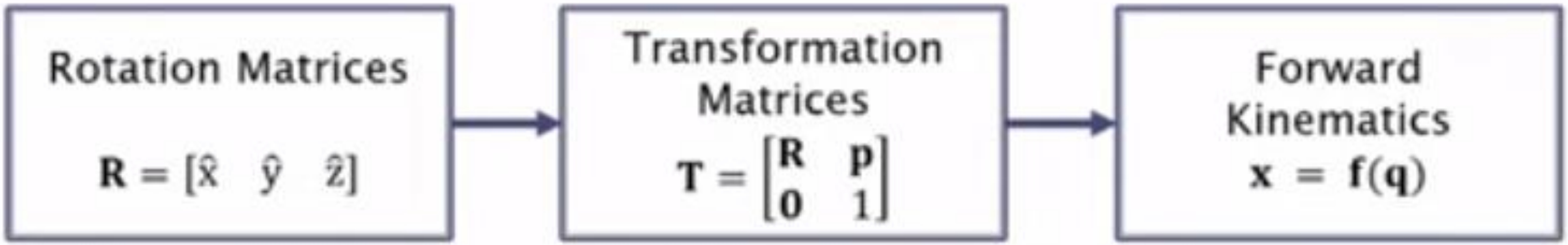
Forward Kinematics



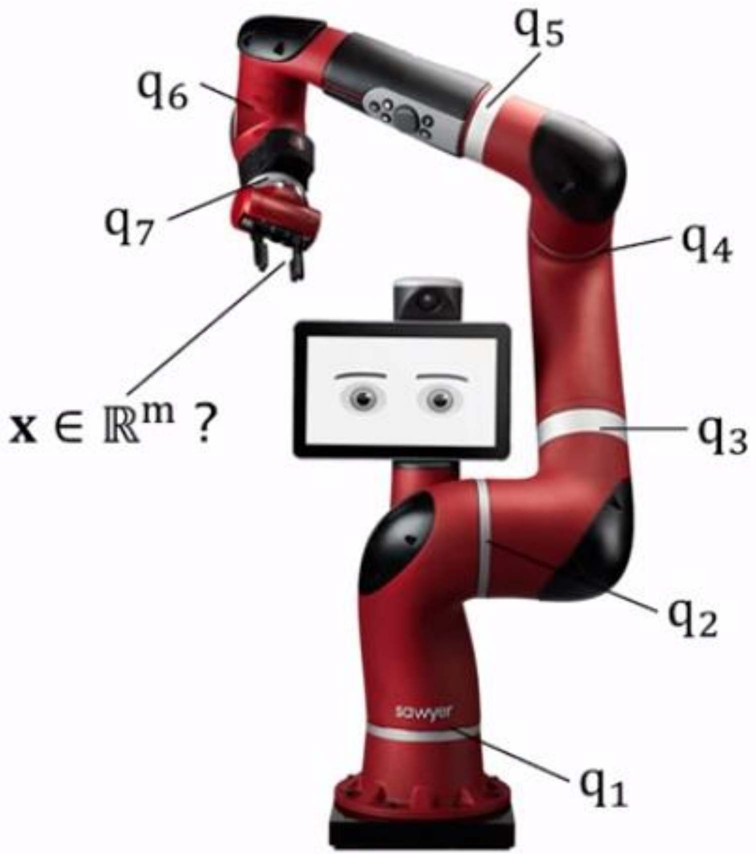
Given a set of joint positions $\mathbf{q}$, what is the pose of the robot tool–tip $\mathbf{x}$?

The Forward Kinematics Problem

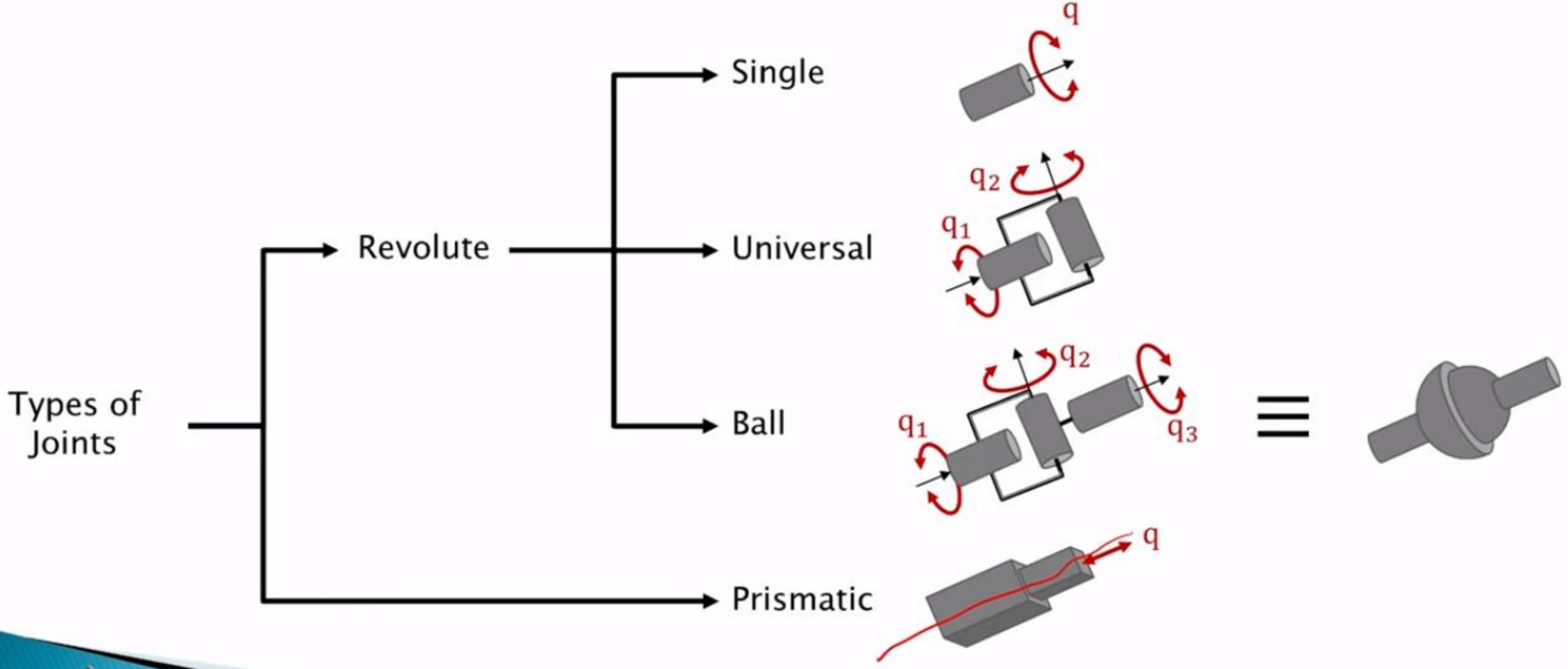
Given a set of joint angles/positions $\mathbf{q} \in \mathbb{R}^n$...

... determine the position and orientation (pose) of the robot tool-tip (end-effector) $\mathbf{x} \in \mathbb{R}^m$.

That is, solve the vector function $\mathbf{x} = \mathbf{f}(\mathbf{q})$, or some equivalent.



