M'sila University
Faculty of Mathematics and Computer
Department of Mathematics
Year 2023/2024
Exam: ALG3 (S3 LMD)
Duration: 1 hour 30 min

Exercise 1. (8 pts)

We consider the following matrix: $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$

1. Show that $P(\lambda) = (\lambda - 1)^2(5 - \lambda)$.
2. Diagonalise A.
3. Find A^n.
4. Solve the system of recurrence $X_{n+1} = AX_n$.

Exercise 2. (10 pts)

We consider the following matrix: $B = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 1 & 5 \\ -4 & 0 & -3 \end{bmatrix}$

1. Show that $P(\lambda) = (1 - \lambda)^3$.
2. Find the minimal polynomial $Q(\lambda)$, what can we deduce?
3. Triangulate B.
4. We put $N = B - I_3$:
   - Verify that N is nilpotent and that $e^{tB} = e^t e^{tN}$.
   - Solve the differential system $X' = BX$.

Exercise 3. (2 pts)

Let A be an $n \times n$ nilpotent matrix:
Show that $\lambda = 0$ is the only eigenvalue of A and deduce that $P(\lambda) = \lambda^n$.

Exam Correction ALG3 2023/2024

Exo 1.

1] $P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{vmatrix} \xrightarrow{L_2+L_3 \to L_3}$

1 $= \begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ 0 & 1-\lambda & 1-\lambda \end{vmatrix} \xrightarrow{R_3 - R_2 \to R_2} \begin{vmatrix} \boxed{\begin{matrix} 2-\lambda & -3 \\ 1 & \lambda-4 \end{matrix}} & \begin{matrix} -1 \\ -1 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{1-\lambda} \end{vmatrix} = (\lambda-1)^2(5-\lambda)$

2] $P(\lambda) = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \ ; \ m_1 = 2 \\ \lambda_2 = 5 \ ; \ m_2 = 1 \end{cases}$

For $\lambda = \lambda_1 = 1$: $(A - \lambda_1 I_3)X = (A - I_3)X = 0 \Rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{cases} x + 2y - z = 0 \\ x + 2y - z = 0 \\ -x - 2y + z = 0 \end{cases} \Rightarrow \boxed{z = x + 2y}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+2y \end{pmatrix} = x\underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{V_1} + y\underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}_{V_2}$

1 $E(\lambda_1) = E(1) = \langle V_1, V_2 \rangle$ and $\boxed{\dim E(\lambda_1) = 2 = m_1}$

For $\lambda = \lambda_2 = 5$: $(A - \lambda_2 I_3)X = (A - 5I_3)X = 0 \Rightarrow \begin{pmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{cases} -3x + 2y - z = 0 \to (1) \\ x - 2y - z = 0 \to (2) \\ -x - 2y - 3z = 0 \to (3) \end{cases}$ $\boxed{x = y = -z}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x \\ -x \end{pmatrix} = x\underbrace{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}_{V_3}$

1 $E(\lambda_2) = E(5) = \langle V_3 \rangle$ and $\boxed{\dim E(\lambda_2) = 1 = m_2}$

$B' = (V_1, V_2, V_3)$: $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} = P^{-1}AP$ and

$P = (V_1 V_2 V_3) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$ 1

we have $X' = PX \iff X = P^{-1}X'$, $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$\Rightarrow \begin{cases} x' = x + z \longrightarrow (1) \\ y' = y + z \longrightarrow (2) \\ z' = x + 2y - z \longrightarrow (3) \end{cases}$ | By (1) − (2): $x' - y' = x - y$
By (1) + (3): $x' + z' = 2x + 2y$. 1

$\Rightarrow \begin{cases} x = \frac{3}{4}x' - \frac{1}{2}y' + \frac{1}{4}z' \\ y = -\frac{1}{4}x' + \frac{1}{2}y' + \frac{1}{4}z' \\ z = \frac{1}{4}x' + \frac{1}{2}y' - \frac{1}{4}z' \end{cases} \Rightarrow P^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{pmatrix}$

3] we have $D = P^{-1}AP \Rightarrow A = PDP^{-1} \Rightarrow \boxed{A^n = PD^nP^{-1}}$

$$A^n = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5^n \end{pmatrix} \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{pmatrix}$$

$$\Rightarrow \boxed{A^n = \begin{pmatrix} \frac{3+5^n}{4} & \frac{5^n-1}{2} & \frac{1-5^n}{4} \\ \frac{5^n-1}{4} & \frac{5^n+1}{2} & \frac{1-5^n}{4} \\ \frac{1-5^n}{4} & \frac{-5^n+1}{2} & \frac{3+5^n}{4} \end{pmatrix}}$$

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4] we have $X_{n+1} = AX_n \Rightarrow X_n = A^n X_0$ 1

$$\Rightarrow \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = A^n \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \Rightarrow \begin{cases} x_n = \left(\frac{3+5^n}{4}\right)\alpha + \left(\frac{5^n-1}{2}\right)\beta + \left(\frac{1-5^n}{4}\right)\gamma \\ y_n = \left(\frac{5^n-1}{4}\right)\alpha + \left(\frac{5^n+1}{2}\right)\beta + \left(\frac{1-5^n}{4}\right)\gamma \\ z_n = \left(\frac{1-5^n}{4}\right)\alpha + \left(\frac{1-5^n}{2}\right)\beta + \left(\frac{3+5^n}{4}\right)\gamma \end{cases}$$

Exo2. $B = \begin{pmatrix} 5 & 0 & 4 \\ 2 & 1 & 5 \\ -4 & 0 & -3 \end{pmatrix}$

1] $P(\lambda) = \text{dét}(B - \lambda I_3) = \begin{vmatrix} 5-\lambda & 0 & 4 \\ 2 & 1-\lambda & 5 \\ -4 & 0 & -3-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda)(-3-\lambda) + 16(1-\lambda)$.

