Contents

Chapter 1

Simple, double and triple Integrals

Section 1.1 General

Definition: Let f be a continuous function on an interval $I \subseteq \mathbb{R}$. We say that a function F is an antiderivative (primitives) of f if and only if $F'(x) = f(x)$ on I or,

$$\int f = F + C \text{ where } C \in \mathbb{R}.$$

$f(x)$	$\int f(x)dx = F(x) + C$	$f(x)$	$\int f(x)dx = F(x) + C$
a (cste)	$ax + C$	$\frac{1}{x^2+a^2}$	$\frac{1}{a} arctg \frac{x}{a} + C \quad a \neq 0$
x^n	$\frac{x^{n+1}}{n+1} + C \quad n \neq -1$		$-\frac{1}{a} arcctg \frac{x}{a} + C \quad a \neq 0$
$\frac{1}{x}$	$ln\|x\| + C$	$\frac{1}{a^2 - x^2}$	$\frac{1}{2a} ln \left\|\frac{a+x}{a-x}\right\| + C \quad a \neq 0$
$\frac{1}{x+a}$	$ln\|x+a\| + C$	$\frac{1}{\sqrt{x^2 \pm a}}$	$ln \left\|x + \sqrt{x^2 \pm a}\right\| + C$ $a \neq 0$
a^x	$\frac{1}{lna} a^x + C \quad a > 0$		
e^x	$e^x + C$	$\frac{1}{\sqrt{a - x^2}}$	$arcsin \frac{x}{a} + C \quad a > 0$
			$-arccos \frac{x}{a} + C \quad a > 0$
e^{ax+b}	$\frac{1}{a} e^{ax+b} + C$	$\frac{1}{sinx}$	$ln \left\|tg \frac{x}{2}\right\| + C$
$sinx$	$-cosx + C$	$\frac{1}{cosx}$	$ln \left\|tg \frac{x}{2} + \frac{\pi}{4}\right\| + C$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + C$	$\frac{1}{sin^2 x}$	$-ctgx + C$
$cosx$	$sinx + C$	$\frac{1}{cos^2 x}$	$tgx + C$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + C$		

Figure 1.1: Integral of f.

Definition: The definite integral of a function f on the interval $[a,b]$ is a real number denoted as

$$\mathbf{S} = \int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

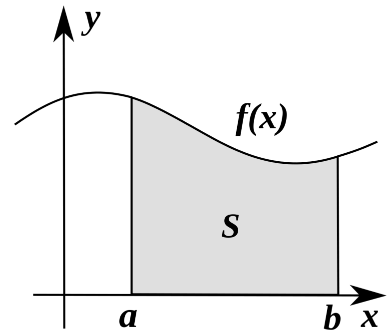

Figure 1.2: Integral of f from a to b.

Properties: Let f and g be two continuous functions on $[a,b]$, and let α and β be real numbers.

1. $\int [\alpha f(x) + \beta g(x)]\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$ for $a \le x \le b$.
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ for $a \le b$.
3. $\int_a^a f(x)\,dx = 0$ for any a.
4. $\int_{-a}^a f(x)\,dx = 0$ if f is an odd function.
5. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ if f is an even function.
6. If $f \ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge 0$.
7. If $f \le g$ on $[a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.
8. $|\int_a^b f(x)\,dx| \le \int_a^b |f(x)|\,dx$.

1.1.1 Integration methods

Integration by Parts: Let u and v be two functions of class C^1 on $[a,b]$. The integration by parts formula is given by:

$$\int_a^b uv' = [uv]_a^b - \int_a^b u'v.$$

Example 1.1 . *Evaluate the integral* $\int_0^\pi x\sin(x)\,dx$.

$$\begin{aligned}
\int_0^\pi x\sin(x)\,dx &= [-x\cos(x)]_0^\pi + \int_0^\pi \cos(x)\,dx \quad \textit{(Integration by parts)} \\
&= -\pi\cos(\pi) - (-0\cos(0)) + \int_0^\pi \cos(x)\,dx \\
&= \pi + \int_0^\pi \cos(x)\,dx \\
&= \pi + [\sin(x)]_0^\pi \\
&= \pi + (\sin(\pi) - \sin(0)) \\
&= \pi + (0 - 0) \\
&= \pi.
\end{aligned}$$

Therefore, $\int x\sin(x)\,dx = \pi$.