Types of Joints



Forward Kinematics of a 2DOF, Planar Manipulator

Task space:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

Joint/control space:

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in \mathbb{R}^2$$

Forward kinematics:

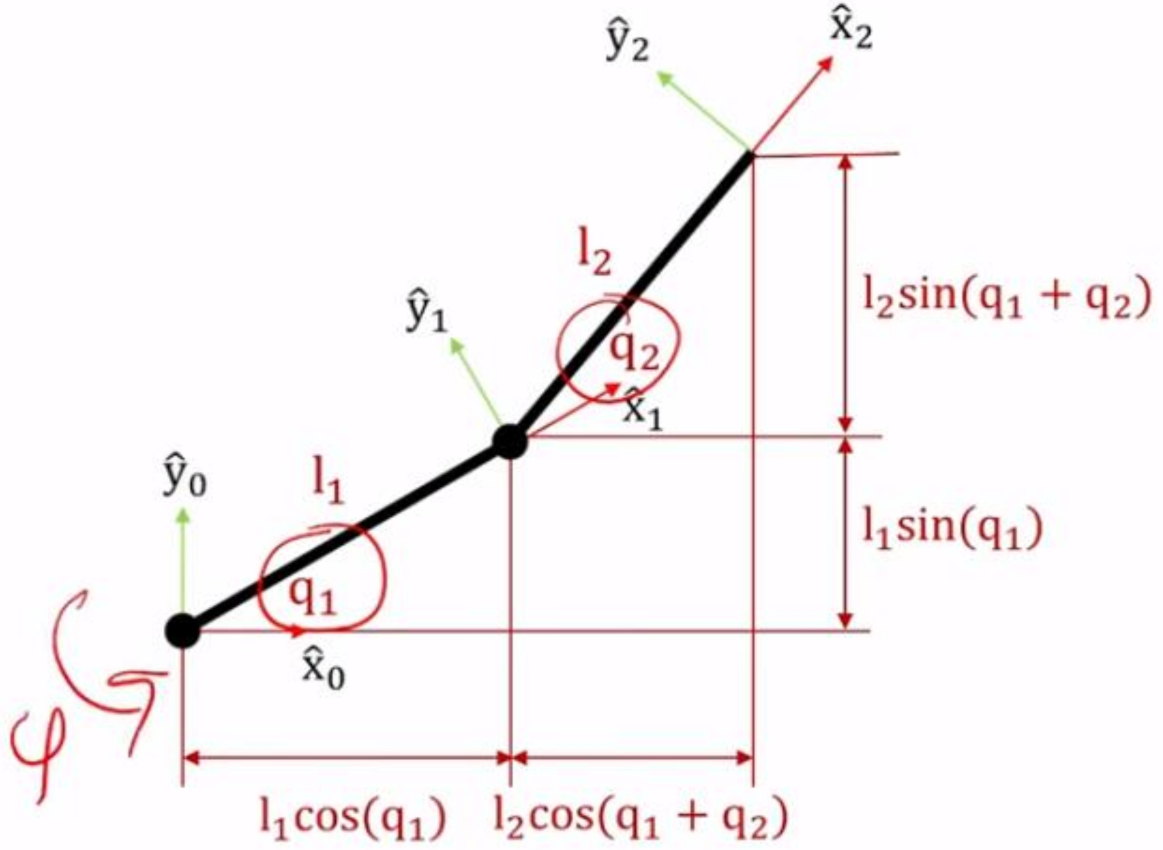
$$\mathbf{x} = \mathbf{f}(\mathbf{q})$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) \\ l_1 \sin(q_1) + l_2 \sin(q_1 + q_2) \end{bmatrix}$$

Simples!

What about orientation?

$$\begin{bmatrix} x \\ y \\ \psi \end{bmatrix} = \begin{bmatrix} l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) \\ l_1 \sin(q_1) + l_2 \sin(q_1 + q_2) \\ q_1 + q_2 \end{bmatrix}$$



Forward Kinematics of a 3DOF Manipulator

Task Space:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

Joint/control space:

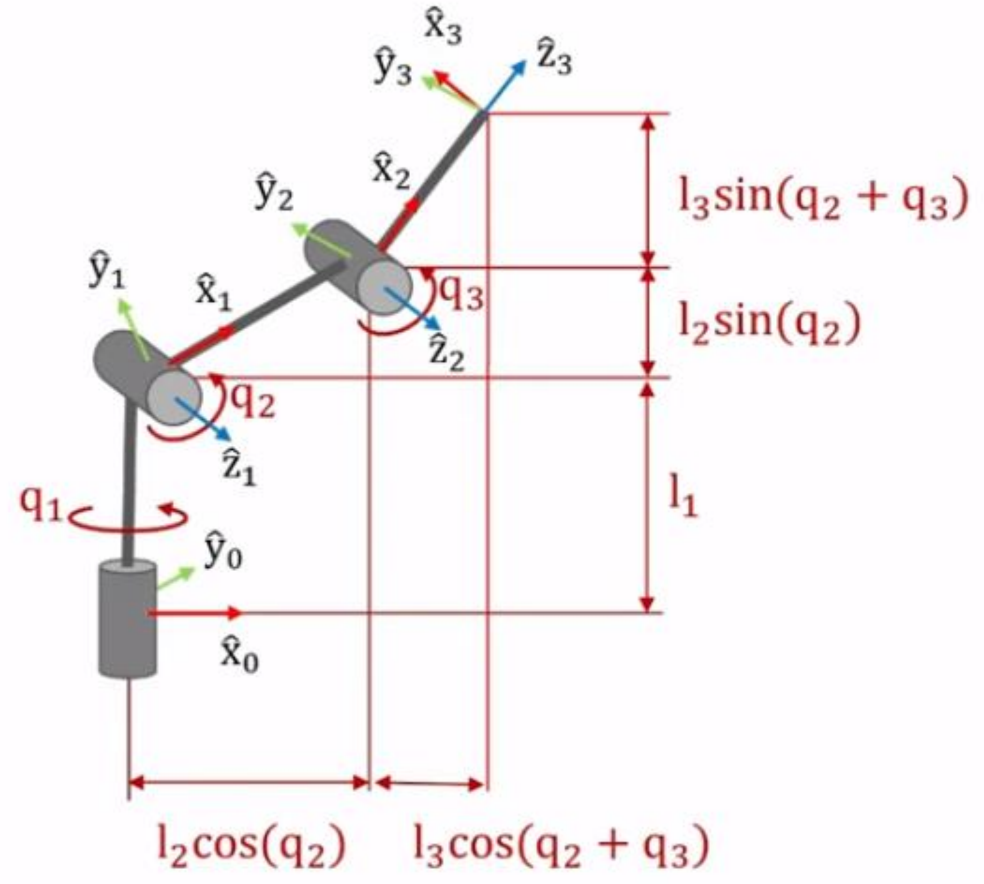
$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \in \mathbb{R}^3$$

