b + I_1 + I_2)\vec{\omega}$$

But $\qquad \theta = \omega t \quad \Longrightarrow \quad \dot{\theta} = \frac{d\theta}{dt} = \omega$

and $\qquad \varepsilon = \frac{d^2\theta}{dt^2} = \dot{\omega}$

So

$$\sum_i \vec{\mathcal{M}}_{i/o} = \vec{\mathcal{M}}_{1/o} + \vec{\mathcal{M}}_{2/o} = \frac{d\vec{L}_O}{dt} = \frac{d}{dt}\big((I_b + I_1 + I_2)\vec{\omega}\big)$$

$$= (I_b + I_1 + I_2)\frac{d\vec{\omega}}{dt} = (I_b + I_1 + I_2).\vec{\varepsilon}$$

$$\Longrightarrow \quad (m_1 - m_2)g\frac{l}{2}cos\theta = \frac{1}{4}l^2\left(\frac{M}{3} + m_1 + m_2\right)\varepsilon$$

$$\Longrightarrow \qquad \varepsilon = \ddot{\theta} = \frac{2.(m_1 - m_2).cos\theta}{l.\left(\frac{M}{3} + m_1 + m_2\right)}g$$

6.3.2 - Conservation of angular momentum

When the system is isolated (free), then the external forces are zero ($\sum_i \vec{\mathcal{M}}_{i/o} = \mathbf{0}$), the angular momentum is constant.

$$\sum_i \vec{\mathcal{M}}_{i/o} = \frac{d\vec{L}_O}{dt} = 0 \qquad \Longrightarrow \qquad \vec{L}_O = Constante$$

6.4- Angular momentum of a deformable body

During its motion, the deformable body undergoes mass redistributions and the moment of inertia changes, but its angular momentum remains constant if it is isolated (free).

$$\vec{L}_O = I\vec{\omega} = Constante$$

The change of "$\boldsymbol{I}$", causes the change of "$\boldsymbol{\omega}$" such that the angular momentum remains constant, which means that, If "$\boldsymbol{I}$" increases, the angular velocity "$\boldsymbol{\omega}$" decreases and inversely.

Note:

This result is valid for a body that rotates with respect to any fixed axis or an axis passing through its center of mass but remains fixed in direction

Example:

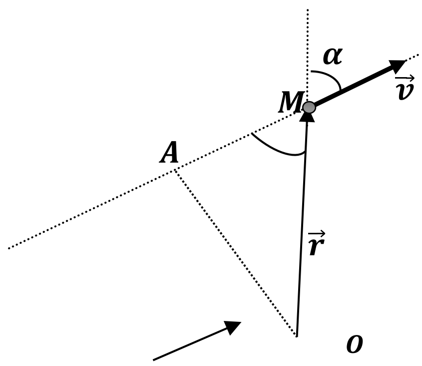
In an ice rink, when skaters (deformable bodies) fold their arms towards their trunks, the moment of inertia decreases but the angular velocity increases (it rotates faster). When they spread out their arms, the moment of inertia increases, but the angular velocity decreases until it stops.

Finally:

If the angular momentum is constant:

$$\vec{L} = \vec{r} \wedge m\vec{v} = \boldsymbol{Constante} \quad \Rightarrow \quad {d\vec{L}}/{dt} = \dot{\vec{L}} = \overrightarrow{\mathcal{M}} = \vec{r} \wedge \vec{F} = \mathbf{0}$$

In this situation, one can have



- $|\vec{F}| = \mathbf{0}$ $\Rightarrow$ The system is free and moves at a constant speed

  $\vec{L} = \vec{r} \wedge m\vec{v} = \boldsymbol{mrsin\alpha}\,\vec{u}$

  $\forall$ the position of the point **M:**

  $\boldsymbol{rsin\alpha} = \boldsymbol{OA}$ is constant

  In addition, the mass is constant

  $\Rightarrow \quad \vec{v} = constant$

  $\Rightarrow$ The motion is uniformly rectilinear.