Integration by Change of Variable: Let f be a continuous function on $I \subseteq \mathbb{R}$, and let $u : J \subseteq \mathbb{R} \to I$ be a C^1 function on J with $u(t) = x$.

Then, $\int_a^b f(x)dx\,dx = \int_{u^{-1}(a)}^{u^{-1}(b)} f(u(t))u'(t)\,dt$.

Example 1.2 . *Evaluate the integral* $\int_0^1 \sqrt{1-x^2}\,dx$. *Let* $x = u(t) = \sin(t)$ *such that* $u'(t) = \cos(t)$ *and* $x = 0 \to t = 0$ *and* $x = 1 \to t = \frac{\pi}{2}$.

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2(t)}\cos(t)\,dt = \int_0^{\frac{\pi}{2}} \cos^2(t)\,dt.$$

Using the identity $\cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1$, *we get* $\cos^2(t) = \frac{1}{2}(1+\cos(2t))$.

$$\int_0^{\frac{\pi}{2}} \cos^2(t)\,dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos(2t))\,dt = \frac{1}{2}\left[t + \frac{1}{2}\sin(2t)\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

Therefore, $\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$.

1.1.2 Primitives of functions

Rational functions

Partial fraction decomposition:

Partial fraction decomposition is a method used to express a rational function as the sum of simpler fractions. The general form of a rational function is

$$\frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials.

The process of partial fraction decomposition involves expressing the rational function as the sum of fractions with simpler denominators. There are three main cases:

1. **Distinct Linear Factors:** If $Q(x)$ can be factored into distinct linear factors, i.e.,

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n),$$

then the partial fraction decomposition is of the form:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n},$$

where $A_1, A_2, \ldots, A_n$ are constants.

2. **Repeated Linear Factors:** If the denominator $Q(x)$ has repeated linear factors, i.e.,

$$Q(x) = (a_1x+b_1)^{m_1}(a_2x+b_2)^{m_2}\cdots(a_nx+b_n)^{m_n},$$

then the partial fraction decomposition includes terms of the form:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_{m_1}}{(a_1x+b_1)^{m_1}} + \cdots,$$

for each repeated factor.

where $Ax+B$, $Cx+D$, $Ex+F$, etc., are linear polynomials.

The next step is to find the constants (such as $A_1, A_2, \ldots, A_n$) by clearing the fractions and equating coefficients.

Example 1.3 *Determination of constants A, B, C, D and E (by identification):*

1. *Distinct Linear Factors*

$$\frac{1+x}{x(x+5)} = \frac{A}{x} + \frac{B}{(x+5)} \Rightarrow \frac{1+x}{x(x+5)} = \frac{(A+B)x+5A}{x(x+5)} \Rightarrow \begin{cases} A+B=1 \\ 5A=1 \end{cases}.$$

Solving the system, we find $B = \frac{4}{5}$ *and* $A = \frac{1}{5}$*. Therefore,*

$$\frac{1+x}{x(x+5)} = \frac{1}{5x} + \frac{4}{5(x+5)}.$$

2. *Repeated Linear Factors*

$$\frac{1+x}{x^3(x-5)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{(x-5)} + \frac{E}{(x-5)^2}.$$

Integration of rational functions: Any rational fraction $P(x)/Q(x)$ can be expressed as the sum of a polynomial and simple elements of two types:

1. First type: $\frac{1}{(x+\alpha)^n}$.

$$\int \frac{1}{(x+\alpha)^n} dx = \begin{cases} \ln|x+a| + c & \text{if } n = 1 \\ -\frac{1}{n-1}\frac{1}{(x+a)^{n-1}} + c & \text{if } n > 1. \end{cases}$$

where a, b, α, β,$\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$ is a positive integer.