Height of end-effector:

$$z = l_1 + l_2 \sin(q_2) + l_3 \sin(q_2 + q_3)$$

Also, distance of end-effector projected on x-y plane:

$$d = l_2 \cos(q_2) + l_3 \cos(q_2 + q_3)$$



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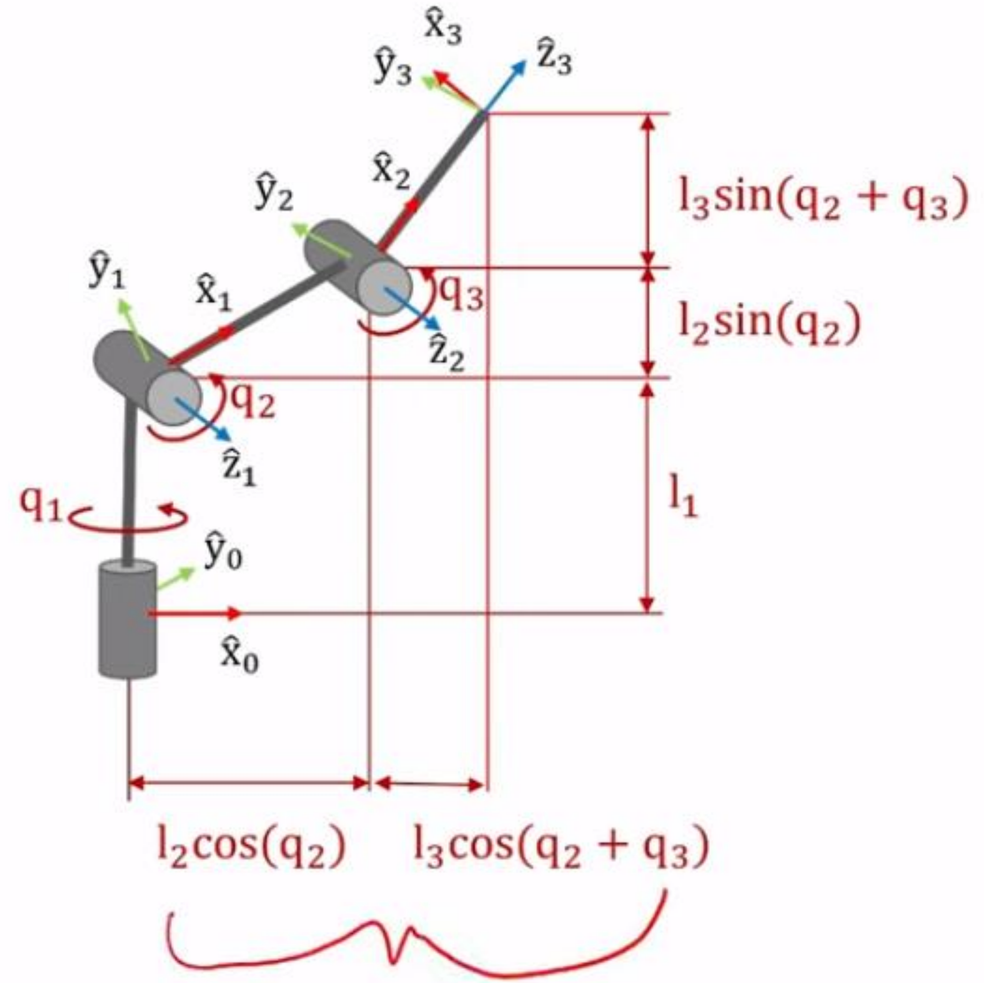


Forward Kinematics of a 3DOF Manipulator

Distance of end-effector projected on x-y plane:

$$d = l_2 \cos(q_2) + l_3 \cos(q_2 + q_3)$$

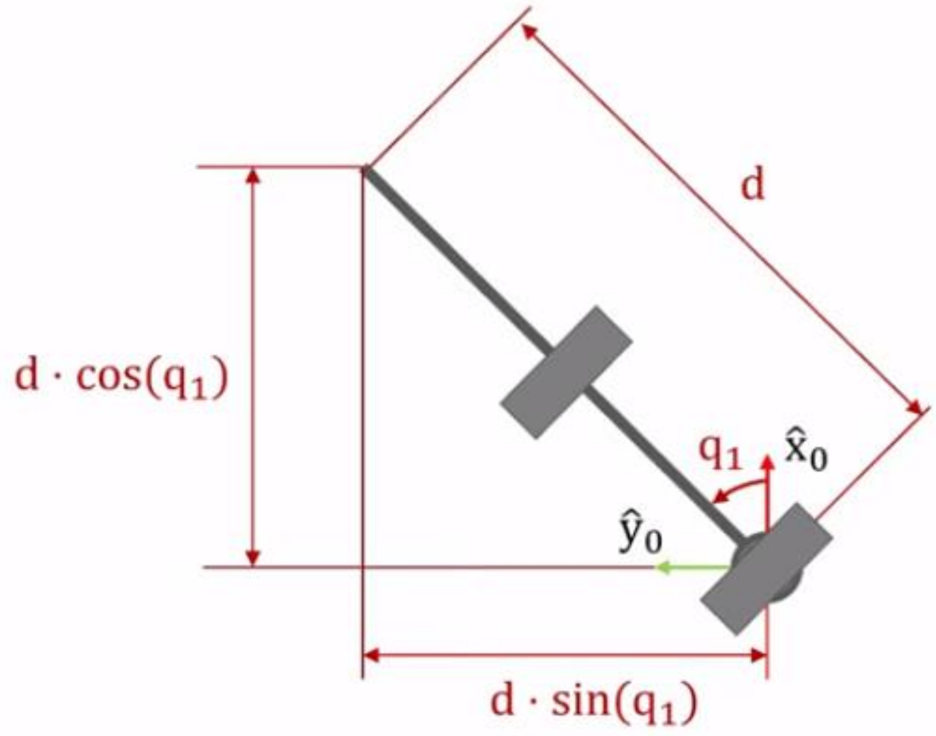
The x and y position of the end-effector is then:

$$\begin{aligned} x &= d \cdot \cos(q_1) \\ &= l_2 \cos(q_1)\cos(q_2) + l_3 \cos(q_1)\cos(q_2 + q_3) \\ y &= d \cdot \sin(q_1) \\ &= l_2 \sin(q_1)\cos(q_2) + l_3 \sin(q_1)\cos(q_2 + q_3) \end{aligned}$$

Forward kinematics:

$$\mathbf{x} = \mathbf{f}(\mathbf{q})$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} l_2 \cos(q_1)\cos(q_2) + l_3 \cos(q_1)\cos(q_2 + q_3) \\ l_2 \sin(q_1)\cos(q_2) + l_3 \sin(q_1)\cos(q_2 + q_3) \\ l_1 + l_2 \sin(q_2) + l_3 \sin(q_2 + q_3) \end{bmatrix}$$