- $|\vec{r}| = \mathbf{0}$ $\Rightarrow$ There is no motion, the particle is at rest or in equilibrium.

- $\vec{r} \parallel \vec{F}$ $\Rightarrow$ The force passes through the center "**O**", i.e., the force is central.

6.5- Area theorems

During the motion, the position of the particle is given by:

$$\overrightarrow{OM} = \rho \vec{u}_\rho$$

Its speed is**:**

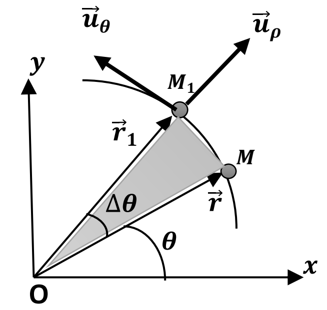
$$\vec{v} = \dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta$$

Its angular momentum will be:

$$\vec{L} = \overrightarrow{OM} \wedge m\vec{v} = \rho\vec{u}_\rho \wedge m\left(\dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta\right) = m\rho^2\dot{\theta}\vec{k}$$

When the point "M" moves to "M_1", the vector "$\vec{r}$" sweeps the area "OMM_1"

$$\Rightarrow \quad \Delta A_{OMM_1} = \frac{1}{2}\widehat{MM_1}r = \frac{1}{2}\rho^2\Delta\theta$$



$$\Rightarrow \quad \frac{\Delta A_{OMM_1}}{\Delta t} = \frac{1}{2}\rho^2\frac{\Delta\theta}{\Delta t}$$

At the limit, we will have: $\Delta\theta \to d\theta$

$$\frac{dA_{OMM_1}}{dt} = \frac{1}{2}\rho^2\frac{d\theta}{dt} = \frac{1}{2}\rho^2\dot{\theta}$$

For central forces, angular momentum is constant.

$$\vec{L} = Const \quad \Rightarrow \quad |\vec{L}| = m\rho^2\dot{\theta}$$

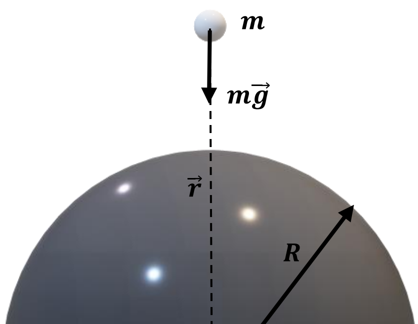
$$\Rightarrow \quad \frac{|\vec{L}|}{2m} = Const = \frac{\rho^2\dot{\theta}}{2} = \frac{dA_{OMM_1}}{dt}$$

During the motion of a particle under the action of central forces, the areas swept in the same time intervals are equal. *This is the area theorem*

7- Types of forces

7.1- Conservative Forces

The forces that derive from a potential and depend on the position of the particle are a conservatives forces.



$$\vec{F}(\vec{r}) = -\vec{\nabla}V(\vec{r})$$

Where is the operator $\vec{\nabla}$ is: $\vec{\nabla} = \frac{\partial}{\partial x}\vec{j} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial \vec{k}}{\partial k}$

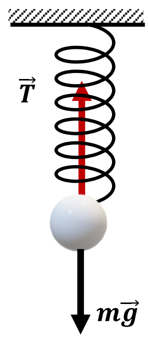
M

$V(\vec{r})$ is a scalar which is the potential energy

Example:

The weight of a body: $V(\vec{r}) = mgr$

The restoring force of a spring: $V(\vec{r}) = \frac{1}{2}kr^2$



7.2- Lorenz Forces

Strength depends on position and speed.

