Exercise 3.1 In terms of the $\hat{x}_s$, $\hat{y}_s$, $\hat{z}_s$ coordinates of a fixed space frame {s}, the frame {a} has its $\hat{x}_a$-axis pointing in the direction $(0,0,1)$ and its $\hat{y}_a$-axis pointing in the direction $(-1,0,0)$, and the frame {b} has its $\hat{x}_b$-axis pointing in the direction $(1,0,0)$ and its $\hat{y}_b$-axis pointing in the direction $(0,0,-1)$.

(a) Draw by hand the three frames, at different locations so that they are easy to see.

(b) Write down the rotation matrices R_{sa} and R_{sb}.

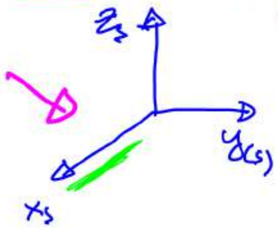
(c) Given R_{sb}, how do you calculate R_{sb}^{-1} without using a matrix inverse? Write down R_{sb}^{-1} and verify its correctness using your drawing.

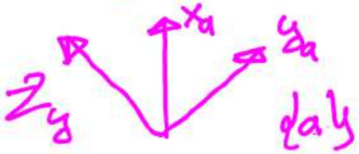
(d) Given R_{sa} and R_{sb}, how do you calculate R_{ab} (again without using matrix inverses)? Compute the answer and verify its correctness using your drawing.

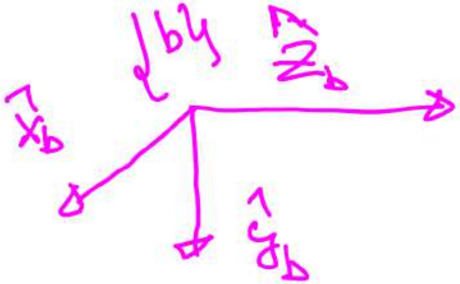
(e) Let $R = R_{sb}$ be considered as a transformation operator consisting of a rotation about $\hat{x}$ by $-90°$. Calculate $R_1 = R_{sa}R$, and think of R_{sa} as a representation of an orientation, R as a rotation of R_{sa}, and R_1 as the new orientation after the rotation has been performed. Does the new orientation R_1 correspond to a rotation of R_{sa} by $-90°$ about the world-fixed $\hat{x}_s$-axis or about the body-fixed $\hat{x}_a$-axis? Now calculate $R_2 = RR_{sa}$.

$$\underset{sa}{}$$

$\{a\}$ $\hat{x}\,(0,0,1)$, $\hat{y}\,(-1,0,0)$, $\{b\}$ $\hat{x}_b\,(1,0,0)$ $\hat{y}_b\,(0,0,-1)$ $\; sb$







b. $R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad \underline{\underline{R_{sb}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

c. $\underline{\underline{R_{sb}^{-1}}} = \underline{\underline{R_{sb}^{T}}} = R_{bs} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

d. $R_{ab} = R_{as} R_{sb} = R_{sa}^{T} R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

e) $R_1 = R_{sa} R = R_{sa} R_{sb} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$R_2 = R R_{sa} = R_{sb} R_{sa}$

(f) Use R_{sb} to change the representation of the point $p_b = (1, 2, 3)$ (which is in {b} coordinates) to {s} coordinates.

(g) Choose a point p represented by $p_s = (1, 2, 3)$ in {s} coordinates. Calculate $p' = R_{sb} p_s$ and $p'' = R_{sb}^{\mathrm{T}} p_s$. For each operation, should the result be interpreted as changing coordinates (from the {s} frame to {b}) without moving the point p or as moving the location of the point without changing the reference frame of the representation?

(h) An angular velocity w is represented in {s} as $\omega_s = (3, 2, 1)$. What is its representation ω_a in {a}?

(i) By hand, calculate the matrix logarithm $[\hat{\omega}]\theta$ of R_{sb}. (You may verify your

f) $p = R_{sb}\, p_b = (1, 3, -2)^T$

g) $$p' = R_{sb}\, p_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$p'' = R_{sb}^T\, p_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

h) $$R_{as}\, w_s = w_a = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$$

$$R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad R_{sa}^T = R_{as}$$

Exercise 3.2 Let p be a point whose coordinates are $p = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right)$ with respect to the fixed frame x̂–ŷ–ẑ. Suppose that p is rotated about the fixed-frame x̂-axis by 30 degrees, then about the fixed-frame ŷ-axis by 135 degrees, and finally about the fixed-frame ẑ-axis by −120 degrees. Denote the coordinates of this newly rotated point by p'.