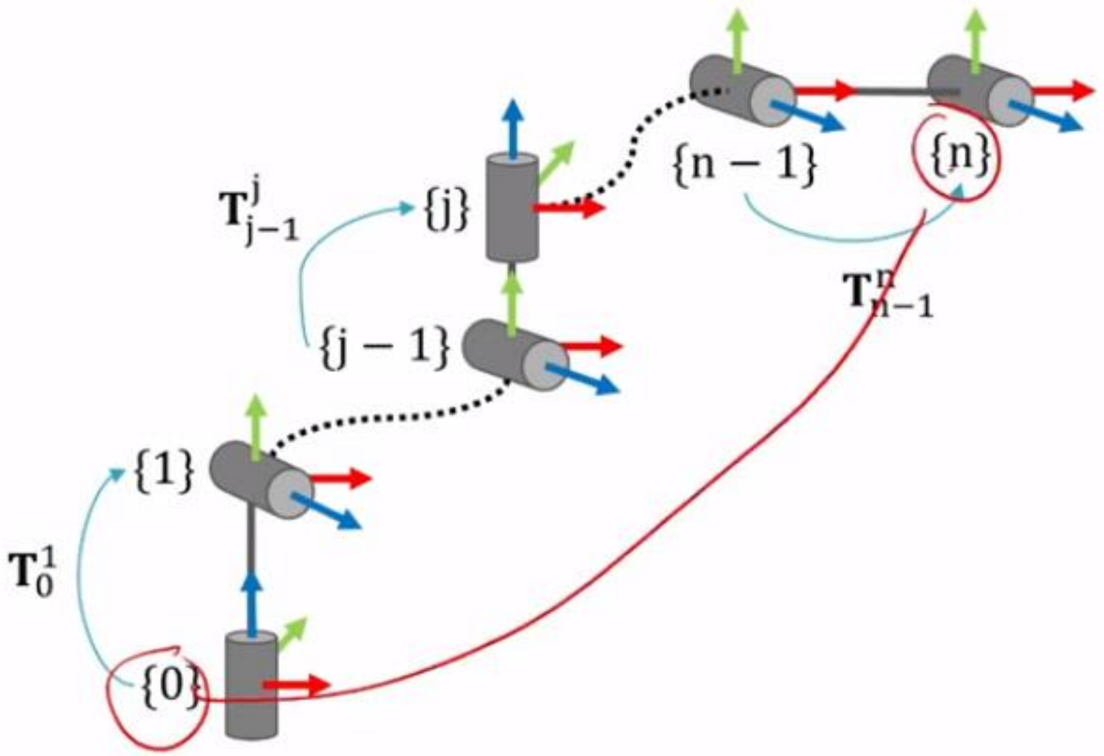
Alternative Approaches to Forward Kinematics

- Problem: Forward kinematics can get tricky
- Need to find a simpler way to derive forward kinematics
- Needs to be applied universally to *any* robot arm

Forward Kinematics Using Transformation Matrices

We can concatenate transformation matrices between joint frames to determine the end-effector pose.

$$\mathbf{T}_0^n = \mathbf{T}_0^1 \times \mathbf{T}_1^2 \times \mathbf{T}_2^3 \times \cdots \times \mathbf{T}_{n-1}^n$$



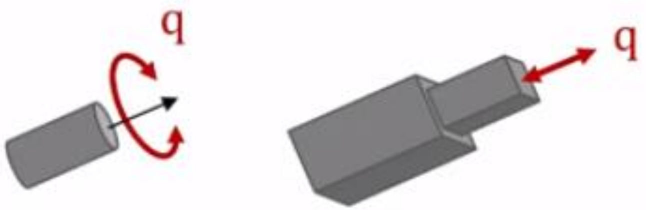
Summary of Forward Kinematics

The general Forward Kinematics (FK) problem expresses the end-effector pose $\mathbf{x} \in \mathbb{R}^m$ as a function of the joint positions $\mathbf{q} \in \mathbb{R}^n$:

$$\mathbf{x} = \mathbf{f}(\mathbf{q})$$

2 types of joints:

- Revolute (single, universal, ball)
- Prismatic



Alternatively, chain the homogeneous transforms from joint-to-joint to get the end-effector pose:

$$\mathbf{T}_0^n = \mathbf{T}_0^1 \times \mathbf{T}_1^2 \times \mathbf{T}_2^3 \times \cdots \times \mathbf{T}_{n-1}^n$$

$$= \prod_{j=1}^{n} \mathbf{T}_{j-1}^{j}$$

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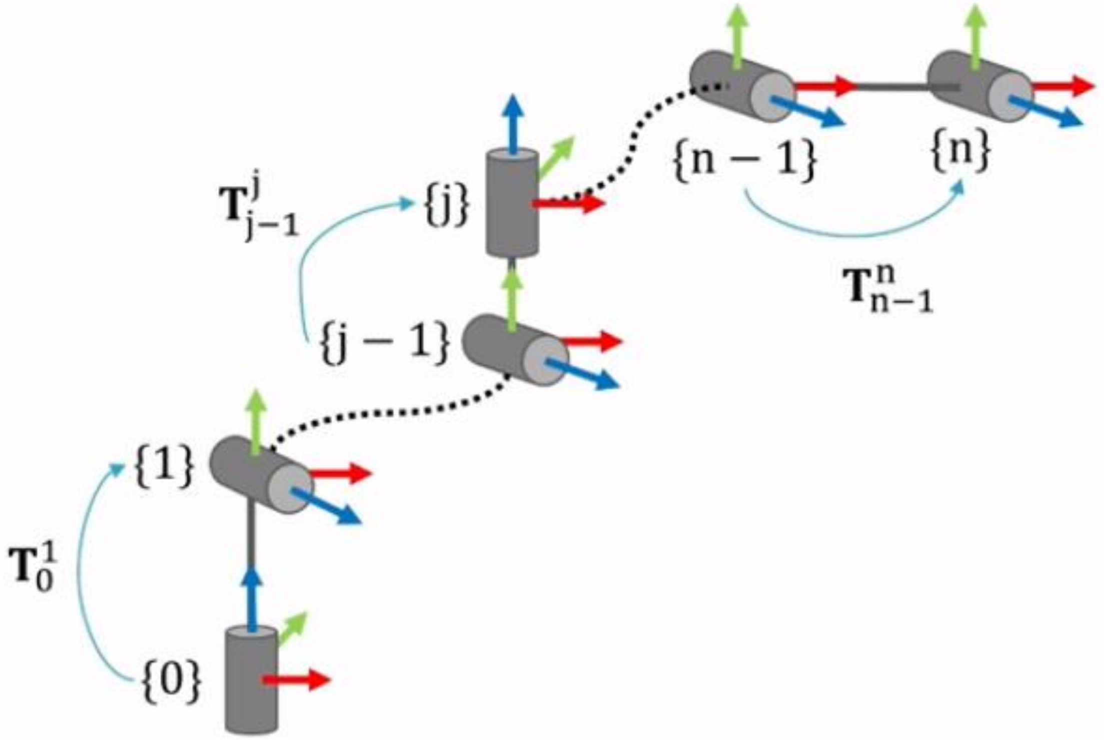
$$= \prod_{j=1}^{n} \mathbf{T}_{j-1}^{j}$$

Need to describe $\mathbf{T}_{j-1}^{j}$ as a function of simple geometry.

