

- For every  $a, b \in \mathbb{R}$  there is a unique real number  $a + b$ , called their *sum*.
- For every  $a, b \in \mathbb{R}$  there is a unique real number  $a \cdot b$ , called their *product*.
- For  $a \in \mathbb{R}$  there is a unique real number  $-a$  called its *negative* or its *additive inverse*.
- For  $a \in \mathbb{R}$  with  $a \neq 0$  there is a unique real number  $\frac{1}{a}$  called its *reciprocal* or its *multiplicative inverse*.
- There is a special element  $0 \in \mathbb{R}$  called *zero* or the *additive identity*.
- There is a special element  $1 \in \mathbb{R}$  called *one* or the *multiplicative identity*.

For all  $a, b, c \in \mathbb{R}$ , we have

- $a + b = b + a$  ( $+$  is commutative)
- $a + (b + c) = (a + b) + c$  ( $+$  is associative)
- $a + 0 = a$  (additive identity)
- $a + (-a) = 0$  (additive inverses)
- $a \cdot b = b \cdot a$  ( $\cdot$  is commutative)
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$  ( $\cdot$  is associative)
- $a \cdot 1 = a$  (multiplicative identity)
- if  $a \neq 0$  then  $a \cdot \frac{1}{a} = 1$  (multiplicative inverses)
- $a \cdot (b + c) = a \cdot b + a \cdot c$  ( $\cdot$  distributes over  $+$ )

- $0 \neq 1$  (to avoid total collapse)

This is a long list of properties, called *axioms*.

I have given a name to each axiom, so that in proofs we can say which property we are using at each stage. Some people prefer to number the axioms, but there is no standard way to do this, and I find it hard to remember the numbers and easier to remember the names.

I think of the axioms as coming in bundles, to help me to remember them. The first four tell us that ‘addition behaves nicely’. The next four tell us that ‘multiplication behaves nicely’. Then there is an axiom that tells us that ‘addition and multiplication interact nicely’, and a final technical detail to clarify that  $\mathbb{R} \neq \{0\}$ .

As you study abstract algebra (such as linear algebra and group theory) this year, you’ll find yourself coming across lists of axioms such as this again.

**Definition.** Let  $\mathbb{F}$  be a set with operations  $+$  and  $\cdot$  that satisfy the axioms above. Then we say that  $\mathbb{F}$  is a *field*.

**Example.** We’ve just said that  $\mathbb{R}$  is a field. The rational numbers  $\mathbb{Q}$  form a field. The complex numbers  $\mathbb{C}$  form a field. You’ll meet other fields too, in other courses. The integers  $\mathbb{Z}$  do not form a field.

In the next section, we’ll use these axioms to deduce the basic properties of arithmetic in  $\mathbb{R}$ . This reasoning would apply equally to any field. We’ll need to make further assumptions about  $\mathbb{R}$ , but we’ll postpone this till we’ve studied basic arithmetic.

### 3 Properties of arithmetic in $\mathbb{R}$

In the last section, we saw the arithmetic axioms for  $\mathbb{R}$ . Now we’ll deduce some properties of arithmetic in  $\mathbb{R}$ .

**Proposition 1.** *Let  $a, b, c, x, y$  be real numbers.*

(i) *If  $a + x = a$  for all  $a$  then  $x = 0$  (uniqueness of 0).*

(ii) *If  $a + x = a + y$  then  $x = y$  (cancellation for +).*

(iii)  $-0 = 0$ .

(iv)  $-(-a) = a$ .

(v)  $-(a + b) = (-a) + (-b)$ .

(vi) *If  $a \cdot x = a$  for all  $a \neq 0$  then  $x = 1$  (uniqueness of 1).*

(vii) *If  $a \neq 0$  and  $a \cdot x = a \cdot y$  then  $x = y$  (cancellation for  $\cdot$ ).*

(viii) *If  $a \neq 0$  then  $\frac{1}{a} = a^{-1}$ .*

(ix)  $(a + b) \cdot c = a \cdot c + b \cdot c$ .

(x)  $a \cdot 0 = 0$ .

(xi)  $a \cdot (-b) = -(a \cdot b)$ . *In particular,  $(-1) \cdot a = -a$ .*

(xii)  $(-1) \cdot (-1) = 1$ .

(xiii) *If  $a \cdot b = 0$  then  $a = 0$  or  $b = 0$ . If  $a \neq 0$  and  $b \neq 0$  then  $\frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b}$ .*

**Remark.**     • (ii) shows the uniqueness of  $-a$ , the additive inverse of  $a$ .

• (vii) shows the uniqueness of  $\frac{1}{a}$ , the multiplicative inverse of  $a$  (if  $a \neq 0$ ).

• As we'll see shortly, (i)–(v) can be proved using only the four axioms about +.

• Similarly, (vi)–(viii) can be proved using only the four axioms about  $\cdot$ .

- (ix)–(xiii) between them use all the axioms.
- It's worth proving results like this in a sensible order! Once we've proved a property, we can add it to the list of properties we can assume in subsequent parts. You'll see that we prove some later parts using earlier parts.

*Proof.* (i) Suppose that  $a + x = a$  for all  $a$ . Then

$$\begin{aligned}
 x &= x + 0 && \text{(additive identity)} \\
 &= 0 + x && \text{(+ is commutative)} \\
 &= 0 && \text{(by hypothesis, with } a = 0\text{)}.
 \end{aligned}$$

(ii) Suppose that  $a + x = a + y$ . Then

$$\begin{aligned}
 y &= y + 0 && \text{(additive identity)} \\
 &= y + (a + (-a)) && \text{(additive inverses)} \\
 &= (y + a) + (-a) && \text{(+ is associative)} \\
 &= (a + y) + (-a) && \text{(+ is commutative)} \\
 &= (a + x) + (-a) && \text{(hypothesis)} \\
 &= (x + a) + (-a) && \text{(+ is commutative)} \\
 &= x + (a + (-a)) && \text{(+ is associative)} \\
 &= x + 0 && \text{(additive inverses)} \\
 &= x && \text{(additive identity)}.
 \end{aligned}$$

(iii) We have  $0 + 0 = 0$  (additive identity)

and  $0 + (-0) = 0$  (additive inverses)

so  $0 + 0 = 0 + (-0)$ , so  $0 = -0$  (cancellation for + (ii)).