Example:

Electric charge "q" immersed in an electromagnetic field:

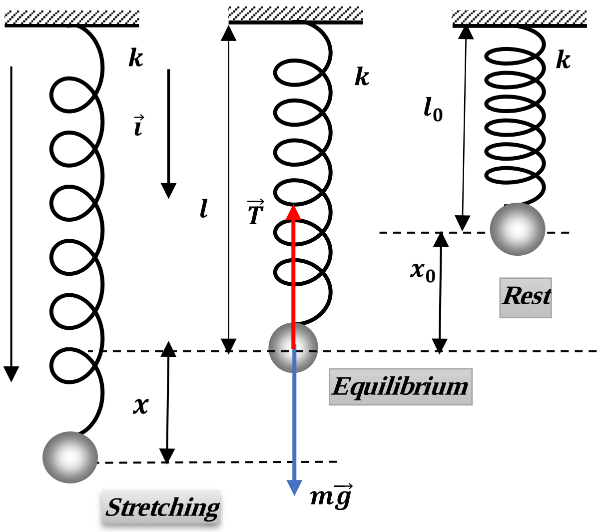
$\vec{F}(\vec{r};\dot{\vec{r}};t) = q(\vec{E} + \vec{v} \wedge \vec{B})$ $\vec{E}$: electric field $\vec{B}$: magnetic field

7.3- Elastic forces

Force due to the elastic deformations of the materials

Example:

A mass "m" suspended from spring of stiffness constant "k" undergoes a so-called restoring force which obeys Hooke's law, i.e., it is proportional to the deformation



- When the spring is unstretched (uncompressed) its length is "l_0".

  When a mass is suspended to the spring but stay in equilibrium. The tension which is the restoring force) is then.

$$\boldsymbol{T = -kx_0}$$

Newton's law gives:

$$\vec{F} = \sum_i \vec{F}_i^{ex} = m\vec{g} + \vec{T} = \vec{0}$$

After projection on: $\overrightarrow{ox}$

$$mg = T = kx_0$$

- Stretching the spring more by "x" and let the mass free to move

The system, executes oscillations.

Newton's law now gives:

$$\vec{F} = \sum_i \vec{F}_i^{ex} = m\vec{g} + \vec{T} = m\vec{a}$$

By projection on :$\overrightarrow{ox}$

$$mg - T = mg - k(x + x_0) = m\ddot{x}$$

$$\Longrightarrow \qquad m\ddot{x} + kx = 0$$

$$\Longrightarrow \qquad \ddot{x} + \omega_0^2 x = 0 \qquad \text{where} \qquad \omega_0^2 = \frac{k}{m}$$

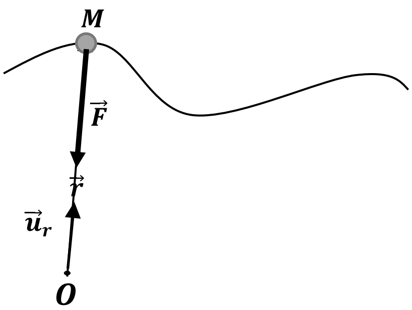
Which is the differential equation of the second order

The Solution of this equation is:

$$x = A\cos(\omega t + \varphi) \quad \rightarrow \quad \text{it's a harmonic motion}$$

7.4- Central Forces

A body is subject to the action of a central force, if it is everywhere directed towards a fixed point of the frame of reference under consideration.

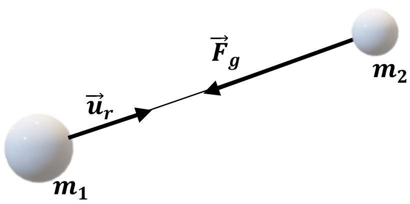


These forces are generally conservative.

$$\vec{F} = -f(\vec{r})\vec{u}_r$$

Example:

<u>Gravitational Forces</u> between two masses m_1 and m_2:

