University of M'sila. 2023/2024
Faculty of Technology
Domaine: Engineering
Level: First year

Tutorial Series 02(Analysis 1)

Exercise 01

Prove (this means give $\varepsilon - N$ proof) that

1. $\dfrac{10}{n} \longrightarrow 0$ as $n \longrightarrow \infty$
2. $\dfrac{2n^2+1}{9n^2+5} \longrightarrow \dfrac{2}{9}$ as $n \longrightarrow \infty$
3. $\dfrac{1-2n}{3n+5} \longrightarrow \dfrac{-2}{3}$ as $n \longrightarrow \infty$
4. $\dfrac{n\cos\left(n^3+1\right)}{5n^2+1} \longrightarrow 0$ as $n \longrightarrow \infty$

Exercise 02

Determine whether each sequence converges or diverges. If the sequence converges, find its limit

1. $\left(\dfrac{n\,(n+1)}{3n^2+7n}\right)_{n\in\mathbb{N}^*}$
2. $\left(\dfrac{\sin n}{n}\right)_{n\in\mathbb{N}^*}$
3. $\left(\dfrac{\sqrt{n+1}}{\sqrt{3n+1}}\right)_{n\in\mathbb{N}^*}$
4. $\dfrac{2^n n!}{(2n+1)!}$

Exercise 03

Determine the limits of the following sequences $(a_n)_{n\in\mathbb{N}}$ whose n^{th} term a_n is given below.

1. $a_n = \dfrac{103n^2-8}{4n^2+99n-3}$
2. $a_n = -n + \sqrt{n^2+3n}$
3. $a_n = \dfrac{5n^3+3n+1}{15n^3+n^2+2}$
4. $a_n = \dfrac{\sin(n^2+1)}{n^2+1}$
5. $a_n = \dfrac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n+1}-\sqrt{n}}$
6. $a_n = \dfrac{n^2+n+1}{3n^2+4}$
7. $a_n = \sqrt{n^4+n^2} - n^2$
8. $a_n = -n + \sqrt{n^2+n}$
9. $a_n = \dfrac{\sin n}{n} + \left(\sqrt{n+1} - \sqrt{n}\right)$

Exercise 04

Show that if (a_n) is a sequence of real numbers which converge to L then the sequence (M_n) whose n^{th} term is

$$M_n = \frac{a_1 + a_2 + ... + a_n}{n}$$

also converges to L

Exercise 05

Which of the following assertions are true? Justify correct ones and give a counter example for each incorrect assertion.

(**a**) If a sequence is monotone and bounded, then it converges.
(**b**) If a sequence converges, then it is monotone and bounded.
(**c**) If a sequence is not bounded, then it is not convergent.
(**d**) If a sequence is not monotone, then it is not convergent.
(**e**) If a sequence has exactly one accumulation point, then it converges.
(**f**) If a sequence converges, then it has exactly one accumulation point.

Exercise 06

Let the sequence (a_n) be given by a starting value $a_0 \in [0; 2]$ and the recursion

$$a_{n+1} = \frac{a_n \left(a_n^2 + 3\right)}{3a_n^2 + 1}, \qquad n = 0, 1, 2, ...$$

1. Show that $a_{n+1} - 1 = \frac{(a_n - 1)^3}{3a_n^2 + 1}, \qquad n = 0, 1, 2, ...$

2. Prove the following two statements:

$$\begin{array}{ll} 0 < a_0 < 1 \Longrightarrow \quad 0 < a_n < 1 & \text{for all } n \in \mathbb{N}; \\ 1 < a_0 < 2 \Longrightarrow \quad 1 < a_n < 2 & \text{for all } n \in \mathbb{N}. \end{array}$$

3. Show that the sequence is strictly montonically increasing for $0 < a_0 < 1$ and strictly monotonically decreasing for $1 < a_0 < 2$.

4. For which $a_0 \in [0; 2]$ does the sequence converge? If so, determine the limit.

Exercice 07

Show that the sequences (x_n) whose n^{th} term is x_n is unbounded

1. $x_n = \frac{n^3 + 3n^2}{n + 1} - n^2$

2. $x_n = \left(n + \frac{1}{n}\right)^3 - n^3$

3. $x_n = \left(n + \frac{1}{n^2}\right)^4 - n^4$

Exercice 08

1. Given that $k! \geq 2^{k-1}$ for all $k \geq 1$, show that the sequence (a_n) whose n^{th} term is $a_n = \sum_{k=1}^{n} \frac{1}{k!}$ is bounded above by 3.

2. Explain why you can deduce that (a_n) converge

Exercice 09

1. Given that $k^k \geq 2^k$ for all $k \geq 2$, show that the sequence (a_n) whose n^{th} term is $a_n = \sum_{k=1}^{n} \frac{1}{k^k}$ is bounded above by $\frac{3}{2}$.
2. Explain why you can deduce that (a_n) converge

Exercice 10

Let $a_k = \frac{1}{k2^k}$, $b_k = \frac{k}{2^k}$, $S_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$

1. Find the limit of the sequence $\left(\frac{a_{k+1}}{a_k}\right)$ and $\left(\frac{b_{k+1}}{b_k}\right)$
2. Given that $a_k < \frac{1}{2^k} < b_k$ and $b_k \leq \left(\frac{3}{4}\right)^{k-2}$, $k \geq 3$. Explain why (S_n) and (t_n) both converge with

$$\lim_{n \longrightarrow \infty} S_n \leq 1 \leq \lim_{n \longrightarrow \infty} t_n < 4$$

Exercice 11

Given that $(1 + 1/n)^n \longrightarrow e = 2.718\cdots$ as $n \longrightarrow \infty$ and for $c > 0$, $c^{\frac{1}{n}} \longrightarrow 1$ as $n \longrightarrow \infty$ show the following.

1. $\left(1 + \frac{1}{n^2}\right) \longrightarrow 1$ as $n \longrightarrow \infty$
2. The sequence (a_n) defined by

$$a_n = \left(1 + \frac{1}{\sqrt{n}}\right)^n$$

   is unbounded
3. If $r = \dfrac{p}{q} \in \mathbb{Q}$ is a rational number and assuming that the sequence (t_n) defined by $t_n = \left(1 + \frac{r}{n}\right)^n$ converges, then the subsequence $(t_{np}) = (t_p, t_{2p}, ...)$ converges to e^r.

Exercice 12

Let (s_n) be the sequence given by $S_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + ... + \dfrac{1}{2n}$

1. Show that the sequence is increasing. Does it converges?
2. By noting that $0 < \int_{k}^{k+1} \frac{1}{k} dx - \frac{1}{k+1} < \frac{1}{k} - \frac{1}{k+1}$. Show that $0 < \ln 2 - S_n < \frac{1}{2n}$, and explain why you can deduce that (S_n) converge.