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Chapter N°4 : Elementary functions

0.1 Inverse trigonometric functions

0.1.1 arccosine

We assume the function **cosine** $\cos: \mathbb{R} \to [-1,1], \quad x \mapsto \cos x$. To obtain a bijection from this function, we must consider the restriction of cosine on the interval $[0,\pi]$. On this interval the function **cosine** is continuous and strictly decreasing, then the restriction

$$\cos: [0,\pi] \to [-1,1], \quad x \mapsto \cos x.$$

is bijection. Its inverse bijection is the function arccosine :

$$\arccos: [-1,1] \to [0,\pi], \quad x \mapsto \arccos x.$$

So we have by definition the inverse bijection :

$$\cos(\arccos x) = x \quad \forall x \in [-1,1]$$

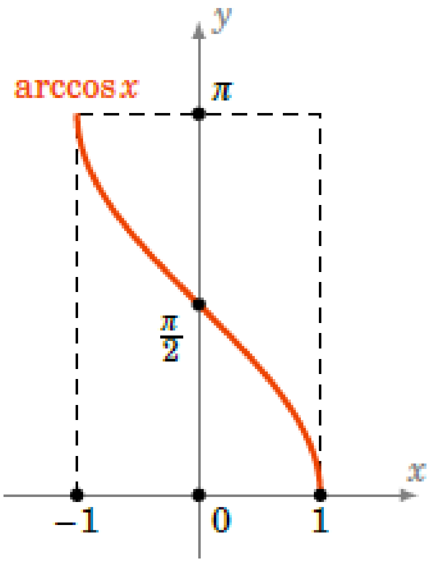
$$\arccos(\cos x) = x \quad \forall x \in [0,\pi]$$

In other words :

$$\text{Si} \quad x \in [0,\pi] \qquad \cos x = y \Leftrightarrow x = \arccos y$$

Let's finish with the derivative of arccos :

$$\arccos'(x) = \frac{-1}{\sqrt{1-x^2}} \qquad \forall x \in]-1,1[$$



0.1.2 arcsine

The restriction

$$\sin: [\frac{-\pi}{2}, \frac{\pi}{2}] \to [-1,1], \quad x \mapsto \sin x.$$

is a bijection. Its inverse bijection is the function arcsine :

$$\arcsin: [-1,1] \to [\frac{-\pi}{2}, \frac{\pi}{2}], \quad x \mapsto \arcsin x.$$

We have by definition :

$$\sin(\arcsin x) = x \quad \forall x \in]-1,1[$$

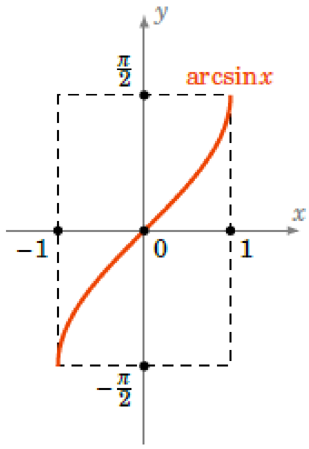
$$\arcsin(\sin x) = x \quad \forall x \in [\frac{-\pi}{2}, \frac{\pi}{2}]$$

In other words :

$$\text{Si} \quad x \in [\frac{-\pi}{2}, \frac{\pi}{2}] \qquad \sin x = y \Leftrightarrow x = \arcsin y$$

Let's finish with the derivative of arcsin is :

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}} \qquad \forall x \in]-1,1[$$



0.1.3 arctangent

The restriction

$$\tan :]\frac{-\pi}{2}, \frac{\pi}{2}[\to \mathbb{R}, \quad x \mapsto \tan x.$$

is a bijection. Its inverse bijection is the function arctangent :

$$\arctan : \mathbb{R} \to]\frac{-\pi}{2}, \frac{\pi}{2}[, \quad x \mapsto \arctan x.$$

$$\tan(\arctan x) = x \quad \forall x \in \mathbb{R}$$

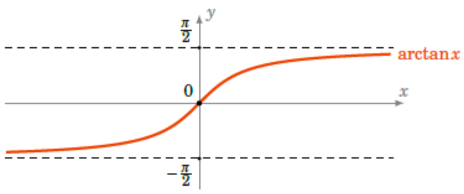
$$\arctan(\tan x) = x \quad \forall x \in]\frac{-\pi}{2}, \frac{\pi}{2}[$$