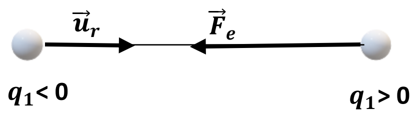


$$\vec{F}_g = -G.\frac{m_1 m_2}{r^2}\vec{u}_r$$

$m_1; m_2$: masses of the 2 bodies

Electrostatic Forces between two chargeses q_1 and q_2:

$$\vec{F}_e = K.\frac{q_1 q_2}{r^2}\vec{u}_r$$



$q_1; q_2$: Charges of the 2 bodies

- A system subjected to central forces has a constant angular momentum $(\vec{L} = Const)$

$$\frac{d\vec{L}}{dt} = \vec{r} \wedge \vec{F} = \vec{r} \wedge f(\vec{r})\vec{u}_r = 0$$

- The executed motion is in a plane, such that "$\vec{L}$" is always perpendicular.

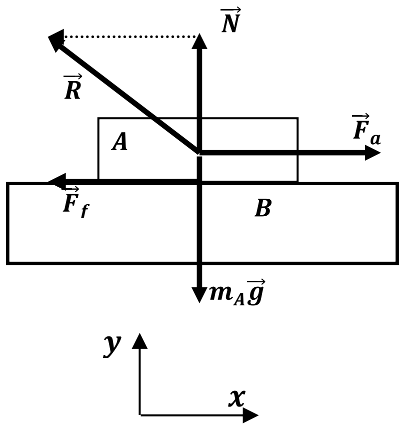
7.5- Frictional force

- Resistance to relative motion between two bodies in contact is caused by friction.

- The origin of this opposition is due to the irregularities of the two surfaces in contact.

7.5.1- Sliding friction (Coulomb friction)

The friction due to sliding of two surfaces of contact of the solids is said to be Coulombian.



A-Static Friction

- By applying force "$\vec{F}_a$" to the body "A", there is opposition to relative motion up to a certain limit. when "A" is ready in beginning to move (starting the motion), we are in a critical equilibrium.
- Applying Newton's Second Law

$$\vec{F} = \sum_i \vec{F}_i^{ex} = m_A\vec{g} + \vec{N} + \vec{F}_f + \vec{F}_a = m\vec{a}$$

$m_A\vec{g}$: body weight "A".

$\vec{N}$: the reaction of "A" to the action of "B".

$\vec{F}_f$: Force of friction: force at the contact surface.

$\vec{F}_a$: Applied force to move "$\boldsymbol{A}$".

At the limit, the body "$\boldsymbol{A}$" is always at rest:

$$\sum_i \vec{F}_i^{ex} = m\vec{a} = 0$$

Let's project the equation onto the axes:

$$\begin{cases} \overrightarrow{ox}: & F_a - F_f = 0 \\ \overrightarrow{oy}: & -m_A g + N = 0 \end{cases} \Rightarrow \begin{cases} F_a = F_f \\ N = m_A g \end{cases}$$

- F_l : The minimum force required to initiate relative motion.
- Experience shows that the frictional force is proportional to the weight

$$F_f = \mu_s . mg \;;$$

μ_s: is the coefficient of static friction

$$\mu_s = \frac{F_f}{mg} = \frac{F_a}{N}$$

At the static limit

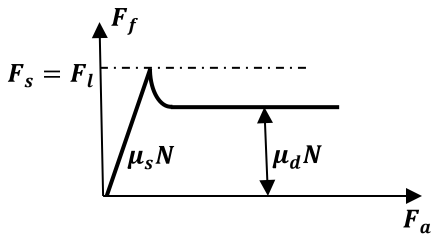
$$F_l = (F_a)_{max} = F_s$$

$$\Rightarrow \quad \mu_s = \frac{F_l}{N} = \frac{F_S}{N}$$

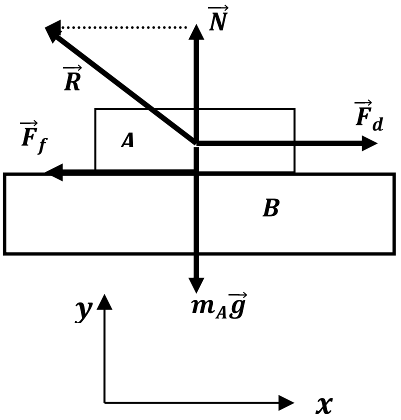
The coefficient of static friction "μ_s" is determined by the ratio of the limiting force "F_l" required to initiate the motion and the normal reaction "$\vec{N}$".