Example 1.4 .

1. $f(x) = \frac{1+x}{x(x+5)}$,

$$\int f(x)dx = \int \frac{1+x}{x(x+5)}dx = \int \frac{1}{5x}dx + \int \frac{4}{5(x+5)}dx = \frac{1}{5}\ln|x| + \frac{4}{5}\ln|x+5| + C.$$

2. $f(x) = \frac{1+x}{x^3(x-5)^2}$,

$$\int f(x)dx = \int \frac{1+x}{x^3(x-5)^2}dx = \int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{(x-5)} + \frac{E}{(x-5)^2}dx$$
$$= A\ln|x| - \frac{B}{x} - \frac{C}{2x^2} + D\ln|x-5| - \frac{E}{(x-5)} + M.$$

Integration of the trigonometric functions

Integration functions (sin), (cos), and (tan).

In this case, let's make the variable change: $t = \tan\left(\frac{x}{2}\right)$. This implies $x = 2\arctan(t)$ and $dx = \frac{2}{1+t^2}dt$.

$$\begin{cases} \sin(x) = \sin\left(2\frac{x}{2}\right) = 2\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) = 2\dfrac{\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)} = \dfrac{2t}{1+t^2} \\ \cos(x) = \dfrac{1-t^2}{1+t^2} \\ \tan(x) = \dfrac{2t}{1-t^2} \end{cases}$$

Example 1.5 *Let*

1. $\int \frac{1}{\sin(x)}\,dx = \int \frac{dt}{t} = \ln|t| + c = \ln|\tan(\frac{x}{2})| + c$
2. $\int_0^{\frac{\pi}{2}} \frac{\tan(x)}{1+\cos(x)}\,dx$, *where* $0 \le x \le \frac{\pi}{3}$. *Let* $t = \tan(\frac{x}{2})$, $dx = \frac{2t}{1+t^2}dt$, $\tan(x) = \frac{2t}{1-t^2}$, *and* $\cos(x) = \frac{1-t^2}{1+t^2}$.

$$\int_0^{\frac{\pi}{2}} \frac{\tan(x)}{1+\cos(x)}\,dx = \int_0^{\frac{1}{\sqrt{3}}} \frac{2t}{1-t^2}\cdot\frac{1+t^2}{1-t^2}\,dt = \int \frac{2t}{\sqrt{3}(1-t^2)}\,dt = -\ln|1-t^2|\Big|_0^{\frac{1}{\sqrt{3}}} = -\ln\left|\frac{2}{3}\right|$$

Integration of exponential functions

Exponential functions: In exponential functions, we use the following substitution: $t = e^x$, $dt = e^x\,dx$, and $dx = \frac{dt}{t}$.

Example 1.6

$$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{t}{1+t}\,dt = \int\left(1 - \frac{1}{1+t}\right)dt = t - \ln|1+t| + c = e^x - \ln|1+e^x| + c.$$

Section 1.2 Double Integral

The general form of a double integral over a region D is written as follows:

$$\iint_D f(x,y)\,dx\,dy,$$

where

1. D represents the domain of integration, specifying the region in the xy-plane over which the integration is performed.
2. $f(x,y)$ is the function being integrated over the specified domain.

Definition: If $D = [a,b]\times[c,d]$ is a rectangle defined by $a \le x \le b$ and $c \le y \le d$, the double integral would be written as:

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx. \tag{1.1}$$

If $f(x,y) = f_1(x)f_2(y)$, we have

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b f_1(x)dx \int_c^d f_2(y)\,dy. \tag{1.2}$$

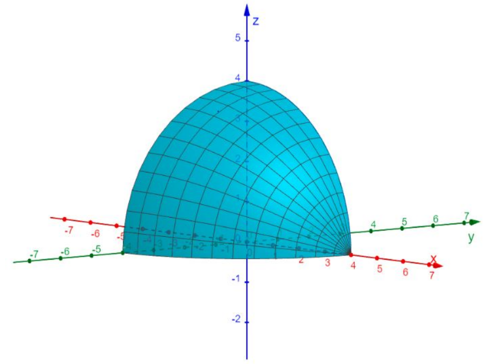

Figure 1.3: Integral of f on D.