In other words :

$$\text{Si} \quad x \in]\frac{-\pi}{2}, \frac{\pi}{2}[\qquad \tan x = y \Leftrightarrow x = \arctan y.$$

The derivative of arctan is :

$$\arctan'(x) = \frac{1}{1+x^2} \qquad \forall x \in \mathbb{R}$$



0.2 Hyperbolic functions and their inverses

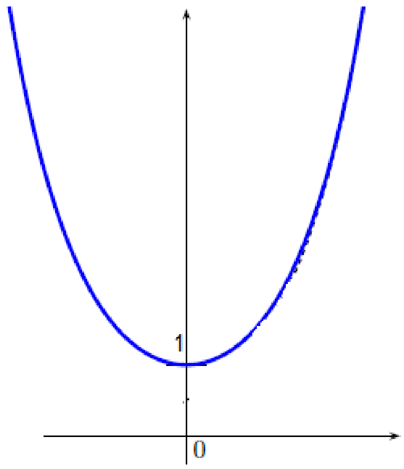
0.2.1 Hyperbolic cosine function and its inverse

For all $x \in \mathbb{R}$ the **hyperbolic cosine function,** written cosh or ch is defined by the relation :

$$chx = \frac{e^x + e^{-x}}{2}.$$

The derivative function is

$$ch'(x) = sh(x)$$

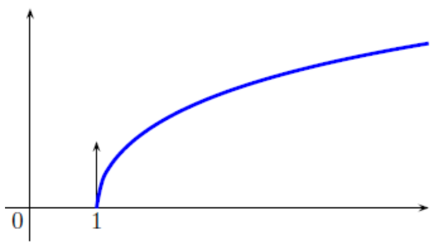


The restriction $ch:\ [0, +\infty[\longrightarrow [1, +\infty[$ is bijective. and its inverse bijection is :

$$argch:\ [1, +\infty[\longrightarrow [0, +\infty[$$

The derivative function is

$$argch'(x) = \frac{1}{\sqrt{x^2 - 1}}, (\quad x > 1)$$



0.2.2 Hyperbolic sine function and its inverse

For all $x \in \mathbb{R}$ le **hyperbolic sine function** is defined by the relation :

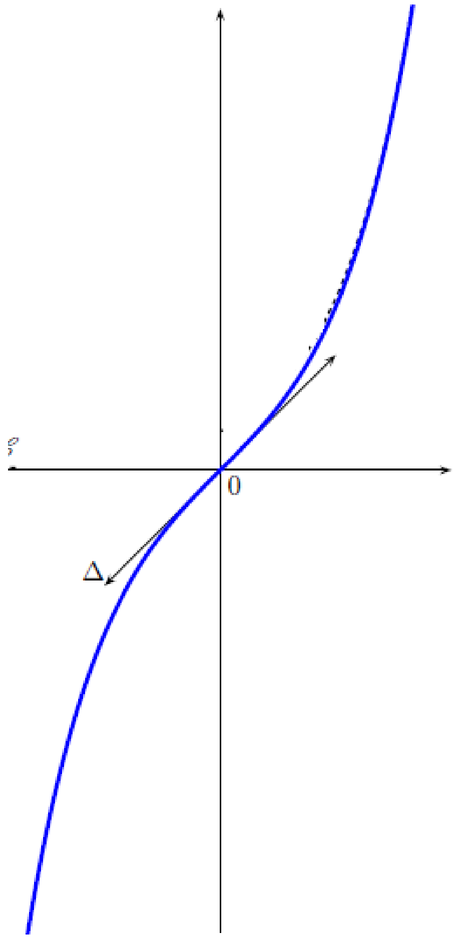
$$shx = \frac{e^x - e^{-x}}{2}.$$

$sh:\ \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function, differentiable and strictly increasing verifying

$$\lim_{x \to -\infty} sh(x) = -\infty \text{ et } \lim_{x \to +\infty} sh(x) = +\infty,$$