(iv) We have

$$\begin{aligned}(-a) + a &= a + (-a) \quad (+ \text{ is commutative}) \\ &= 0 \quad (\text{additive inverses})\end{aligned}$$

and  $(-a) + (-(-a)) = 0$  (additive inverses),

so  $(-a) + a = (-a) + (-(-a))$ ,

so  $a = -(-a)$  (cancellation for  $+$  (ii)).

(v) Exercise (see Sheet 1).

(vi)–(viii) Exercise — similar to (i), (ii), (iv).

(ix) (This is another form of distributivity, similar to the axiom but different!)

We have

$$\begin{aligned}(a + b) \cdot c &= c \cdot (a + b) \quad (+ \text{ is commutative}) \\ &= c \cdot a + c \cdot b \quad (\cdot \text{ distributes over } +) \\ &= a \cdot c + b \cdot c \quad (\cdot \text{ is commutative - twice}).\end{aligned}$$

(x) We have  $a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$  ( $\cdot$  distributes over  $+$ ),

and also

$$\begin{aligned}a \cdot (0 + 0) &= a \cdot 0 \quad (\text{additive identity}) \\ &= a \cdot 0 + 0 \quad (\text{additive identity})\end{aligned}$$

so  $a \cdot 0 = 0$  (cancellation for  $+$  (ii))

(xi) We have

$$\begin{aligned} a \cdot b + a \cdot (-b) &= a \cdot (b + (-b)) \quad (\cdot \text{ distributes over } +) \\ &= a \cdot 0 \quad (\text{additive inverses}) \end{aligned}$$

and  $a \cdot b + -(a \cdot b) = 0$  (additive inverses),

so  $a \cdot (-b) = -(a \cdot b)$  (cancellation for  $+$  (ii)).

(xii) We have

$$\begin{aligned} (-1) \cdot (-1) &= -((-1) \cdot 1) \quad ((\text{xi}) \text{ with } a = -1, b = 1) \\ &= -(-1) \quad (\text{multiplicative identity}) \\ &= 1 \quad ((\text{iv})). \end{aligned}$$

(xiii) Suppose, for a contradiction, that  $a \neq 0$ ,  $b \neq 0$  but  $a \cdot b = 0$ . Then

$$\begin{aligned} 0 &= \left(\frac{1}{a} \cdot \frac{1}{b}\right) \cdot 0 \quad ((\text{x})) \\ &= 0 \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) \quad (\cdot \text{ is commutative}) \\ &= (a \cdot b) \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) \quad (\text{hypothesis}) \\ &= (b \cdot a) \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) \quad (\cdot \text{ is commutative}) \\ &= \left((b \cdot a) \cdot \frac{1}{a}\right) \cdot \frac{1}{b} \quad (\cdot \text{ is associative}) \\ &= \left(b \cdot \left(a \cdot \frac{1}{a}\right)\right) \cdot \frac{1}{b} \quad (\cdot \text{ is associative}) \\ &= (b \cdot 1) \cdot \frac{1}{b} \quad (\text{multiplicative inverses}) \\ &= b \cdot \frac{1}{b} \quad (\text{multiplicative identity}) \\ &= 1 \quad (\text{multiplicative inverses}) \end{aligned}$$

and this is a contradiction ( $0 \neq 1$ ).

So if  $a \cdot b = 0$  then  $a = 0$  or  $b = 0$ .

Note that on the way we showed that if  $a \neq 0$  and  $b \neq 0$  then  $a \cdot b \neq 0$  and  $(a \cdot b) \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) = 1$  so  $\frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b}$  (cancellation for  $\cdot$  (vii)).

□

From now on, we can use all of these properties. We shan't give such detailed, one-axiom-at-a-time, derivations in the remainder of the course — but we could, if we needed to!

**Remark.** (This remark is not part of the course!) You might be concerned that if we don't go back to the axioms every time then we might overlook an unproved step, or might make a mistake. But going back to the axioms every time is not practical: a research paper might take tens of pages to give a proof, referring back to other results, which themselves build on results, which build on results, . . . . This is where *proof verification* comes in: computers can take care of this detailed checking, leaving humans to focus on things that humans are good at (like having creative ideas for proofs).

One interesting project in this area (there are others, not just this one) is Xena <https://xenaproject.wordpress.com/what-is-the-xena-project/> — this is aimed at undergraduates, which is why I'm mentioning it. Some of you might find it interesting. There's an article about proof verification and Xena in the London Mathematical Society Newsletter, pages 32–36 of [https://www.lms.ac.uk/sites/lms.ac.uk/files/files/NLMS\\_484-forweb2.pdf](https://www.lms.ac.uk/sites/lms.ac.uk/files/files/NLMS_484-forweb2.pdf).

Now back to the course . . . .

**Notation.** From now on, we use more familiar notation. We write

$$a - b \text{ for } a + (-b)$$

$$ab \text{ for } a \cdot b$$

$$\frac{a}{b} \text{ for } a \cdot \left(\frac{1}{b}\right)$$

$$a^{-1} \text{ sometimes for } \frac{1}{a}.$$

The associativity of addition and multiplication means that we can write expressions like  $a + b + c$  and  $xyz$ , without needing to write brackets.

**Definition.** Take  $a \in \mathbb{R} \setminus \{0\}$ .

Define  $a^0 = 1$ .

We define positive powers of  $a$  inductively: for integers  $k \geq 0$ , we define  $a^{k+1} = a^k \cdot a$ .

For integers  $l \leq -1$ , we define  $a^l = \frac{1}{a^{-l}}$ .

**Remark.** Note that with this definition  $a^1 = a$  and  $a^2 = a \cdot a$  (as we'd want).

**Lemma 2.** For  $a \in \mathbb{R} \setminus \{0\}$  we have  $a^m a^n = a^{m+n}$  for  $m, n \in \mathbb{Z}$ .

*Proof.* Exercise (see Sheet 1). □

This finishes this section on arithmetic in  $\mathbb{R}$ . In the next section, we'll go on to explore more key properties of  $\mathbb{R}$ , in addition to it being a field.

## 4 Ordering the real numbers

When we picture  $\mathbb{R}$  in our minds, typically it is not as a scattered collection of numbers. Rather, we often picture them as lying along a number line, usually running from left to right, with 0 in the 'middle'; positive numbers on the right, increasing as we move away from 0; and negative numbers on



the left, getting smaller as we move left away from 0. Even in writing that description, I have made assumptions about  $\mathbb{R}$ : I have assumed that we have notions of ‘positive’ and ‘negative’, and that we can compare the sizes of two real numbers. This section is about formalising those assumptions.