Example 1.7 *Let* $I = \int\int (2x+y)\,dx\,dy$ =?.

1. $I = \int_0^1 \left(\int_0^1 (2x+y)\,dx\right) dy = \int_0^1 (1+y)\,dy = \dfrac{3}{2}$.
2. $I = \int_0^1 \left(\int_0^1 (2x+y)\,dy\right) dx = \int_0^1 (2x+12)\,dx = \dfrac{3}{2}$.

Double Integral over a Non-Rectangular Domain: Let f be a continuous function on a domain $D \subseteq \mathbb{R}^2$. The domain D can be represented in one of the following forms:

1. Case 1: $D = \{(x,y) \in \mathbb{R}^2 \,|\, a \le x \le b \text{ and } \phi(x) \le y \le \psi(x)\}$.
   The double integral is given by:
   $$\iint_D f(x,y)\,dx\,dy = \int_a^b \int_{\phi(x)}^{\psi(x)} f(x,y)\,dy\,dx$$
2. Case 2: $D = \{(x,y) \in \mathbb{R}^2 \,|\, c \le y \le d \text{ and } \phi(y) \le x \le \psi(y)\}$. The double integral is given by:
   $$\iint_D f(x,y)\,dx\,dy = \int_c^d \int_{\phi(y)}^{\psi(y)} f(x,y)\,dx\,dy$$

Example 1.8 *Consider the double integral:*

$$I = \iint_D y\,dx\,dy,$$

where the domain D is defined as:

$$D = \{(x,y) \in \mathbb{R}^2 \,|\, 1 \ge x \ge 0, 1 \ge y \ge 0, \text{ and } x+y \le 1\}.$$

1. *Expression 1:*
   $$I = \int_0^1 \int_0^{1-x} y\,dy\,dx = \frac{1}{2}\int_0^1 (x^2 - 2x + 1)\,dx = \frac{1}{6}.$$
2. *Expression 2:*
   $$I = \int_0^1 y \int_0^{1-y} dx\,dy = \int_0^1 (y - y^2)\,dy = \frac{1}{6}.$$

1.2.1 Integration methods

Double Integration with Change of Variables

Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation given by

$$\phi(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}.$$

The Jacobian matrix of ϕ is given by the matrix of partial derivatives:

$$J_\phi(u,v) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}.$$

Affine change:

Consider a continuous function f on a domain $D \subseteq \mathbb{R}^2$. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a bijective affine transformation defined by $\phi(u,v) = (x,y)$. We have the following expression for the double integral:

$$\iint_D f(x,y)\, dx\, dy = \iint_\Delta f(\phi(u,v), \phi(u,v))\, J\, du\, dv,$$

where $\Delta = \phi^{-1}(D)$ and $J = \left|J_\phi(u,v)\right|$.

Example 1.9 *Consider the double integral:*

$$I = \iint_D (x+y)\, dx\, dy,$$

where the domain D is defined as:

$$D = \{(x,y) \in \mathbb{R}^2 \,|\, 1 \leq x-y \leq 2, -1 \leq x+3y \leq 1\}.$$

Let's introduce the following change of variables:

$$\begin{cases} u = x - y \\ v = x + 3y \end{cases} \Rightarrow \begin{cases} x = \frac{1}{4}(3u+v) \\ y = \frac{1}{4}(-u+v) \end{cases}$$

The Jacobian determinant is given by $J = \dfrac{1}{4}$*, where* Δ *satisfies* $1 \leq u \leq 2$ *and* $-1 \leq v \leq 1$.