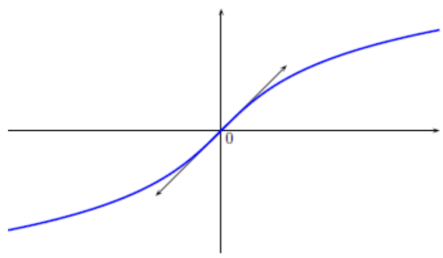
The derivative function is

$$sh'(x) = ch(x)$$



then is a bijection. And its inverse bijection is : *argsh* : $\mathbb{R} \longrightarrow \mathbb{R}$. The derivative function is :

$$argsh'(x) = \frac{1}{\sqrt{x^2+1}},$$



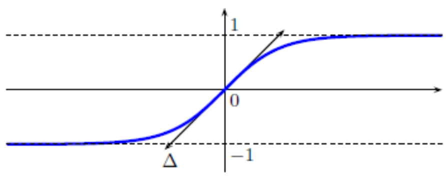
Remark 1 *The name of these two hyperbolic functions suggest that the have similar properties to the trigonometric functions and some of these will be investigated.*

0.2.3 Hyperbolic tangent function and its inverse

By definition the **tangent hyperbolic** is :

$$th(x) = \frac{sh(x)}{ch(x)}.$$

The function $th : \mathbb{R} \longrightarrow]-1,+1[$ is a bijection.
The derivative function is : $th'(x) = 1 - th^2(x) = \dfrac{1}{ch^2(x)}$

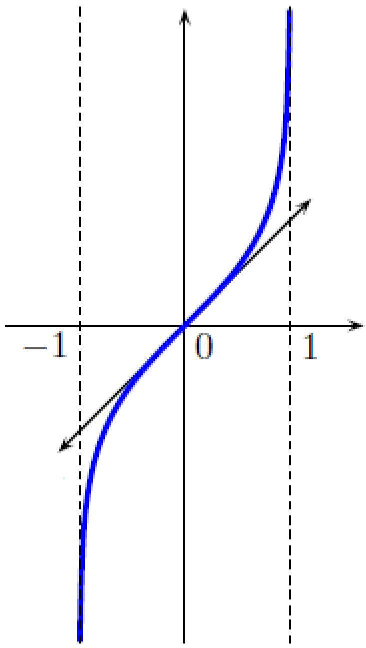


we notice

$$\textcolor{blue}{argth}:]-1,+1[\longrightarrow \mathbb{R}$$

is inverse bijection.
The derivative function is

$$argth'(x) = \frac{1}{1-x^2}, \quad (|x|<1)$$



0.3 Hyperbolic trigonometry

$ch(a+b) = ch(a).ch(b) + sh(a).sh(b), ...(1)$
$ch(a-b) = ch(a).ch(b) - sh(a).sh(b), ...(2)$
$sh(a+b) = sh(a).ch(b) + ch(a).sh(b), ...(3)$
$sh(a-b) = sh(a).ch(b) - ch(a).sh(b)...........(4)$
By multiplying the expression for $(chx + shx)$ and $(chx - shx)$ together, we have

$$ch^2(x) - sh^2(x) = 1$$

obvious,

$$chx + shx = e^x$$

Put $a = b$ in first formulate (1)

$$ch(2a) = ch^2(a) + sh^2(a)$$

Put $a = b$ in third formulate (3)

$$sh(2a) = 2.sh(a).ch(a)$$

$$th(a+b) = \frac{th(a) + th(b)}{1 + th(a).th(b)}.....(5)$$

Put $a = b$ in third formulate (5)