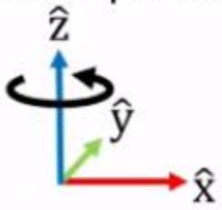


Denavit–Hartenberg (DH) Parameters

Minimum of 4 parameters, applied in sequence:

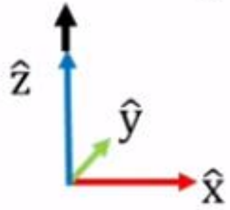
1. Rotate about z-axis by θ

$$\mathbf{T}_{Rz}(\theta) = \begin{bmatrix} \mathbf{R}_z(\theta) & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$$



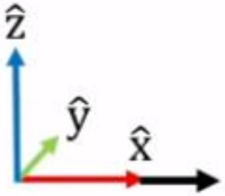
2. Translate across z-axis by d

$$\mathbf{T}_z(d) = \begin{bmatrix} & & & 0 \\ & \mathbf{I} & & 0 \\ & & & d \\ & \mathbf{0}_{1\times3} & & 1 \end{bmatrix}$$



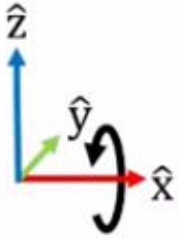
3. Translate across x-axis by a

$$\mathbf{T}_x(a) = \begin{bmatrix} & & & a \\ & \mathbf{I} & & 0 \\ & & & 0 \\ & \mathbf{0}_{1\times3} & & 1 \end{bmatrix}$$



4. Rotate about x-axis by α

$$\mathbf{T}_{Rx}(\alpha) = \begin{bmatrix} \mathbf{R}_x(\alpha) & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$$



For joint {j−1} to {j}:

$$\mathbf{T}_{j-1}^{j} = \mathbf{T}_{Rz}(\theta_j)\mathbf{T}_z(d_j)\mathbf{T}_x(a_j)\mathbf{T}_{Rx}(\alpha_j)$$

Then for the end-effector frame {n}:

$$\mathbf{T}_0^n = \prod_{j=1}^{n} \mathbf{T}_{j-1}^{j}$$

Why DH Parameters?

Minimum number of parameters to describe forward kinematics

Universal nomenclature

- Knowing DH parameters gives complete knowledge of kinematics
- Easily understood by anyone

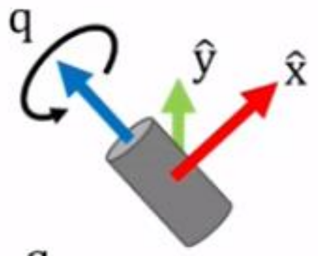
Computationally efficient for differential kinematics, dynamics

- Need to minimize no. of calculations for high-frequency feedback control

Rules for DH Parameters

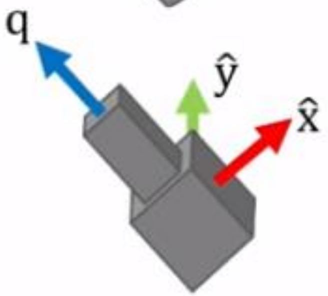
1. Actuate about z-axis
   - Rotate about z for revolute joints

$$\mathbf{T}_{Rz}(q) = \begin{bmatrix} \mathbf{R}_z(q) & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$$

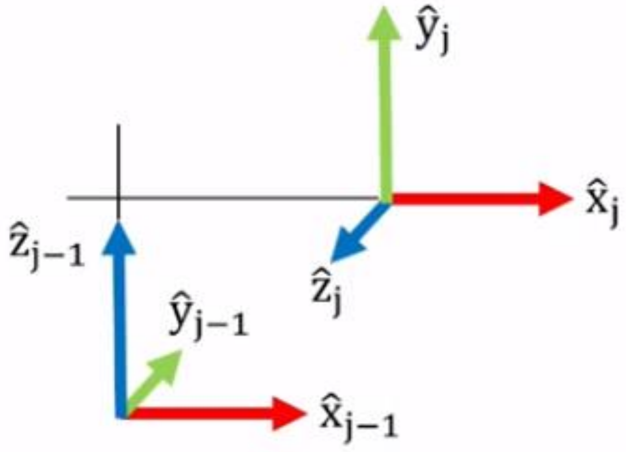


   - Translate about z for prismatic joints

$$\mathbf{T}_z(q) = \begin{bmatrix} & & & 0 \\ & \mathbf{I} & & 0 \\ & & & q \\ & \mathbf{0}_{1\times3} & & 1 \end{bmatrix}$$



2. Axis $\hat{z}_{j-1}$ is perpendicular to, and intersects, $\hat{x}_j$
3. The y-axis is solved implicitly: $\hat{y}_j = \hat{z}_j \times \hat{x}_j$

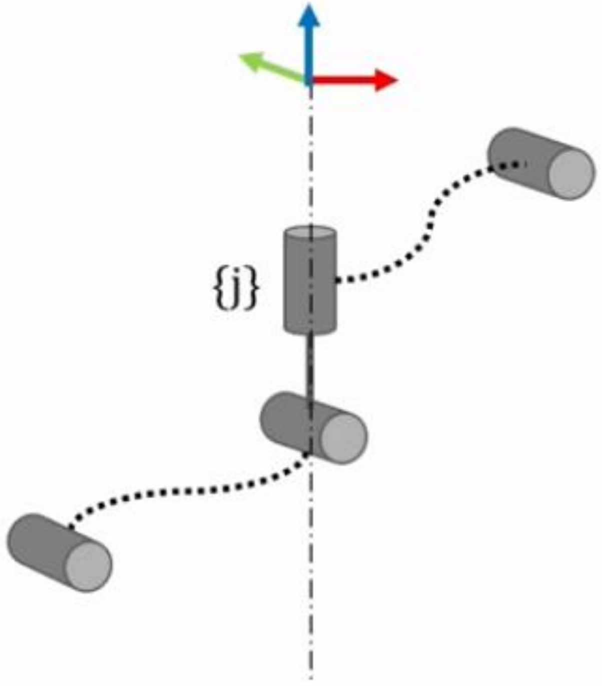


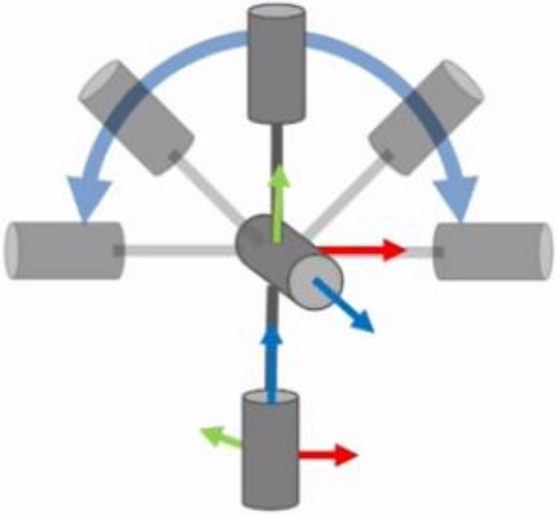
Tips for DH Parameters

- The joint frame does not need to physically coincide with the actual joint.