The integral is then transformed into:

$$I = \frac{1}{8} \iint_\Delta (u+v)\, du\, dv = \frac{1}{8}\left(\int_1^2 u\, du \int_{-1}^1 dv + \int_{-1}^1 dv \int_1^2 v\, du\right) = \frac{3}{8}.$$

Change to Polar Coordinates: Consider a change to polar coordinates given by:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

where the Jacobian determinant is $J = r$.

Let Δ be the region in the xy-plane and D be its image under the transformation. The double integral is then transformed as follows:

$$\iint_D f(x,y)\, dx\, dy = \iint_\Delta f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta.$$

Example 1.10 *Consider the double integral:*

$$I = \iint_D xy\,dx\,dy,$$

where the domain D is defined as:

$$D = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ y > 0,\ \text{and } x^2 + y^2 \leq 1\}.$$

Now, let's perform a change to polar coordinates:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

with Jacobian determinant $J = r$. *The region* Δ *in polar coordinates is defined as* $0 \leq r \leq 1$ *and* $0 \leq \theta \leq \frac{\pi}{2}$.

The integral becomes:

$$I = \int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin(\theta)\cos(\theta)\,dr\,d\theta = \left(\int_0^1 r^3\,dr\right)\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2\theta)\,d\theta\right) = \frac{1}{8}.$$

Section 1.3 Triple Integral

Let f be a continuous function on a domain $D \subseteq \mathbb{R}^3$.

Definition 01: The triple integral of f over D is denoted by I and is defined as:

$$I = \iiint_D f(x,y,z)\,dx\,dy\,dz.$$

Fubini's Theorem: Let $D = [a\ b] \times [c\ d] \times [p\ q] \subseteq \mathbb{R}^3$. The triple integral of f over D can be expressed as iterated integrals:

$$\begin{aligned} \iiint_D f(x,y,z)\,dx\,dy\,dz &= \int_a^b \int_c^d \int_p^q f(x,y,z)\,dz\,dy\,dx \\ &= \int_a^b \int_c^d \int_p^q f(x,y,z)\,dx\,dz\,dy \\ &= \int_a^b \int_c^d \int_p^q f(x,y,z)\,dy\,dz\,dx. \end{aligned}$$

1.3.1 Triple Integration with Change of Variables

Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a transformation given by

$$\phi(u,v,w) = \begin{bmatrix} x(u,v,w) \\ y(u,v,w) \\ z(u,v,w) \end{bmatrix}.$$

The Jacobian matrix of ϕ is given by the matrix of partial derivatives:

$$J_\phi(u,v,w) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}.$$

Cylindrical Coordinates: The cylindrical coordinates of a point (x, y, z) in $\mathbb{R}^3$ are obtained by representing the x and y coordinates using polar coordinates (or potentially the y and z coordinates or x and z coordinates) and letting the third coordinate remain unchanged

$$\phi := \begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}, \quad \Delta = \phi(D), \quad 0 \le \theta \le 2\pi, \quad 0 \le r < \infty, \quad J = |J_\phi(u, v, w)| = r,$$

Integral of f on D:

$$\iiint_D f(x, y, z)\,dx\,dy\,dz. = \iiint_\Delta f(r\cos(\theta), r\sin(\theta), z) J\,dr\,d\theta\,dz.$$

Example 1.11

$$\iiint_D zdxdydz, \quad D = \{(x, y, z) \in \mathbb{R}^3, 0 \le z \le 1, x^2 + y^2 \le z^2\},$$

$I =$ *Switch to cylindrical coordinates:* $\{x = r\cos(\theta), y = r\sin(\theta), z = z, J = r, 0 \le r \le z \le 1, 0 \le \theta \le 2\pi\}$,

$$I = \int_0^1 \int_0^{2\pi} \int_0^z zr\,dr\,d\theta\,dz = 2\pi \int_0^1 \frac{z^3}{3}\,dz = \frac{2}{3}\pi.$$