B-Dynamic Friction

The experience shows that during relative motion, the force dynamic friction $\vec{F}_d$ is less than the force of static friction $\vec{F}_s$. This is how the coefficient of kinetic friction μ_d is defined.



In such way: $\mu_d = \frac{F_d}{N}$



$$\vec{F} = \sum_i \vec{F}_i^{ex} = m_A\vec{g} + \vec{N} + \vec{F}_f + \vec{F}_a = m\vec{a}$$

Uniform Motion:

$$\sum_i \vec{F}_i^{ex} = m\vec{a} = 0$$

Let's project the equation onto the axes:

$$\begin{cases} \overrightarrow{ox}: \quad F_d - F_f = 0 \\ \overrightarrow{oy}: \quad -m_A g + N = 0 \end{cases} \Rightarrow \begin{cases} F_d = F_f \\ N = m_A g \end{cases}$$

- F_d: Force required to have a uniform motion.

$$\mu_d = \frac{F_f}{mg} = \frac{F_d}{N}$$

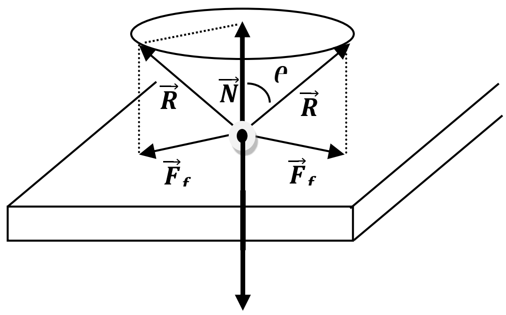
The kinetic coefficient of friction "μ_d" ***is determined by the ratio of the force*** "F_d" ***required to have a uniform motion and the normal reaction*** "$\vec{N}$".

- ***Friction angle***

  $N = mg = Const$: always perpendicular to the surface

  $\vec{F}_f$: depends on the direction in the contact surface.

  $\vec{R} = \vec{N} + \vec{F}_f$: All reaction forces belong to the surface of the so-called friction cone.



  If the angle at the top of the cone is "θ", then for a defined direction

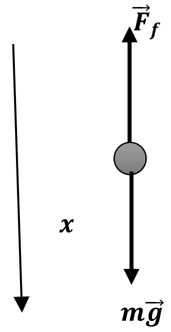
  The angle of friction is: $tg(\theta) = \frac{F_f}{N} \Rightarrow \mu = tg(\theta)$

7.5.2- Viscous friction

These are the frictions caused by contact with a solid and a fluid (liquid or gas)

A- Stock Friction ($\vec{F} = -\alpha\vec{v}$): Drag force

The force in this type is proportional to the speed:



$$\vec{F} = -\alpha\vec{v} = -K\eta\vec{v}$$

K : Form constant;

η: viscosity.

Examples

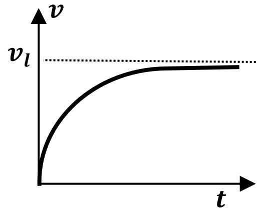
01: What is the terminal (limit) speed of a spherical ball in free fall.