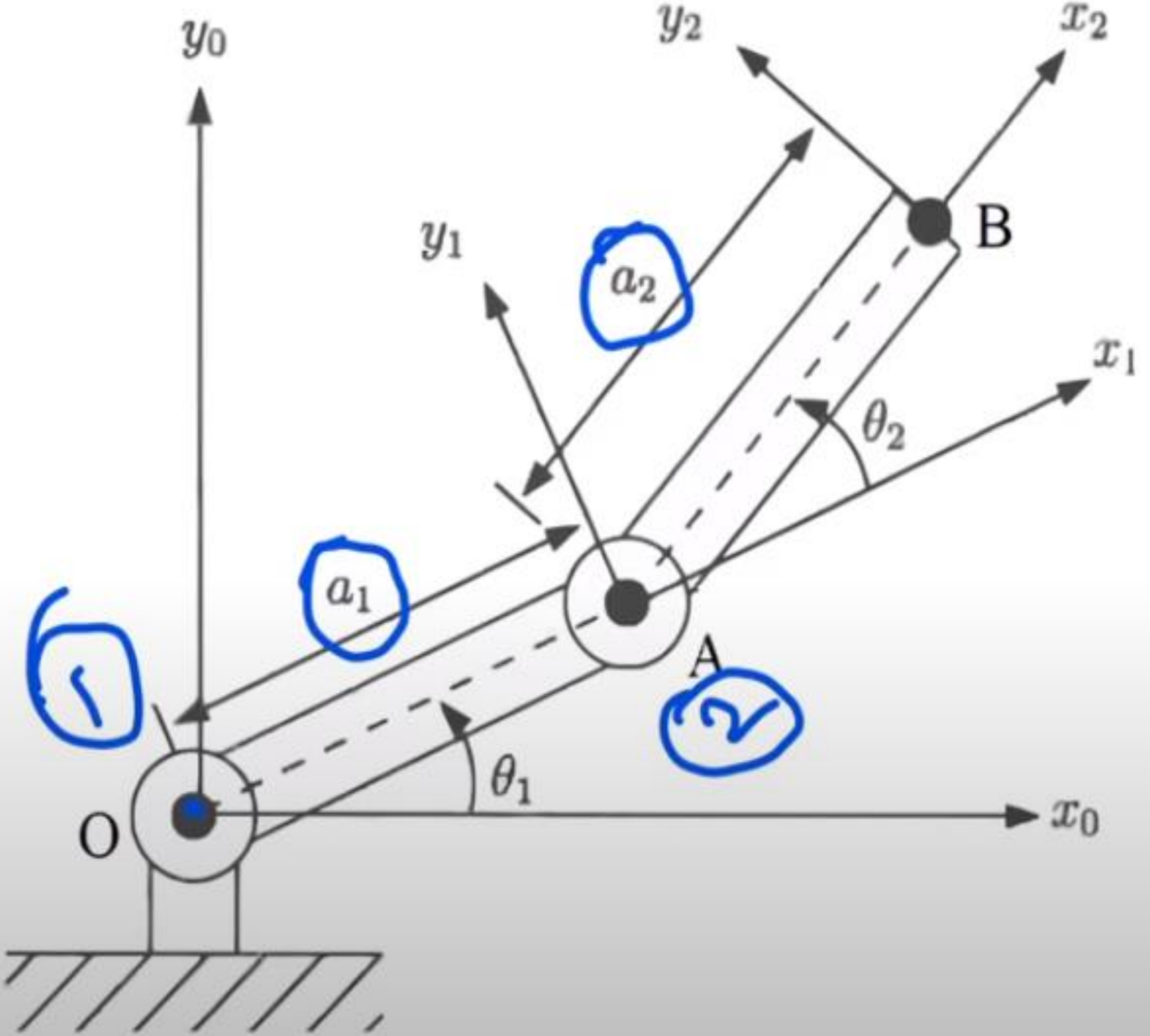
  It only needs to align with the axis of actuation.

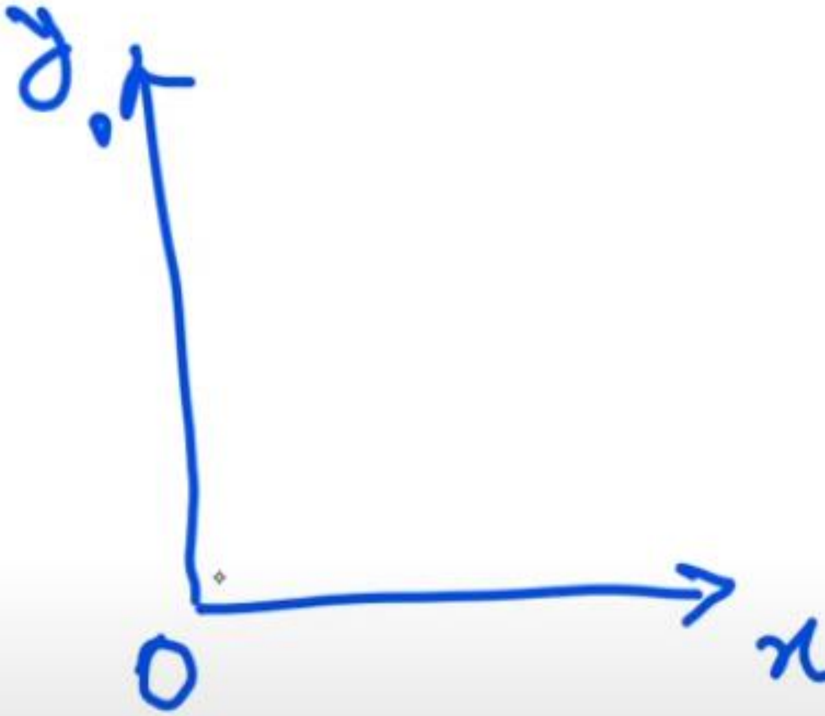
- The robot arm can be arranged in any configuration that suits the DH parameters.





Example 1: Two link planar manipulator (2-DoF)