Spherical Coordinates: To evaluate the triple integral of a function f over D using spherical coordinates, we need to express the integral in terms of the spherical coordinates r, ϕ, θ. The spherical coordinates are related to Cartesian coordinates by the following transformations:

$$\phi : \begin{cases} x = r\sin(\phi)\cos(\theta) \\ y = r\sin(\phi)\sin(\theta) \\ z = r\cos(\phi) \end{cases}, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi,$$

$$\iiint_D f(x, y, z)\,dx\,dy\,dz = \iiint_\Delta f(r\sin(\phi)\cos(\theta), r\sin(\phi)\sin(\theta), r\cos(\phi)) J\,dr\,d\theta\,d\phi.$$

The Jacobian determinant of the spherical coordinate transformation is $J = r^2\sin(\phi)$.

Example 1.12

$$\iiint_D zdxdydz, \quad D = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 \le 1\}.$$

Using spherical coordinates, we get

$$\begin{aligned} \iiint_\Delta r\cos(\phi) r^2 \sin(\phi)\,dr\,d\theta\,d\phi &= \int_0^1 \int_0^{2\pi} \int_0^\pi r^3 \sin(\phi)\cos(\phi)\,d\phi\,d\theta\,dr \\ &= \frac{1}{2}\int_0^1 \int_0^{2\pi} \int_0^\pi r^3 \sin(2\phi)\,d\phi\,d\theta\,dr = \frac{\pi}{8}. \end{aligned} \tag{1.3}$$

Section 1.4 Application

Exercise *Calculate the following integrals:*

1. $\int_0^2 \frac{x^3}{x^2 + 2}\,dx.$
2. $\int (\ln(x))^2\,dx.$

3. $\int_1^2 \frac{1}{\frac{1}{2}x+\sqrt{x}}\,dx.$

4. $\int \frac{1}{x^2+x+1}\,dx.$

5. $\int \frac{1}{x+x(\ln(x))^2}\,dx.$

6. $\int \frac{1}{\cos(x)+1}\,dx.$ ■

Exercise *Calculate the area of D and the following double integrals:*

Area of D:

$$\iint_D 1\,dA, \quad D=\{(x,y)\in\mathbb{R}^2 : 0\le x\le 2, 1\le y\le 2\}$$

Double integral of e^x+y:

$$\iint_D (e^x+y)\,dxdy, \quad D=\{(x,y)\in\mathbb{R}^2 : 0\le x\le 2, 1\le y\le 2\}$$

Double integral of $\sqrt{x^2+y^2}$:

$$\iint_D \sqrt{x^2+y^2}\,dxdy, \quad D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\ge 4, x^2+y^2\le 9\}$$

Double integral of x^2+y^2:

$$\iint_D (x^2+y^2)\,dxdy, \quad D=\{(x,y)\in\mathbb{R}^2 : x>0, y>0, \text{ and } x\le x^2+y^2\le 1\}$$

■

Exercise *Volume of D:*

$$\iiint_D 1\,dV, \quad D=\{(x,y,z)\in\mathbb{R}^3 : x>0, y>0, z>0 \text{ with } z\le 1-y^2 \text{ and } x+y\le 1\}$$

Triple integral of z:

$$\iiint_D z\,dxdydz, \quad D=\{(x,y,z)\in\mathbb{R}^3 : 0\le z\le 1 \text{ and } x^2+y^2\le z^2\}$$

Triple integral of xy:

$$\iiint_D xy\,dxdydz, \quad D=\{(x,y,z)\in\mathbb{R}^3 : z>0 \text{ and } 1\le x^2+y^2+z^2\le 4\}$$

Triple integral of $x^2+y^2+z^2$:

$$\iiint_D (x^2+y^2+z^2)\,dxdydz, \quad D=\{(x,y,z)\in\mathbb{R}^3 : z>0 \text{ and } 1\le x^2+y^2+z^2\le 4\}$$