$K = 6\pi R$

$\sum_i \vec{F}_i^{ex} = m\vec{g} + \vec{F}_f = m\vec{a}$

Projection on $\overrightarrow{ox}$ gives:

$mg - \alpha v = m\ddot{x} \quad \Rightarrow \quad \ddot{x} = g - \frac{\alpha}{m}v$



Solving the Differential Equation

$g - \frac{\alpha}{m}v = \frac{dv}{dt} \quad \Rightarrow \quad \frac{dv}{g - \frac{\alpha}{m}v} = dt$

We put:

$$g - \frac{\alpha}{m}v = u \quad \Rightarrow \quad -\frac{\alpha}{m}dv = du \quad \Rightarrow \quad \frac{du}{u} = -\frac{\alpha}{m}dt$$

$\Rightarrow \quad \int\frac{du}{u} = -\frac{\alpha}{m}\int dt \quad \Rightarrow \quad Ln(u) = -\frac{\alpha}{m}t \quad \Rightarrow \quad u = Ce^{-\frac{\alpha}{m}t}$

$\Rightarrow \quad g - \frac{\alpha}{m}v = Ce^{-\frac{\alpha}{m}t}$ If at $t = 0, \qquad v_0 = 0 \Rightarrow C = g$

$\Rightarrow \quad v = \frac{mg}{\alpha}\left(1 - e^{-\frac{\alpha}{m}t}\right)$

02:

What is the terminal speed of a spherical ball falling in oil of viscosity η ?

The mass of the ball is:

$$m = \rho V = \frac{4}{3}\rho\pi R^3$$

ρ: density; V:volume

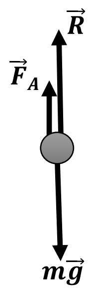
If "ρ_f" is the density of the oil (fluid), then its mass corresponding to the ball is:

$$m_f = \rho_f V = \frac{4}{3}\rho_f\pi R^3$$

The Bouncing force (Archimedean thrust) is:

$$\vec{F}_A = -m_f\vec{g}$$

The forces of weight, bouncing force and drag forces act on the ball



$$\sum_i \vec{F}_i^{ex} = m\vec{g} + \vec{F}_A + \vec{R} = m\vec{a}$$

For terminal speed: $\vec{a} = \vec{0}$

$\Longrightarrow \quad mg - F_A - R = mg - m_f g - K\eta v_l = 0, \qquad K = 6\pi R$

$\Longrightarrow \quad v_l = \frac{(m-m_f)}{K\eta} g = \frac{\frac{4}{3}\pi R^3(\rho-\rho_f)}{6\pi R\eta} g$

$$v_l = \frac{2}{9}\frac{(\rho - \rho_f)}{\eta} R^2 g$$

- If the terminal velocity is measured " v_l" and a copper bead is used, the viscosity of the fluid (oil or other immersion liquid) can be determined.

B- Newtonian friction: $\vec{F} = -\beta v^2$

The force in this type is proportional to the square of the velocity: $\vec{F} = -\beta v^2 \vec{u}$. This relationship is applicable in the case of high speeds compared to the speed of sound and higher (supersonic):$v \approx 1\, mac$ $v > 1\, mac$.

$$\beta = \frac{1}{2} CA\rho$$

C : constant :**0.4 … 2**, :

A : cross-sectional area

ρ : density of fluid

Example:

A raindrop, supposed to be sphere of radius $R = 1.5mm$, falling under the action of its weight from a height $h = 1200m$.

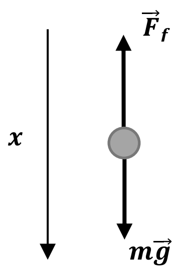
What is its terminal speed?

$\rho_w = 1000kg/m^3 \qquad \rho_a = 1.2kg/m^3$

We have: $\beta = \frac{1}{2} CA\rho$

$\rho = \rho_{aire} = 1.2kg/m^3; \qquad C = 0.6$

cross-sectional area: $A = \pi R^2$



If there is no resistance from the air, the drop will reach the ground at speed:

$$v = \sqrt{2gh} \approx 550 km/h$$

It's a high speed, it approaches the speed of a bullet coming out of a gun.