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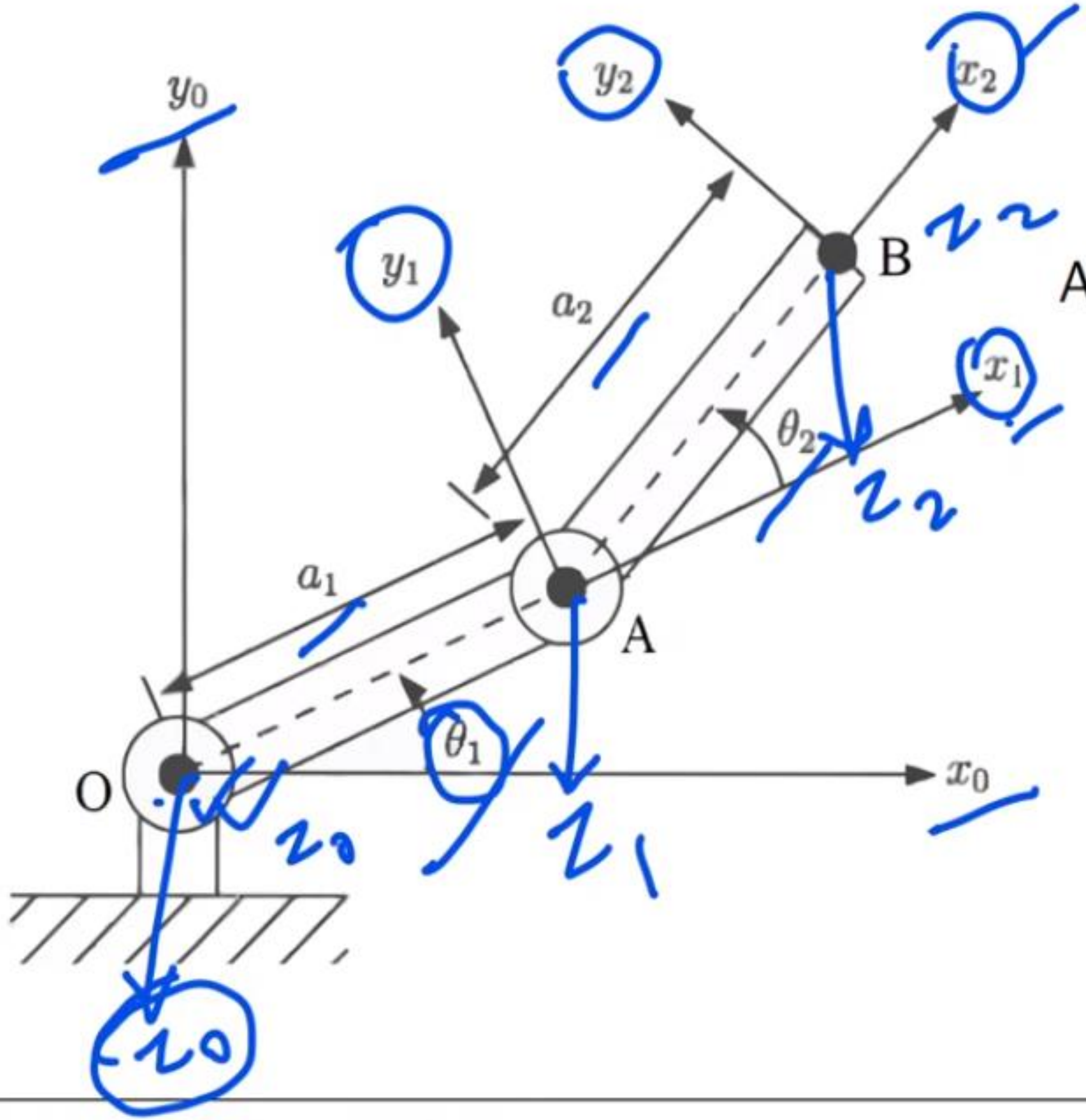


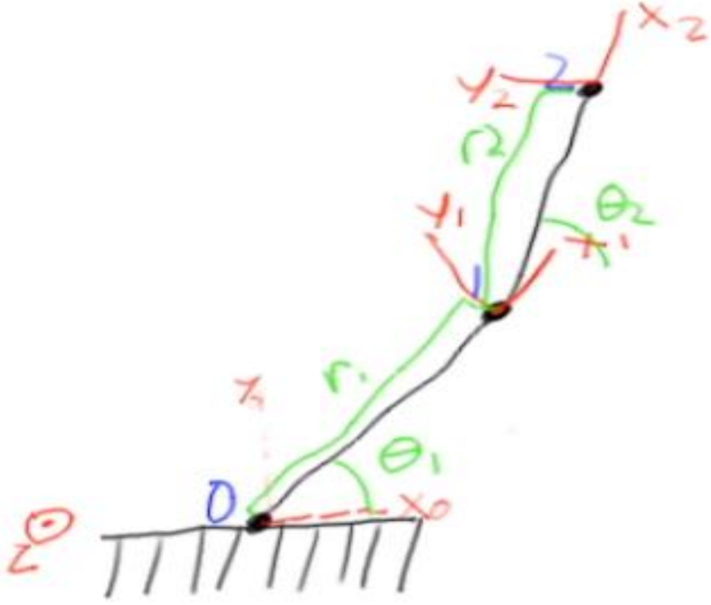
Assigning frames:

- z_0 and z_1 are axis of rotation of joint 1 and 2
- axes are parallel
- choose x_i to intersect 0_{i-1}
- y_i is normal to form a R.H coordinate system
- The links are planar (no twist) $\alpha_i = 0$

Link	r_i	α_i	d_i	θ_i
1	r_1	0	0	θ_1
2	r_2	0	0	θ_2

$$A_1 = \begin{bmatrix} C_{\theta_1} & -S_{\theta_1} & 0 & r_1 C_{\theta_1} \\ S_{\theta_1} & C_{\theta_1} & 0 & r_1 S_{\theta_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

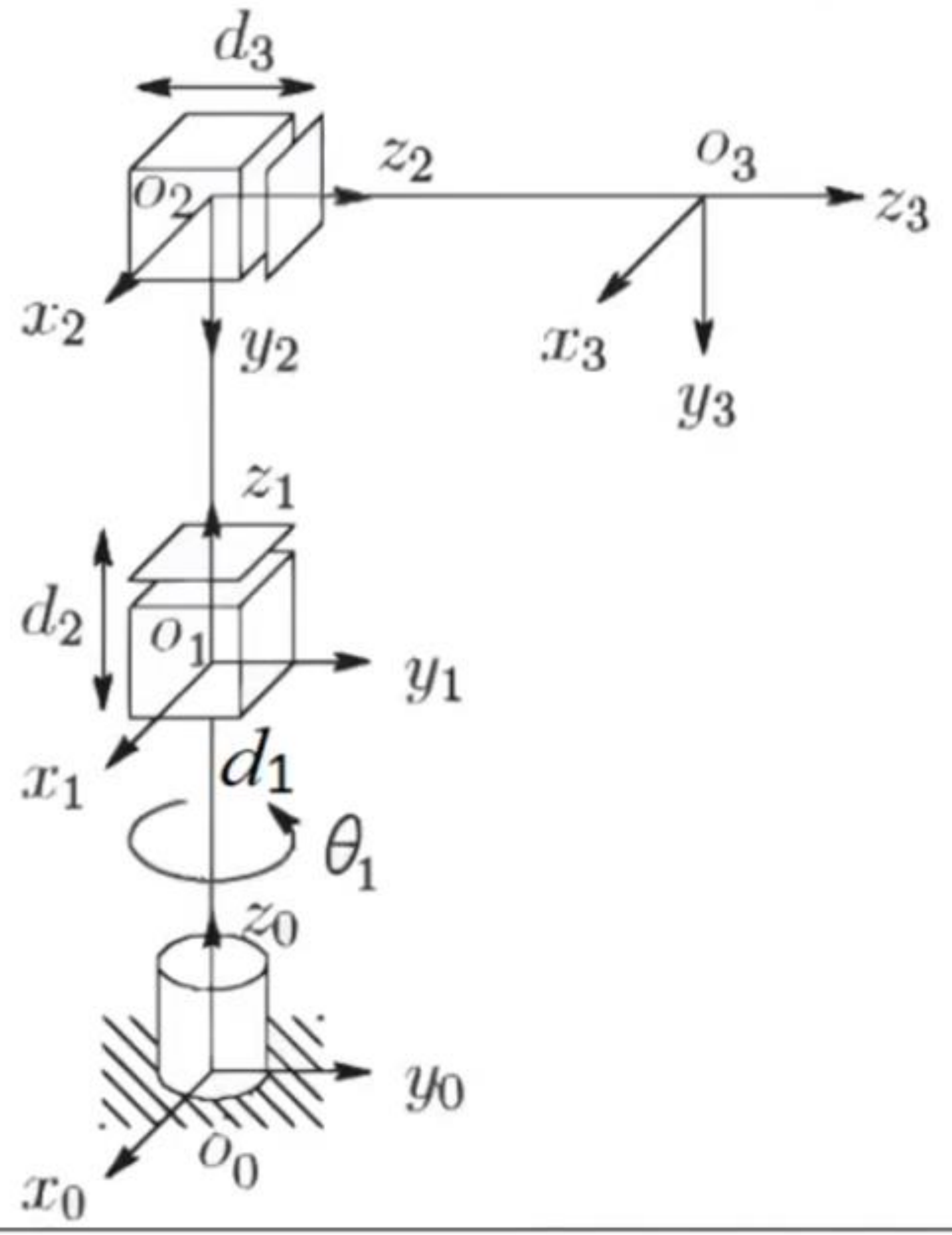
$$A_2 = \begin{bmatrix} C_{\theta_2} & -S_{\theta_2} & 0 & r_2 C_{\theta_2} \\ S_{\theta_2} & C_{\theta_2} & 0 & r_2 S_{\theta_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$${}^0_2T = A_1 A_2$$