Here are our axioms for the usual ordering on  $\mathbb{R}$ .

There is a subset  $\mathbb{P}$  of  $\mathbb{R}$  such that for  $a, b \in \mathbb{R}$

- if  $a, b \in \mathbb{P}$  then  $a + b \in \mathbb{P}$  (+ and ordering)
- if  $a, b \in \mathbb{P}$  then  $a \cdot b \in \mathbb{P}$  ( $\cdot$  and ordering)
- exactly one of  $a \in \mathbb{P}$ ,  $a = 0$  and  $-a \in \mathbb{P}$  holds (positive, negative or 0).

The elements of  $\mathbb{P}$  are called the *positive numbers*. The elements of  $\mathbb{P} \cup \{0\}$  are called the *non-negative numbers*.

We write  $a < b$ , or  $b > a$ , exactly when  $b - a \in \mathbb{P}$ .

We write  $a \leq b$ , or  $b \geq a$ , exactly when  $b - a \in \mathbb{P} \cup \{0\}$ .

Now, just as with the axioms for arithmetic, we can infer useful properties from these axioms. First, some important properties of the ordering.

**Proposition 3.** *Take  $a, b, c \in \mathbb{R}$ . Then*

- (i)  $a \leq a$ ; (*reflexivity*)
- (ii) if  $a \leq b$  and  $b \leq a$  then  $a = b$ ; (*antisymmetry*)
- (iii) if  $a \leq b$  and  $b \leq c$  then  $a \leq c$ , and similarly with  $<$  in place of  $\leq$ ;  
(*transitivity*)
- (iv) exactly one of  $a < b$ ,  $a = b$  and  $a > b$  holds. (*trichotomy*)

*Proof.* (i) We have  $a - a = 0 \in \mathbb{P} \cup \{0\}$  (additive inverses).

(ii) Suppose that  $a \leq b$  and  $b \leq a$ .

If  $a - b = 0$  or  $b - a = 0$  then  $a = b$  (properties of  $+$ ) and we are done.

If not, then  $b - a \in \mathbb{P}$  and  $a - b \in \mathbb{P}$ .

But  $b - a = -(a - b)$  (properties of  $+$ ),

so then  $a - b \in \mathbb{P}$  and  $-(a - b) \in \mathbb{P}$ , contradicting ‘positive, negative or 0’.

(iii) Note that  $c - a = c + (-a) = c + 0 + (-a) = c + (-b) + b + (-a) = (c - b) + (b - a)$  (properties of  $+$ )

so if  $a < b$  and  $b < c$  then  $a < c$  ( $+$  and ordering).

The cases where  $a = b$  and/or  $b = c$  are straightforward, and give the result for  $\leq$ .

(iv) This follows from ‘positive, negative or 0’.

□

In the next section, we’ll explore the interaction between the ordering and basic arithmetic.

## 5 Inequalities and arithmetic

The next result has some useful results about inequalities and arithmetic, which will save us from having to go back to the axioms every time.

**Proposition 4.** *Take  $a, b, c \in \mathbb{R}$ .*

(i)  $0 < 1$ .

(ii)  $a < b$  if and only if  $-b < -a$ . In particular,  $a > 0$  if and only if  $-a < 0$ .

(iii) If  $a < b$  then  $a + c < b + c$ .

(iv) If  $a < b$  and  $0 < c$  then  $ac < bc$ .

(v)  $a^2 \geq 0$ , with equality if and only if  $a = 0$ .

(vi)  $a > 0$  if and only if  $\frac{1}{a} > 0$ .

(vii) If  $a, b > 0$  and  $a < b$  then  $\frac{1}{b} < \frac{1}{a}$ .

Furthermore, (ii), (iii) and (iv) hold with  $\leq$  in place of  $<$ .

*Proof.* (i) By trichotomy, we have  $0 < 1$  or  $0 = 1$  or  $0 > 1$ .

But ‘to avoid total collapse’  $0 \neq 1$ . So it suffices to rule out  $0 > 1$ .

Suppose, for a contradiction, that  $0 > 1$ .

Then  $-1 \in \mathbb{P}$  (by definition of  $>$ ) so  $(-1) \cdot (-1) \in \mathbb{P}$  ( $\cdot$  and ordering).

But  $(-1) \cdot (-1) = 1$  (Proposition 1 (xii)),

so  $0 < 1$  — but this contradicts trichotomy.

So  $0 < 1$ .

(ii) Using properties of addition, we have

$$\begin{aligned} a < b &\Leftrightarrow b - a \in \mathbb{P} \\ &\Leftrightarrow (-a) - (-b) \in \mathbb{P} \\ &\Leftrightarrow -a > -b. \end{aligned}$$

(iii) Assume that  $a < b$ .

Then  $(b + c) - (a + c) = b - a > 0$  so  $a + c < b + c$ .

(iv) Assume that  $a < b$  and  $0 < c$ .

Then  $bc - ac = (b - a)c > 0$  ( $\cdot$  and ordering).

(v) Certainly  $a^2 = 0$  if and only if  $a = 0$  (Proposition 1 (x) and (xiii)).

If  $a \neq 0$ , then exactly one of  $a$  and  $-a$  is positive, and either way  $a^2 = a \cdot a = (-a) \cdot (-a) > 0$  ( $\cdot$  and ordering).

(vi) Suppose, for a contradiction, that  $a > 0$  and  $\frac{1}{a} < 0$ , so  $a > 0$  and  $-\frac{1}{a} > 0$ .

Then  $-1 = -\left(a \cdot \frac{1}{a}\right) = a \cdot \left(-\frac{1}{a}\right) > 0$ . But this contradicts (i).

Similarly if  $a < 0$  and  $\frac{1}{a} > 0$ .

(vii) Suppose that  $a, b > 0$  and  $a < b$ .

Then  $\frac{1}{a}, \frac{1}{b} > 0$  by (vi),

so  $a \cdot \frac{1}{a} \cdot \frac{1}{b} < b \cdot \frac{1}{a} \cdot \frac{1}{b}$  by (iv),

so  $\frac{1}{b} < \frac{1}{a}$ .

□

Now we can prove a useful inequality (you'll have an opportunity to apply it on Sheet 1).