Because $m_w = \rho_w V = \frac{4}{3}\rho_w \pi R^3$

So, the second Newton's law give:

$$mg - F_f = \frac{4}{3}\rho_w \pi R^3 g - \frac{1}{2}CA\rho v^2 = m\vec{a}$$

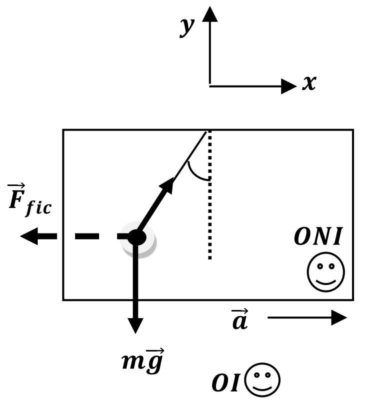
For a limit speed: $\vec{a} = \vec{0}$

$$\Longrightarrow \qquad v_l = \sqrt{\frac{8}{3}\frac{\rho_w}{\rho_a}.\frac{R}{C}g}$$

$$\Longrightarrow \qquad v_l = 7.4\frac{m}{s} \approx 27\frac{km}{h}$$

7.6- Pseudo-forces (Inertial force)

- During the study of relative motion, it was found that the acceleration of a body with respect to the inertial (fixed) frame of reference is: $\vec{a} = \vec{a}_r + \vec{a}_e + \vec{a}_c$.



- Each acceleration induces a force. Then Newton 2nd law gives:

$$\sum_i \vec{F}_i^{ex} = m\vec{a} = m(\vec{a}_r + \vec{a}_e + \vec{a}_c)$$

- In the non-inertial frame of reference Newton's law is:

$$m\vec{a}_r = \sum_i \vec{F}_i^{ex} - m(\vec{a}_e + \vec{a}_c) = \vec{F} - \vec{F}_{in}$$

$\vec{F} = \sum_i \vec{F}_i^{ex}$ Net external force. These are the physical (actual) forces

$\vec{F}_{in}$ The forces of inertia or pseudo-forces. These are the fictitious (virtual) forces

7.6.1- Translational Motion

- For the inertial observer "OI", the mass of the pendulum is subject to the action of the weight and tension of the wire. Its acceleration is the same than that of the car that is produced by the component "T_x" of the tension in the wire

  Newton's law gives:

$$\sum_i \vec{F}_i^{ex} = m\vec{g} + \vec{T} = m\vec{a} \Longrightarrow \begin{cases} T\sin\theta = ma & (1) \\ T\cos\theta = mg & (2) \end{cases}$$

- For the non-inertial observer "ONI" , the wire that suspends the mass which is at rest, despite being in equilibrium, deviates from the vertical "y" by an angle "θ".

  To solve this paradox, he introduces a fictitious (virtual) force in the horizontal direction and asserts that the sum of the acting forces is zero. Newton's law gives:"x "

$$\sum_i \vec{F}_i^{ex} = m\vec{g} + \vec{T} = m\vec{a} = 0 \quad \Longrightarrow \begin{cases} T\sin\theta - F_{fic} = 0 & (3) \\ T\cos\theta - mg = 0 & (4) \end{cases}$$

The two systems of equations (1; 2) and (3; (4) are equivalent only if: $F_{fic} \equiv ma$

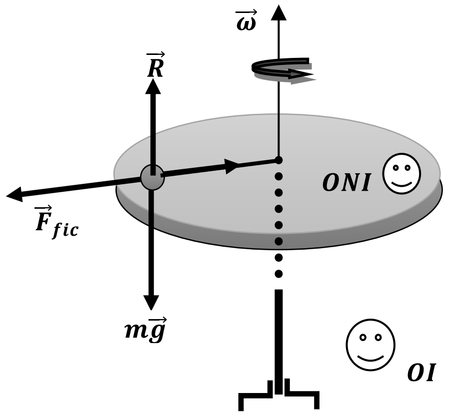
The two systems are mathematically equivalent, but the physical interpretation of the string deflection is different for the two observers (reference frames).