$$th2a = \frac{2tha}{1+th^2a}$$

0.3.0.1 expression of inverse hyperbolic functions with natural logatithm

$$argch(x) = \ln(x + \sqrt{x^2-1}), (\quad x \geq 1)$$

$$argsh(x) = \ln(x + \sqrt{x^2+1}), (\quad x \in \mathbb{R})$$

$$argth(x) = \frac{1}{2}\ln(\frac{1+x}{1-x}), (\quad -1 < x < 1)$$

Exercise : Use the definition of chx and shx in term of exponential functions to prove that :

$$chx = 2ch^2x - 1, \qquad\qquad chx = 1 + 2sh^2x$$

0.4 Solution of the series "4"

Exercise 1 :

1. For all $x \in \mathbb{R}$, $\frac{-\pi}{2} \leq \arctan(\frac{x}{2}) \leq \frac{\pi}{2}$, therefore the equation does not solutions.
2. $\arcsin x = \arccos x \Leftrightarrow \sin(\arcsin x) = \sin(\arccos x) \Leftrightarrow x = \sqrt{1-x^2} \Leftrightarrow x = \frac{\sqrt{2}}{2}$
3. The derivative : direct compute
4. $a)$ $\sin(\arccos x) = \sqrt{1-\cos^2(\arcsin x)} = \sqrt{1-x^2}$
5. $b)$ $\tan(\arcsin x) = \frac{\sin(\arcsin x)}{\cos(\arcsin x)} = \frac{x}{\sqrt{1-x^2}}$
6. $c)$ $\arcsin x + \arccos x = \frac{\pi}{2}$, we can prove this by two methods
   First, consider the function $f(x) = \arcsin x + \arccos x$, continuous on $[-1,1]$, differentiable on $]-1,1[$ and $f'(x) = 0$. Then f is constant over $[-1,1]$, hence
   $f(0) = \arcsin(0) + \arccos(0) = \frac{\pi}{2}$. Therefore $\forall x \in [-1,1],\ f(x) = \frac{\pi}{2}$
   Or, we have : $\arccos x = \frac{\pi}{2} - \arcsin x$, because cos is bijective on $[0,\pi]$ so $\cos(\arccos x) = \cos(\frac{\pi}{2} - \arcsin x)$. So $x = x$, then, $\arcsin x + \arccos x = \frac{\pi}{2}$.
7. d) We have :
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a . \tan b}$$
   we put $a = \arctan \alpha,\ b = \arctan x$, so
$$\tan(\arctan \alpha + \arctan x) = \frac{\tan \arctan \alpha + \tan \arctan x}{1 - \tan \arctan \alpha . \tan \arctan x} = \frac{\alpha + x}{1 - \alpha . x}$$
   because arctan is a bijective, then : $\arctan \alpha + \arctan x = \arctan(\frac{\alpha + x}{1 - \alpha . x})$
8. e) we put $\alpha = \frac{1}{x}$ in precedent formula (d)
   $\arctan \frac{1}{x} + \arctan x = \arctan(\frac{\frac{1}{x} + x}{1 - \frac{1}{x}.x})$
$$\arctan x + \arctan \frac{1}{x} = \frac{2x+1}{0} = \arctan(\infty) = \frac{\pi}{2}$$

Exercise 2 :

1. If $x \neq 0$, f is quotient of two differentiable functions, then differentiable.
   If $x = 0$, $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\frac{\arctan x}{x} - 1}{x} = \lim_{x \to 0} \frac{\arctan x - x)}{x^2} = \frac{0}{0}$. I.F By hospital theorem $\lim_{x \to 0} \frac{\frac{1}{1-x^2} - 1}{2x} = \lim_{x \to 0} \frac{-x}{\acute{e}(1+x^2)} = 0$. The f is differentiable at 0, therefore differentiable on $\mathbb{R}$.
$$f(x) = \begin{cases} \frac{1}{x(1+x^2)} - \frac{\arctan x}{x^2}, & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0 \end{cases}$$
2. we know, $\arctan : \mathbb{R} \to]-\frac{\pi}{2}, \frac{\pi}{2}[$ is continuous on $\mathbb{R}$ then over $[0,x]$, $\quad x > 0$ and differentiable over $]0,x[$, according to mean value theorem there exists $c \in]0,x[$ such that
$$\arctan'(c) = \lim_{x \to 0} \frac{\arctan x - \arctan 0}{x - 0} = \frac{\arctan x}{x} = \frac{1}{1+c^2}$$