**Theorem 5** (Bernoulli's Inequality). *Let  $x$  be a real number with  $x > -1$ . Let  $n$  be a positive integer. Then  $(1 + x)^n \geq 1 + nx$ .*

*Proof.* By induction on  $n$ . Fix  $x > -1$ .

$n = 1$ : clear.

induction step: suppose the result holds for some  $n \geq 1$ , that is,  $(1 + x)^n \geq 1 + nx$ .

Note that  $1 + x > 0$ , and  $nx^2 \geq 0$  (since  $n > 0$  and  $x^2 \geq 0$  by Proposition 4 (v)).

Then

$$\begin{aligned}(1+x)^{n+1} &= (1+x)(1+x)^n \quad (\text{by definition}) \\ &\geq (1+x)(1+nx) \quad (\text{induction hypothesis and Prop 4 (iv)}) \\ &= 1 + (n+1)x + nx^2 \quad (\text{properties of arithmetic}) \\ &\geq 1 + (n+1)x \quad (\text{since } nx^2 \geq 0).\end{aligned}$$

So, by induction, the result holds. □

In the next section, we'll move on to consider the modulus of a real number.

## 6 The modulus of a real number

**Definition.** Let  $a \in \mathbb{R}$ . The *modulus*  $|a|$  of  $a$  is defined to be

$$|a| := \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0. \end{cases}$$

(It is also sometimes called the *absolute value* of  $a$ .)

**Remark.** The modulus is well defined (that is, this is a legitimate definition) thanks to the ‘positive, negative or 0’ property (essentially trichotomy).

Here are some basic properties of the modulus.

**Proposition 6.** Take  $a, b, c \in \mathbb{R}$ . Then

(i)  $|-a| = |a|;$

(ii)  $|a| \geq 0;$

(iii)  $|a|^2 = a^2$ ;

(iv)  $|ab| = |a||b|$ ;

(v)  $-|a| \leq a \leq |a|$ ;

(vi) if  $c \geq 0$ , then  $|a| \leq c$  if and only if  $-c \leq a \leq c$ ; and similarly with weak inequalities ( $\leq, \geq$ ) replaced by strict ( $<, >$ ).

*Proof*(i), (ii) Immediate from the definition, since  $a > 0$  if and only if  $-a < 0$ .

(iii) Check using the definition and trichotomy – go through the cases and also use  $(-a)(-a) = a^2$ .

(iv) Check the cases using the definition and trichotomy.

(v) If  $a \geq 0$ , then  $-|a| \leq 0 \leq a = |a|$ .

If  $a < 0$ , then  $-|a| = a < 0 \leq |a|$ .

(vi) Assume that  $c \geq 0$ .

( $\Rightarrow$ ) Suppose that  $|a| \leq c$ . Then, by (v),  $-c \leq -|a| \leq a \leq |a| \leq c$ , and we're done by transitivity (Proposition 3).

( $\Leftarrow$ ) Suppose that  $-c \leq a \leq c$ . Then  $-a \leq c$  and  $a \leq c$ . But  $|a|$  is  $a$  or  $-a$ , so  $|a| \leq c$ .

Similarly for the version with strict inequalities.

□

**Theorem 7** (Triangle Inequality). *Take  $a, b \in \mathbb{R}$ . Then*

(i)  $|a + b| \leq |a| + |b|$ ;

(ii)  $|a + b| \geq ||a| - |b||$ .

**Remark.** (ii) is called the *Reverse Triangle Inequality*.

*Proof.* (i) We have  $-|a| \leq a \leq |a|$  and  $-|b| \leq b \leq |b|$ , by Proposition 6.

We can add these (see Sheet 1 Q2); using properties of addition, we get  $-(|a| + |b|) \leq a + b \leq |a| + |b|$ .

By Proposition 6 (vi) (with  $c = |a| + |b| \geq 0$ ), this gives  $|a + b| \leq |a| + |b|$ .

(ii) By (i), we have  $|a| = |a + b + (-b)| \leq |a + b| + |-b| = |a + b| + |b|$ ,

so  $|a + b| \geq |a| - |b|$ .

Similarly (swap  $a$  and  $b$ ),  $|a + b| \geq |b| - |a|$ .

Now  $||a| - |b||$  is  $|a| - |b|$  or  $|b| - |a|$ , so  $|a + b| \geq ||a| - |b||$ .

□

## 7 The complex numbers

You met (or renewed your acquaintance with) the set  $\mathbb{C}$  of complex numbers in the course *Introduction to Complex Numbers* ???, so we shan't repeat much of that material here. We'll just focus on revisiting the complex numbers with the new perspective of having considered axioms for  $\mathbb{R}$ .

In particular, as we said earlier,  $\mathbb{C}$  is a field (under the usual addition and multiplication). If you are feeling enthusiastic, then check the axioms!

But  $\mathbb{C}$  is fundamentally different from  $\mathbb{R}$  because there is no ordering on  $\mathbb{C}$  that satisfies the ordering axioms.

**Exercise.** Prove this!

As you saw in the Complex Numbers course, the Triangle Inequality holds in  $\mathbb{C}$ , and so does the Reverse Triangle Inequality.

So far, we know that  $\mathbb{R}$  is an ordered field. But we still haven't captured everything that's important about  $\mathbb{R}$ . The rational numbers  $\mathbb{Q}$  also form an ordered field, but there are fundamental differences between  $\mathbb{Q}$  and  $\mathbb{R}$ . Intuitively, in  $\mathbb{Q}$  there are 'gaps' in the number line (such as at  $\sqrt{2}$ ), whereas this is not the case in  $\mathbb{R}$ . In the next chunk of the course, we'll explore this in detail, to try to identify what additional property or properties we need to assume hold in  $\mathbb{R}$ .

## 8 Upper and lower bounds

In the next few sections of the course, we going to explore the difference between  $\mathbb{Q}$  and  $\mathbb{R}$ , as discussed in the last paragraph of the previous section.

One helpful example to have in mind is the square root of two. We know that  $\sqrt{2}$  is not in  $\mathbb{Q}$  (you have probably seen a proof of this elsewhere). But there *is* a positive real number that squares to give 2. One goal of these sections of the course is to prove the existence of this positive real number (which we call  $\sqrt{2}$ ).

The key property, which  $\mathbb{R}$  has and  $\mathbb{Q}$  does not, is called *completeness*. But before we get there, we need a few preliminary definitions.