(a) What are the coordinates p'?

(b) Find the rotation matrix R such that $p' = Rp$ for the p' you obtained in (a).

Exercise 3.3 Suppose that $p_i \in \mathbb{R}^3$ and $p_i' \in \mathbb{R}^3$ are related by $p_i' = Rp_i$, $i = 1, 2, 3$, for some unknown rotation matrix R. Find, if it exists, the rotation R for the three input–output pairs $p_i \mapsto p_i'$, where

$$
\begin{aligned}
p_1 = (\sqrt{2}, 0, 2) \quad &\mapsto \quad p_1' = (0, 2, \sqrt{2}), \\
p_2 = (1, 1, -1) \quad &\mapsto \quad p_2' = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right), \\
p_3 = (0, 2\sqrt{2}, 0) \quad &\mapsto \quad p_3' = (-\sqrt{2}, \sqrt{2}, -2).
\end{aligned}
$$

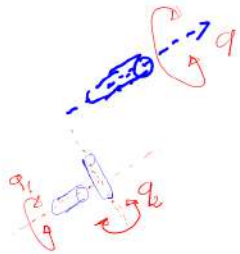
a) $$p' = \vec{Rot}_z\, \vec{Rot}_y\, \vec{R}_x\, p = \begin{bmatrix} \cos(-120) & -\sin(-120 & 0 \\ \sin(-120 & \cos(-120) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(135) & 0 & \sin(135) \\ 0 & 1 & 0 \\ -\sin(135) & 0 & \cos(135) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & \sin 30 & \cos 30 \end{bmatrix} p = \begin{bmatrix} -0{,}55 \\ 0{,}46 \\ -0{,}696 \end{bmatrix}$$

b) $$R = Rot_z\, Rot_y\, R_x = \begin{bmatrix} 0.35 & 0{,}57 & -0{,}74 \\ 0{,}61 & -0{,}74 & -0{,}28 \\ -0{,}7 & -0{,}35 & -0{,}61 \end{bmatrix}$$

Forward Kinematics

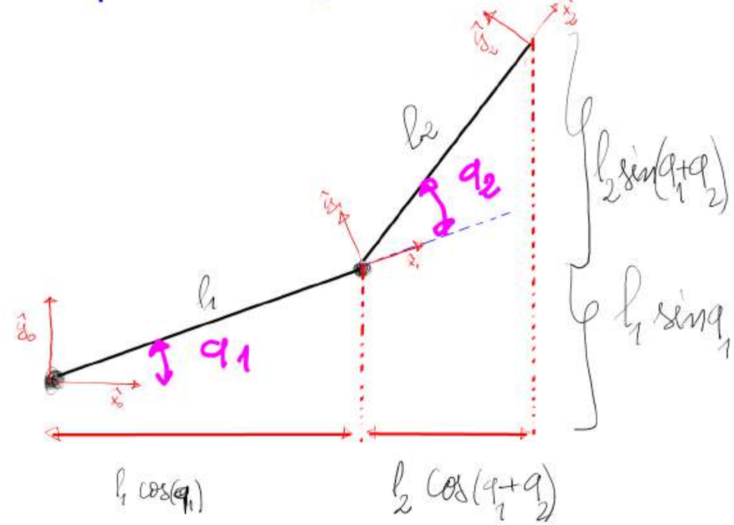
Rotation Matrices $R = [\hat{x}\ \ \hat{y}\ \ \hat{z}]$ → Transformation Matrices $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$ → Forward Kinematics $x = f(q)$

Types of Joints →
- Revolute →
  - Single
  - Universal
  - Ball
- Prismatic



Forward Kinematics : $x = f(q)$ of 2 DOF Planar Manipulator Robot

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$



$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{matrix} l_1 \cos(q_1) + l_2 \cos(q_1+q_2) \\ l_1 \sin(q_1) + l_2 \sin(q_1+q_2) \end{matrix}$$

$$\varphi = q_1 + q_2$$

Forward Kinematics of 3DOF Manipulator