7.6.2- Rotational Movement

A mass "m"hung by means of a wire to the axis of rotation, a smooth plate that rotates at the angular velocity "$\vec{\omega}$". If the speed is constant:

$$\vec{a}_e = [\vec{\omega} \wedge (\vec{\omega} \wedge \vec{r}_1)] = -\omega^2 \vec{r}_1 \qquad \text{and} \qquad \vec{a}_c = 2\vec{\omega} \wedge \vec{v}_r$$

A - Centrifugal force:



- For the inertial observer "OI", the mass subjected to the forces $(m\vec{g}; \vec{R}; \vec{T})$ and rotates at a constant speed, it has a centripetal acceleration: $a = \frac{v^2}{R}$.

- Newton's law gives:

$$\sum_i \vec{F}_i^{ex} = m\vec{g} + \vec{R} + \vec{T} = m(\vec{a}_N + \vec{a}_T)$$

$$\Longrightarrow \begin{cases} \vec{u}_\rho : T = ma_N = m\frac{v^2}{R} & (1) \\ \vec{k} : \; R = mg & (2) \end{cases}$$

- For the non-inertial observer "ONI" attached to the platform, the mass is at rest, but the thread is taut. To resolve this paradox, the observer introduces a fictitious force (virtual) in the horizontal direction and asserts that the sum of the forces is null. Newton's law gives:

$$\sum_i \vec{F}_i^{ex} = m\vec{g} + \vec{T} + \vec{R} + \vec{F}_{fic} = m\vec{a} = 0 \quad \Longrightarrow \quad \begin{cases} T - F_{fic} = 0 & (3) \\ R - mg = 0 & (4) \end{cases}$$

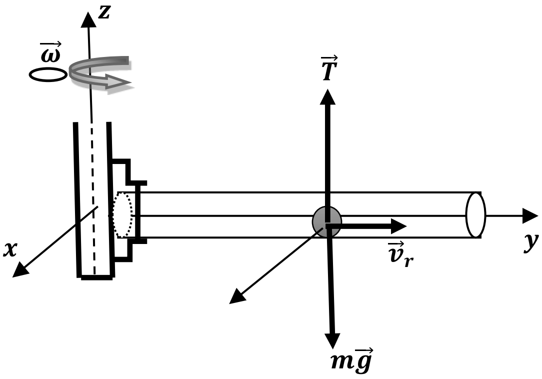
The two systems of equations (1; 2) and (3; (4) are equivalent only if:

$$\boldsymbol{F_{fic} \equiv ma_N = m\frac{v^2}{R}}$$

The force of inertia $\boldsymbol{F_{fic}} = \boldsymbol{F_{in}}$ is equal to the centrifugal force: $\boldsymbol{F_{in}} = \boldsymbol{m}\frac{\boldsymbol{v^2}}{\boldsymbol{R}}$

B – Coriolis's Force

A ball of mass "$\boldsymbol{m}$" moves inside a tube that rotates about the vertical axis at a constant angular velocity "$\vec{\boldsymbol{\omega}}$".



- The inertial observer "$\boldsymbol{OI}$" sees the ball carried away in the tube and it are subject only to its weight and the reaction exerted by the tube.

- The inertial observer "$\boldsymbol{NOI}$" sees that the ball going outwards, tends to stick to the side wall of the tube. The outward movement is explained by the effect of the centrifugal force already mentioned. Its deviation towards the wall is explained by the effect of another force called the Coriolis force $\vec{\boldsymbol{F}}_{\boldsymbol{c}} = \boldsymbol{2}\vec{\boldsymbol{\omega}} \wedge \vec{\boldsymbol{v}}_{\boldsymbol{r}}$