**Definition.** Let  $S \subseteq \mathbb{R}$ . Take  $b \in \mathbb{R}$ . We say that

- $b$  is an *upper bound* of  $S$  if  $s \leq b$  for all  $s \in S$ ;
- $b$  is a *lower bound* of  $S$  if  $s \geq b$  for all  $s \in S$ ;
- $S$  is *bounded above* if  $S$  has an upper bound;
- $S$  is *bounded below* if  $S$  has a lower bound;
- $S$  is *bounded* if  $S$  is bounded above and below.



**Example.** I could have put various examples and non-examples here, but instead I've put them in a short Moodle quiz for you to try. You'll get immediate feedback on your answers, to help you explore and check your understanding of the definitions. Please go to the Moodle course page for Analysis I, and try quiz 8.1, before you read on to the next section.

## 9 The (nearly) empty section

Sorry, this is here just to resolve section/video numbering issues!

## 10 Supremum, infimum and completeness

Some upper bounds are more interesting than others. The set  $[0, 1]$  has upper bounds including 15, 1, 1.7 and infinitely many more. Of these, 1 feels special. This is the focus of our next definition.

**Definition.** Let  $S \subseteq \mathbb{R}$ . We say that  $\alpha \in \mathbb{R}$  is the *supremum* of  $S$ , written  $\sup S$ , if

- (i)  $s \leq \alpha$  for all  $s \in S$ ; ( $\alpha$  is an upper bound of  $S$ )
- (ii) if  $s \leq b$  for all  $s \in S$  then  $\alpha \leq b$  ( $\alpha$  is the least upper bound of  $S$ ).

**Remark.** If  $S$  has a supremum, then  $\sup S$  is unique. (Check you can show this!)

Now that we have defined the supremum, we can state our final key property of  $\mathbb{R}$  (in addition to the properties that make it an ordered field).

**Completeness axiom for the real numbers** Let  $S$  be a non-empty subset of  $\mathbb{R}$  that is bounded above. Then  $S$  has a supremum.

**Remark.** There are two conditions on  $S$  here: non-empty, and bounded above. They are both crucial!

It is easy to forget the non-empty condition, but it has to be there: the empty set does not have a supremum, because every real number is an upper bound for the empty set — there is no *least* upper bound.

The condition that  $S$  is bounded above is also necessary: a set with no upper bound certainly has no supremum.

**Example.** • Let  $S = [1, 2)$ . Then 2 is an upper bound, and is the least upper bound: if  $b < 2$  then  $b$  is not an upper bound because  $\max(1, 1 + \frac{b}{2}) \in S$  and  $\max(1, 1 + \frac{b}{2}) > b$ . Note that in this case  $\sup S \notin S$ .

• Let  $S = (1, 2]$ . Then we again have  $\sup S = 2$ , and this time  $\sup S \in S$ .

The supremum is the least upper bound of a set. There's an analogous definition for lower bounds.

**Definition.** Let  $S \subseteq \mathbb{R}$ . We say that  $\alpha \in \mathbb{R}$  is the *infimum* of  $S$ , written  $\inf S$ , if

- $s \geq \alpha$  for all  $s \in S$ ; ( $\alpha$  is a lower bound of  $S$ )
- if  $s \geq b$  for all  $s \in S$  then  $\alpha \geq b$  ( $\alpha$  is the greatest lower bound of  $S$ ).

Let's explore some useful properties of sup and inf.

**Proposition 8.** (i) Let  $S, T$  be non-empty subsets of  $\mathbb{R}$ , with  $S \subseteq T$  and with  $T$  bounded above. Then  $S$  is bounded above, and  $\sup S \leq \sup T$ .

(ii) Let  $T \subseteq \mathbb{R}$  be non-empty and bounded below. Let  $S = \{-t : t \in T\}$ . Then  $S$  is non-empty and bounded above. Furthermore,  $\inf T$  exists, and  $\inf T = -\sup S$ .

**Remark.** (ii) and a similar result with sup and inf swapped essentially tell us that we can pass between sups and infs. Any result we prove about sup will have an analogue for inf. Also, we could have phrased the Completeness Axiom in terms of inf instead of sup. Proposition 8(ii) tells us that we don't need separate axioms for sup and inf.

*Proof.* (i) Since  $T$  is bounded above, it has an upper bound, say  $b$ .

Then  $t \leq b$  for all  $t \in T$ , so certainly  $t \leq b$  for all  $t \in S$ , so  $b$  is an upper bound for  $S$ .

Now  $S, T$  are non-empty and bounded above, so by completeness each has a supremum.

Note that  $\sup T$  is an upper bound for  $T$  and hence also for  $S$ , so  $\sup T \geq \sup S$  (since  $\sup S$  is the *least* upper bound for  $S$ ).

(ii) Since  $T$  is non-empty, so is  $S$ .

Let  $b$  be a lower bound for  $T$ , so  $t \geq b$  for all  $t \in T$ .

Then  $-t \leq -b$  for all  $t \in T$ , so  $s \leq -b$  for all  $s \in S$ , so  $-b$  is an upper bound for  $S$ .

Now  $S$  is non-empty and bounded above, so by completeness it has a supremum.

Then  $s \leq \sup S$  for all  $s \in S$ , so  $t \geq -\sup S$  for all  $t \in T$ , so  $-\sup S$  is a lower bound for  $T$ .

Also, we saw before that if  $b$  is a lower bound for  $T$  then  $-b$  is an upper bound for  $S$ .

Then  $-b \geq \sup S$  (since  $\sup S$  is the *least* upper bound),

so  $b \leq -\sup S$ .

So  $-\sup S$  is the greatest lower bound.

So  $\inf T$  exists and  $\inf T = -\sup S$ .

□

You might be wondering how all this relates to familiar notions of maximum and minimum so let's explore that.

**Definition.** Let  $S \subseteq \mathbb{R}$  be non-empty. Take  $M \in \mathbb{R}$ . We say that  $M$  is the *maximum* of  $S$  if

- (i)  $M \in S$ ; ( $M$  is an element of  $S$ )
- (ii)  $s \leq M$  for all  $s \in S$  ( $M$  is an upper bound for  $S$ ).

**Remark.** • If  $S$  is empty or  $S$  is not bounded above then  $S$  does not have a maximum. (Check this!)

- Let  $S \subseteq \mathbb{R}$  be non-empty and bounded above, so (by completeness)  $\sup S$  exists.

Then  $S$  has a maximum if and only if  $\sup S \in S$ .