$$= \begin{bmatrix} C_{\theta_1+\theta_2} & -S_{\theta_1+\theta_2} & 0 & r_1 C_{\theta_1} + r_2 C_{\theta_1+\theta_2} \\ S_{\theta_1+\theta_2} & C_{\theta_1+\theta_2} & 0 & r_1 S_{\theta_1} + r_2 S_{\theta_1+\theta_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2: Three link cylindrical robot



- Choose $\mathbf{z}_i$ as axis for rotation or translation.
- Follow R.H co-ordinate system
- First define $\mathbf{a}_i$, α_i constants
- define constants θ_i and $\mathbf{d}_i$

Table: DH parameters

Link	$\mathbf{a}_i$	α_i	$\mathbf{d}_i$	θ_i
1	0	0	d_1	θ_1^*
2	0	-90^0	d_2^*	0
3	0	0	d_3^*	0

*Joint variable

$$
{}^{0}\mathbf{A}_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}\mathbf{A}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}\mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$\because \sin(\alpha_2) = -1$

$$
{}^{0}\mathbf{T}_3 = {}^{0}\mathbf{A}_1 \, {}^{1}\mathbf{A}_2 \, {}^{2}\mathbf{A}_3
$$

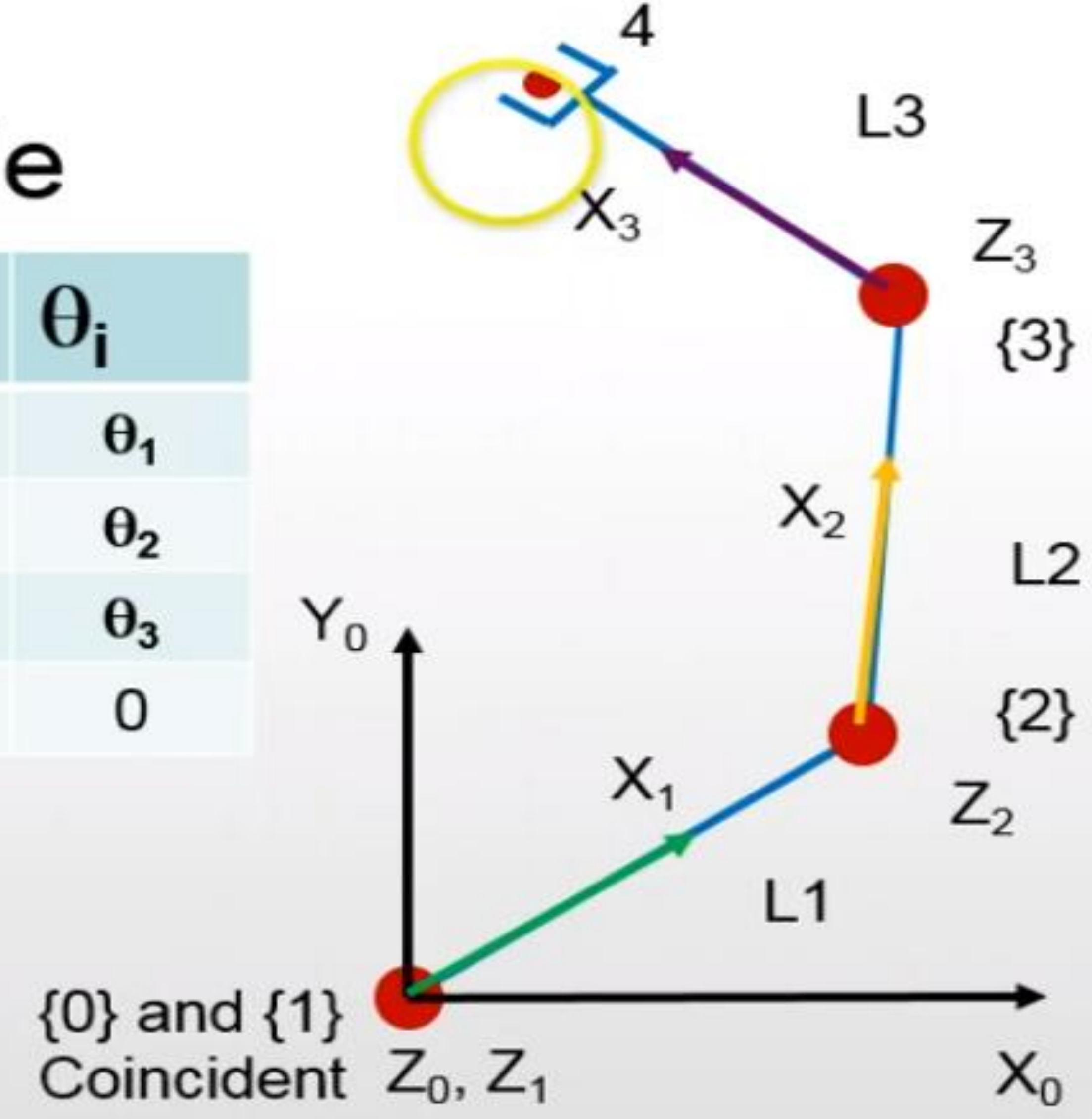
u v w

$$
= \begin{bmatrix} C_1 & 0 & -S_1 & -d_3 S_1 \\ S_1 & 0 & C_1 & d_3 C_1 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

3 DOF Planer robotic arm

DH Parameter Table

Joint i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	L1	0	0	θ_2
3	L2	0	0	θ_3
4	L3	0	0	0



$$
{}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & L1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{2}_{3}T = \begin{bmatrix} c_3 & -s_3 & 0 & L_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{3}_{4}T = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{4}T = \begin{bmatrix} c_{123} & -s_{123} & 0 & L_3 c_{123} + L_2 c_{12} + L_1 c_1 \\ s_{123} & c_{123} & 0 & L_3 s_{123} + L_2 s_{12} + L_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$