$P(\lambda) = (1-\lambda)[(5-\lambda)(-3-\lambda)+16] = (1-\lambda)(\lambda^2 - 2\lambda + 1) = (1-\lambda)^3$.

$\boxed{P(\lambda) = (1-\lambda)^3}$ 1

2] 2
$$\begin{cases} Q(\lambda) = \lambda - 1 \longrightarrow N = B - I_3 = \begin{pmatrix} 4 & 0 & 4 \\ 2 & 0 & 5 \\ -4 & 0 & -4 \end{pmatrix} \neq O_3 \\ \text{or} \\ Q(\lambda) = (\lambda-1)^2 \longrightarrow N^2 = (B - I_3)^2 = \begin{pmatrix} 0 & 0 & 0 \\ -12 & 0 & -12 \\ 0 & 0 & 0 \end{pmatrix} \neq O_3 \\ \text{or} \\ Q(\lambda) = (\lambda-1)^3 \longrightarrow N^3 = (B - I_3)^3 = O_3 \text{ and } Q(A) = 0. \end{cases}$$

Then $\boxed{Q(\lambda) = (\lambda - 1)^3 = -P(\lambda)}$

We deduce B is not diagonalizable $\boxed{2}$

3] $P(\lambda)=0 \Rightarrow \lambda_1=1$, $m_1=3$

For $\underline{\lambda=\lambda_1=1}$: $(B-\lambda_1 I_3)X=0 \Rightarrow (\overbrace{B-I_3}^{N})X=0$
$\Rightarrow NX=0$

$$\Rightarrow \begin{pmatrix}4&0&4\\2&0&5\\-4&0&-4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{cases}2x+2z=0 \longrightarrow (1)\\ 2x+5z=0 \longrightarrow (2)\end{cases}$$

By (2)-(1): $\boxed{z=0} \Rightarrow \boxed{x=0}$ 1

$$X=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\y\\0\end{pmatrix}=\underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{V_1}$$

$E(\lambda_1)=\langle V_1\rangle$ and $\boxed{\dim E(\lambda_1)=1 \neq m_1=3}$

$$B'=(V_1,V_2,V_3): \begin{cases}V_1=e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}\\ V_2=e_1-e_3=\begin{pmatrix}1\\0\\-1\end{pmatrix}\\ V_3=e_3=\begin{pmatrix}0\\0\\1\end{pmatrix}\end{cases} \Rightarrow \begin{cases}e_1=V_2+V_3\\ e_2=V_1\\ e_3=V_3\end{cases}$$ 1

$$\begin{cases}f(V_1)=\lambda_1 V_1=V_1\\ f(V_2)=f(e_1-e_3)=f(e_1)-f(e_3)=5e_1+2e_2-4e_3-4e_1-5e_2+3e_3\\ \qquad =-3V_1+V_2\\ f(V_3)=f(e_3)=4e_1+5e_2-3e_3=4(V_2+V_3)+5V_1-3V_3\\ \qquad =5V_1+4V_2+V_3\end{cases}$$ 1

$$T=\begin{array}{c} \begin{array}{ccc} f(V_1) & f(V_2) & f(V_3)\end{array} \\ \begin{pmatrix}1&-3&5\\0&1&4\\0&0&1\end{pmatrix}\end{array}\begin{array}{c}V_1\\V_2\\V_3\end{array}=P^{-1}BP$$ 1

$$P=(V_1V_2V_3)=\begin{pmatrix}0&1&0\\1&0&0\\0&-1&1\end{pmatrix},\quad P^{-1}=(e_1e_2e_3)=\begin{pmatrix}0&1&0\\1&0&0\\1&0&1\end{pmatrix}$$

4] By 2) $N^3=O_3$, N is nilpotent, 3 is the index of N.

We have $e^{tB}=e^{t(N+I_3)}=e^{tN+tI_3}=e^{tN}\cdot e^{tI_3}$

or $e^{tI_3}=I_3e^t \Rightarrow e^{tB}=e^{tN}\cdot I_3e^t$

$$\Rightarrow \boxed{e^{tB}=e^t\cdot e^{tN}}$$ 1

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We have $X' = BX \Rightarrow X = e^{tB} X_0$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $X_0 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

and $e^{tB} = e^{t} \cdot e^{tN} = e^{t} \cdot \left(I_3 + tN + \frac{(tN)^2}{2} \right)$

$$e^{tB} = e^{t}\left(I_3 + tN + \frac{t^2}{2} N^2 \right)$$

$$= e^{t}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 4 & 0 & 4 \\ 2 & 0 & 5 \\ -4 & 0 & -4 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 0 & 0 & 0 \\ -12 & 0 & -12 \\ 0 & 0 & 0 \end{pmatrix} \right]$$

a)

$$= e^{t} \begin{pmatrix} 1+4t & 0 & 4t \\ 2t-6t^2 & 1 & 5t-6t^2 \\ -4t & 0 & 1-4t \end{pmatrix}$$

$$\Rightarrow \begin{cases} x = e^{t}\left[(1+4t)c_1 + 4t c_3 \right] \\ y = e^{t}\left[(2t-6t^2)c_1 + c_2 + (5t-6t^2)c_3 \right] \\ z = e^{t}\left[(-4t)c_1 + (1-4t)c_3 \right] \end{cases}$$

Exo3. We have A is nilpotent, then $\exists m \,/\, A^m = 0$ and m is the index of A, then $\boxed{Q(\lambda) = \lambda^m}$ is

a) the minimal polynomial and $\lambda = 0$ is the only eigenvalue of A

A is $n \times n$ matrix, Then $\boxed{P(\lambda) = \lambda^n}$