Also, if  $S$  has a maximum then  $\max S = \sup S$ .

(Check this!)

**Definition.** Let  $S \subseteq \mathbb{R}$  be non-empty. Take  $m \in \mathbb{R}$ . We say that  $m$  is the *minimum* of  $S$  if

- (i)  $m \in S$ ; ( $m$  is an element of  $S$ )
- (ii)  $s \geq m$  for all  $s \in S$  ( $m$  is a lower bound for  $S$ ).

Here is a key result about the supremum, which we'll use *a lot*. It is a quick consequence of the definition, but it will be useful to have formulated it in this way.

**Proposition 9** (Approximation Property). *Let  $S \subseteq \mathbb{R}$  be non-empty and bounded above. For any  $\varepsilon > 0$ , there is  $s_\varepsilon \in S$  such that  $\sup S - \varepsilon < s_\varepsilon \leq \sup S$ .*

*Proof.* Take  $\varepsilon > 0$ .

Note that by definition of the supremum we have  $s \leq \sup S$  for all  $s \in S$ .

Suppose, for a contradiction, that  $\sup S - \varepsilon \geq s$  for all  $s \in S$ .

Then  $\sup S - \varepsilon$  is an upper bound for  $S$ , but  $\sup S - \varepsilon < \sup S$ . Contradiction.

So there is  $s_\varepsilon \in S$  with  $\sup S - \varepsilon < s_\varepsilon$ . □

## 11 Existence of roots

Now that we have identified the completeness property of  $\mathbb{R}$ , we are ready to prove that  $\mathbb{R}$  contains a square root of 2.

**Theorem 10.** *There exists a unique positive real number  $\alpha$  such that  $\alpha^2 = 2$ .*

*Proof. Existence* Let  $S = \{s \in \mathbb{R} : s > 0, s^2 < 2\}$ .

Idea: argue that  $S$  has a supremum, and show that  $\sup S$  has the required properties.

Note that  $S$  is non-empty (eg  $1 \in S$ )

and  $S$  is bounded above, because if  $x > 2$  then  $x^2 > 4$  (properties of ordering) so  $x \notin S$ , so 2 is an upper bound for  $S$ .

So, by completeness,  $S$  has a supremum. Let  $\alpha = \sup S$ .

Note that certainly  $\alpha > 0$  (since  $1 \in S$  so  $\alpha \geq 1$ ).

By trichotomy, we have  $\alpha^2 < 2$  or  $\alpha^2 = 2$  or  $\alpha^2 > 2$ .

Idea: show that if  $\alpha^2 < 2$  or  $\alpha^2 > 2$  then we get a contradiction.

**Case 1** Suppose, for a contradiction, that  $\alpha^2 < 2$ .

Then  $\alpha^2 = 2 - \varepsilon$  for some  $\varepsilon > 0$ .

Idea: consider  $\alpha + h$  for a small  $h > 0$ . Later on, we'll choose  $h$  small enough that  $(\alpha + h)^2 < 2$ , and that will be a contradiction because  $\alpha + h \in S$  and  $\alpha + h > \sup S$ .

Note that  $\alpha \leq 2$  (we said earlier that 2 is an upper bound for  $S$ ).

For  $h \in (0, 1)$  we have

$$\begin{aligned}(\alpha + h)^2 &= \alpha^2 + 2\alpha h + h^2 \\ &= 2 - \varepsilon + 2\alpha h + h^2 \\ &\leq 2 - \varepsilon + 4h + h \\ &\leq 2 - \varepsilon + 5h\end{aligned}$$

so let  $h = \min(\frac{\varepsilon}{10}, \frac{1}{2})$  and then  $(\alpha + h)^2 < 2$ .

Now  $\alpha + h \in S$  and  $\alpha + h > \sup S$ . This is a contradiction.

So it is not the case that  $\alpha^2 < 2$ .

**Case 2** Suppose, for a contradiction, that  $\alpha^2 > 2$ .

Then  $\alpha^2 = 2 + \varepsilon$  for some  $\varepsilon > 0$ .

Idea: consider  $\alpha - h$  for a small  $h > 0$ . Later on, we'll choose  $h$  small enough that  $(\alpha - h)^2 > 2$ , and that will lead to a contradiction because  $\alpha - h < \sup S$ .

For  $h \in (0, 1)$  we have

$$\begin{aligned}(\alpha - h)^2 &= \alpha^2 - 2\alpha h + h^2 \\ &= 2 + \varepsilon - 2\alpha h + h^2 \\ &\geq 2 + \varepsilon - 4h\end{aligned}$$

so choose  $h = \min(\frac{\varepsilon}{8}, \frac{1}{2}, \frac{\alpha}{2})$  and then  $(\alpha - h)^2 > 2$  (and also  $\alpha - h > 0$ ).

Now  $\alpha - h < \sup S$ , so by the Approximation property there is  $s \in S$  with  $\alpha - h < s$ .

But then  $2 < (\alpha - h)^2 < s^2 < 2$ , which is a contradiction.

So it is not the case that  $\alpha^2 > 2$ .

Hence, by trichotomy,  $\alpha^2 = 2$ .

**Uniqueness** Suppose that  $\beta$  is also a positive real number such that  $\beta^2 = 2$ .

Aim:  $\alpha = \beta$ .

Then  $0 = \alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta)$

and  $\alpha + \beta > 0$ , so  $\alpha = \beta$ . □

**Proposition 11.**  $\mathbb{Q}$  is not complete (with the ordering inherited from  $\mathbb{R}$ ).

*Proof.* If  $\mathbb{Q}$  were complete, then the proof of Theorem 10 would work just as well in  $\mathbb{Q}$ . But we know that there is not an element of  $\mathbb{Q}$  that squares to 2. So  $\mathbb{Q}$  is not complete. □

**Theorem 12.** Let  $n$  be an integer with  $n \geq 2$ , and take a positive real number  $r$ . Then  $r$  has a real  $n^{\text{th}}$  root.

*Proof.* Exercise. (See Sheet 2 for the case of the cube root of 2.) □

## 12 More consequences of completeness

In this course, we write  $\mathbb{N}$  for the set of positive integers, so  $\mathbb{N} = \mathbb{Z}^{>0}$ .

**Theorem 13** (Archimedean property of  $\mathbb{N}$ ).  $\mathbb{N}$  is not bounded above.

*Proof.* Idea: if there's an upper bound then we can find a natural number just less than it, and add 1.