other words, $0 < c < x \Rightarrow 1 < 1+c^2 < 1+x^2 \Rightarrow \dfrac{1}{1+x^2} < 1+c^2 < 1$, hence, $\dfrac{1}{1+x^2} < \dfrac{\arctan x}{x}$

$$\text{then,} \quad \frac{x}{1+x^2} < \arctan x.$$

Exercise 3 :

1. $\forall x \in \mathbb{R} : \dfrac{shx}{1+chx} = \dfrac{2sh\frac{x}{2}.ch\frac{x}{2}}{2ch^2\frac{x}{2}} = \dfrac{sh\frac{x}{2}}{ch\frac{x}{2}} = th\dfrac{x}{2}.$
2. $\forall x \neq 0 : \dfrac{2}{th2x} - \dfrac{1}{thx} = \dfrac{2}{\frac{2thx}{1+th^2x}} - \dfrac{1}{thx} = \dfrac{th^2x}{thx} = thx.$
3. We have $thx = \dfrac{2th\frac{x}{2}}{1+th^2\frac{x}{2}}$ and

   $chx-1 = ch^2\frac{x}{2}+sh^2\frac{x}{2}-ch^2\frac{x}{2}+sh^2\frac{x}{2} = 2sh^2\frac{x}{2}, \quad chx+1 = ch^2\frac{x}{2}+sh^2\frac{x}{2}+ch^2\frac{x}{2}-sh^2\frac{x}{2} = 2ch^2\frac{x}{2}.$

   So
   $$\sqrt{\frac{chx-1}{chx+1}} = \sqrt{th^2\frac{x}{2}} \quad \text{then} \quad argth(th\frac{x}{2}) = \frac{x}{2}.$$

Exercise 4 :

For $sh(x) \geq x$ let : $f(x) = sh(x) - x$ so $f'(x) = ch(x) - 1, \quad x \geq 0$
for every $x \geq 0$, $f'(x) \geq 0$ then f is increasing and $f(0) = 0$, then
$f(x) \geq 0 \Leftrightarrow f(x) - x \geq 0 \Leftrightarrow f(x) \geq x$.

For : $ch(x) \geq 1 + \dfrac{x^2}{2}$

Let $g(x) = ch(x) - 1 - \dfrac{x^2}{2}$ we have $g(0) = 0$ et $g'(x) = sh(x) - x \geq 0$ then g is increasing and $g(0) = 0$, therefore $g(x) \geq 0 \Leftrightarrow ch(x) \geq 1 + \dfrac{x^2}{2}$

Exercise 5 :

1. For $\sqrt{1-x^2} \leq x$ has meaning if, $x \geq 0$ and $1 - x^2 \geq 0$ if $0 \leq x \leq 1$, so $1 - x^2 \leq x^2 \Leftrightarrow x^2 \geq \dfrac{1}{2} \Leftrightarrow x \in [\dfrac{\sqrt{2}}{2}, 1]$.
2. f defined on $[-1, 1]$ and $f'(x) = \dfrac{-x+\sqrt{1-x^2}}{\sqrt{1-x^2}} e^{\arcsin x}$

   $f'(x) = 0 \Leftrightarrow x = \dfrac{\sqrt{2}}{2}, \quad f'(x) > 0,$ si $x \in]-1, \dfrac{\sqrt{2}}{2}[, \quad f'(x) < 0,$ si $x \in]\dfrac{\sqrt{2}}{2}, 1[$