Suppose, for a contradiction, that  $\mathbb{N}$  is bounded above.

Then  $\mathbb{N}$  is non-empty and bounded above, so by completeness (of  $\mathbb{R}$ )  $\mathbb{N}$  has a supremum.

By the Approximation property with  $\varepsilon = \frac{1}{2}$ , there is a natural number  $n \in \mathbb{N}$  such that  $\sup \mathbb{N} - \frac{1}{2} < n \leq \sup \mathbb{N}$ .

Now  $n + 1 \in \mathbb{N}$  and  $n + 1 > \sup \mathbb{N}$ . This is a contradiction.  $\square$

**Corollary 14.** *Let  $\varepsilon > 0$ . Then there is  $n \in \mathbb{N}$  such that  $0 < \frac{1}{n} < \varepsilon$ .*

*Proof.* If not, then  $\frac{1}{\varepsilon}$  would be an upper bound for  $\mathbb{N}$ . This would contradict Theorem 13.  $\square$

**Theorem 15.** *Let  $S$  be a non-empty subset of  $\mathbb{Z}$ .*

(i) *If  $S$  is bounded below, then  $S$  has a minimum.*

(ii) *If  $S$  is bounded above, then  $S$  has a maximum.*

*Proof.* (i) Assume that  $S$  is bounded below.

Then, by completeness (applied to  $\{-s : s \in S\}$ ),  $S$  has an infimum.

Secret aim:  $\inf S \in S$ .

By the Approximation property (with  $\varepsilon = 1$ ), there is  $n \in S$  such that  $\inf S \leq n < \inf S + 1$ . Aim:  $\inf S = n$ .

Suppose, for a contradiction, that  $\inf S < n$ .

Write  $n = \inf S + \delta$ , where  $0 < \delta < 1$ .

By the Approximation property (with  $\varepsilon = \delta$ ), there is  $m \in S$  such that  $\inf S \leq m < \inf S + \varepsilon = n$ .

Now  $m < n$  so  $n - m > 0$

but  $n - m$  is an integer, so  $n - m \geq 1$ .

Now  $n \geq m + 1 \geq \inf S + 1$ . This is a contradiction.

So  $n = \inf S \in S$  so  $\inf S = \min S$ .



(ii) Similar.

□

**Proposition 16.** *Take  $a, b \in \mathbb{R}$  with  $a < b$ . Then*

(i) *there is  $x \in \mathbb{Q}$  such that  $a < x < b$  (the rationals are dense in the reals); and*

(ii) *there is  $y \in \mathbb{R} \setminus \mathbb{Q}$  such that  $a < y < b$  (the irrationals are dense in the reals).*

*Proof.* Exercise (see Sheet 2).

□

## Summary of our work so far

$\mathbb{R}$  is a complete ordered field.

This sums up the key properties we have identified as our assumptions about  $\mathbb{R}$ . From this, we shall develop the theory of real analysis.

## 13 Countability

The concept of countability gives us a way to distinguish between sets by comparing their ‘sizes’. This will be a quick introduction. You can find more information in the supplementary notes by Dr Hilary Priestley, on the Moodle Analysis I course.

Our main tool for comparing the ‘sizes’ of two sets is to ask whether there is a bijection between them. In this section and the next, we’ll often be using the notions of bijection, injection and surjection. If you don’t feel confident with the definitions of these, then I recommend you remind yourself of the definitions before you continue with this section.

**Definition.** Let  $A$  be a set. We say that  $A$  is *finite* if  $A = \emptyset$  or there exists  $n \in \mathbb{N}$  such that there is a bijection  $f : A \rightarrow \{1, 2, \dots, n\}$ . We say that  $A$  is *infinite* if it is not finite

**Remark.**     • A subset of a finite set is finite.

- A non-empty finite subset of  $\mathbb{R}$  is bounded above (in fact, has a maximum) and so a subset of  $\mathbb{R}$  that is not bounded above is infinite.
- $\mathbb{N}$  is not bounded above (by the Archimedean property) so is infinite.

**Definition.** Let  $A$  be a set. We say that  $A$  is

- *countably infinite* if there is a bijection  $f : A \rightarrow \mathbb{N}$ ;
- *countable* if there is an injection  $f : A \rightarrow \mathbb{N}$ ;
- *uncountable* if  $A$  is not countable.

**Remark.** There are variations on the details of these definitions, so it's worth checking carefully if you're looking at a book or other source. For example, some people say 'countable' where we are using 'countably infinite'.

Here are a couple of useful properties.

**Proposition 17.** *Let  $A$  be a set.*

- (i)  *$A$  is countable if and only if  $A$  is countably infinite or finite.*
- (ii) *If there is an injection  $f : A \rightarrow B$  and an injection  $g : B \rightarrow A$ , then there is a bijection  $h : A \rightarrow B$ .*

*Proof.* Not in this course. See Priestley's supplementary notes on countability. □

**Proposition 18.** *Each of the following sets is countably infinite.*

(i)  $\mathbb{N}$

(ii)  $\mathbb{N} \cup \{0\}$

(iii)  $\{2k - 1 : k \in \mathbb{N}\}$

(iv)  $\mathbb{Z}$

(v)  $\mathbb{N} \times \mathbb{N}$ .

**Remark.** It might feel surprising that the set of odd natural numbers ‘has the same size as’ the set of all natural numbers!

*Proof.* (i) Clear.

(ii) Define  $f : \mathbb{N} \cup \{0\} \rightarrow \mathbb{N}$  by  $f(n) = n + 1$ . This is a bijection.

(iii) Define  $f : \mathbb{N} \rightarrow \{2k - 1 : k \in \mathbb{N}\}$  by  $f(n) = 2n - 1$ .

(iv) Idea: line up the integers as  $0, 1, -1, 2, -2, 3, -3, \dots$

Define  $f : \mathbb{Z} \rightarrow \mathbb{N}$  by

$$f(k) = \begin{cases} 2k & \text{if } k \geq 1 \\ 1 - 2k & \text{if } k \leq 0. \end{cases}$$

This is a bijection.

(v) Define  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  by  $f((m, n)) = 2^{m-1}(2n - 1)$ .

**Claim**  $f$  is a bijection.

**Proof of claim**

**injective:** If  $f((m_1, n_1)) = f((m_2, n_2))$  then

$$2^{m_1-1}(2n_1 - 1) = 2^{m_2-1}(2n_2 - 1),$$

so, by uniqueness of prime factorisation in  $\mathbb{N}$ ,  $2^{m_1-1} = 2^{m_2-1}$  and  $2n_1 - 1 = 2n_2 - 1$ ,

so  $m_1 = m_2$  and  $n_1 = n_2$ .

**surjective:** Take  $k \in \mathbb{N}$ .

Then  $k = 2^r(2s + 1)$  for some  $r, s \geq 0$

(consider the set  $T = \{t \in \mathbb{Z}^{\geq 0} : \frac{k}{2^t} \in \mathbb{N}\}$  — this is non-empty and bounded above so has a maximum).

Then  $k = f(r + 1, s + 1)$ .

□

## 14 More on countability

We can build new countable sets from old. This is a very helpful way to prove that a set is countable!

**Proposition 19.** *Let  $A, B$  be countable sets.*

(i) *If  $A$  and  $B$  are disjoint, then  $A \cup B$  is countable.*

(ii)  *$A \times B$  is countable.*

**Remark.** In (i), we don't need the condition that  $A$  and  $B$  are disjoint, but it makes life easier for our proof.

*Proof.* Since  $A$  and  $B$  are countable, there are injections  $f : A \rightarrow \mathbb{N}$  and  $g : B \rightarrow \mathbb{N}$ .

(i) Idea: to list elements of  $A \cup B$ , alternate taking elements from  $A, B$ .

Define  $h : A \cup B \rightarrow \mathbb{N}$  by

$$h(x) = \begin{cases} 2f(x) - 1 & \text{if } x \in A \\ 2g(x) & \text{if } x \in B. \end{cases}$$

This is an injection (because  $f$  and  $g$  are).

(ii) Define  $h : A \times B \rightarrow \mathbb{N}$  by  $h((a, b)) = 2^{f(a)}3^{g(b)}$ .

By the uniqueness of prime factorisation in  $\mathbb{N}$ , this is an injection.

□

**Theorem 20.**  $\mathbb{Q}^{>0}$  is countable.

*Proof.* Define  $f : \mathbb{Q}^{>0} \rightarrow \mathbb{N}$  by  $f(\frac{p}{q}) = 2^p 3^q$  where  $p, q \in \mathbb{Z}^{>0}$  and  $\text{hcf}(p, q) = 1$ .

This is an injection (by uniqueness of prime factorisation in  $\mathbb{N}$ ).

□

**Corollary 21.**  $\mathbb{Q}$  is countable.

*Proof.* We can write  $\mathbb{Q} = \mathbb{Q}^{>0} \cup \{0\} \cup \mathbb{Q}^{<0}$ . This is a disjoint union.

We have just seen that  $\mathbb{Q}^{>0}$  is countable, and similarly so is  $\mathbb{Q}^{<0}$ , and  $\{0\}$  is finite and hence countable.

Hence, by Proposition 19,  $\mathbb{Q}$  is countable.

□

Our next task will be to show that  $\mathbb{R}$  is uncountable. We're going to do this using decimal expansions. That means that we need to know that decimal expansions exist. Ideally we'd study sequences and series for a bit, then look at decimal expansions, then prove that  $\mathbb{R}$  is uncountable. But I'd like to wrap up this section on countability now. So we're going to assume a fact about decimal expansions, and deduce that  $\mathbb{R}$  is uncountable. I encourage you to revisit the fact later in the course — I'll let you know when we've covered the relevant theory. You could also (now) look at Dr Hilary Priestley's

supplementary notes on decimal expansions and the uncountability of  $\mathbb{R}$  (on the Moodle Analysis I page).

**Fact** Every real number has a decimal expansion, and if we require that we choose a non-terminating expansion (such as  $0.24999\dots$  for  $\frac{1}{4}$ ) rather than a terminating one (such as  $0.25$  for  $\frac{1}{4}$ ) where there is a choice, then this decimal expansion is unique.

**Theorem 22.**  $\mathbb{R}$  is uncountable.

**Remark.** The proof strategy we are going to use is called *Cantor's diagonal argument*.

*Proof.* It suffices to show that  $(0, 1]$  is uncountable.

Note that certainly  $(0, 1]$  is not finite (by Corollary 14 of the Archimedean property).

Suppose, for a contradiction, that  $(0, 1]$  is countably infinite. List the elements as  $x_1, x_2, x_3, \dots$ . (If you like, we have a bijection  $\mathbb{N} \rightarrow (0, 1]$ , and  $x_1 = f(1), x_2 = f(2), x_3 = f(3), \dots$ )

Each has a non-terminating decimal expansion (where relevant choosing the non-terminating option):

$$x_1 = 0.a_{11}a_{12}a_{13}a_{14}\dots$$

$$x_2 = 0.a_{21}a_{22}a_{23}a_{24}\dots$$

$$x_3 = 0.a_{31}a_{32}a_{33}a_{34}\dots$$

$$\vdots$$

$$x_k = 0.a_{k1}a_{k2}a_{k3}a_{k4}\dots$$

$$\vdots$$

Construct a real number  $x \in (0, 1]$  with decimal expansion  $0.b_1b_2b_3\dots$

where

$$b_k = \begin{cases} 5 & \text{if } a_{kk} = 6 \\ 6 & \text{if } a_{kk} \neq 6. \end{cases}$$

Then  $x \neq x_k$  for all  $k$ , because  $x$  differs from  $x_k$  in the  $k^{\text{th}}$  decimal place, so  $x$  is not on our list, which supposedly contained all elements of  $(0, 1]$ . This is a contradiction.  $\square$

**Remark.** The only significance of the choice of 5 and 6 as the key digits when defining  $x$  was that we didn't involve 0 or 9, to avoid issues with non-unique decimal expansions.

There are many further interesting things to say about countability, but not in this course. We need to move on to consider sequences. Before you continue to the next section, try to come up with a collection of examples of sequences that you can use as your personal repertoire for testing the definitions and results we'll look at. Try to find some 'typical' and some 'extreme' examples of sequences that you think converge, and of sequences that you think don't converge.

## 15 Introduction to sequences

**Remark.** You are already familiar with the sine and cosine, exponential and logarithm functions, and with raising a number to a non-integer power. At the moment, though, we haven't formally defined these. We'll do so later in the course (using infinite series—that's why it needs to wait till later), and this term and later in other Analysis courses you'll explore and prove the familiar properties of these function.