MOHAMED BOUDIAF UNIVERSITY OF MSILA
Faculty of Mathematics and Computer Science
Computer Science Department
University year :2023/2024

1.st year bachelor's degree - Semester 1
Final Exam in Analysis 1
Date : 15/01/2024 Duration : 1 h 30 m

Course questions : (05 Pt)

1. Using the definition of limit, verify that $\lim_{n\to\infty}[1+\frac{(-1)^n}{n}]=1$.
2. Write Rolle's theorem and applying it on the function $x \mapsto \sin x$ in the interval $[0,\pi]$.
3. Write the Intermediate value theorem.

Exercise 1 : (05,5 Pt)
(U_n) numerical sequence defined by

$$\begin{cases} U_0 = 1, \\ U_{n+1} = \sqrt{6+U_n}, \quad \forall n \in \mathbb{N}. \end{cases}$$

1. By induction, show that $\forall n \in \mathbb{N} : U_n \in]0, 10[$.
2. By induction, show that (U_n) is increasing.
3. Deduce that (U_n) is convergent and find its limit.

Exercise 2 : (04,5 Pt)
f real function defined by

$$\begin{cases} \cos^2(\pi x) \quad \text{if} \quad x \leq 1 \\ 1+\frac{\ln(x)}{x} \quad \text{if} \quad x > 1. \end{cases}$$

1. Find the definition domain of f.
2. Study the continuity and the differentiability of f on their domain of definition.

Exercise 3 (05 Pt) :
Let f real function defined on $]-2,+\infty[$ by the following relation :

$$f(x) = -x + \ln(x+2)$$

Prove that the equation $f(x) = 0$ has exactly two solutions c_1 and c_2 such that $-2 < c_1 < 0 < c_2$.

0.1 Solution

Course questions

1. We have

$$\lim_{n\to\infty} u_n = l \Leftrightarrow \forall\epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}(n > N) : |u_n - l| < \epsilon. \quad (0.5Pt)$$

So $|1 + \frac{(-1)^n}{n} - 1| = \frac{1}{n} < \epsilon,$ when $n > N = [\frac{1}{\epsilon}]$ (01 Pt)

2. **Theorem 1** ***Rolle's theorem****. If f is continuous on a closed interval $[a, b]$, and differentiable on the open interval $]a, b[$, and $f(a) = f(b)$, then there exists $c \in]a, b[$ such that $f'(c) = 0$. (01 Pt)*

   Application : sin is continuous on $[0, \pi]$, differentiable on $]0, \pi[$ and $\sin(0) = \sin(\pi) = 0$, then according to the Rolle's theorem there exists $c \in]0, \pi[$ such that

$$\cos c = 0 \Leftrightarrow c = \frac{\pi}{2}. \quad (01Pt)$$

3. **Theorem 2** ***Intermediate value theorem*** *: If the function f is continuous in the bounded and closed $[a, b]$ interval, then every value y between $f(a)$ and $f(b)$ is attained c in $[a, b]$, such that $y = f(c)$ (1.5 Pt)*

   In logical symbolism this theorem has the following expression :

$$f \in \mathcal{C}([a, b]), \text{ and } f(a).f(b) < 0 \Rightarrow \exists c \in]a, b[, \text{ such that } f(c) = 0.$$

Exercise 1 :

1. For $n = 0, U_0 = 1 \in]0, 10[$. true. We suppose that $U_n \in]0, 10[$ i.e $0 < U_n < 10$. (0.5 Pt) So

$$6 < U_n + 6 < 16 \Rightarrow \sqrt{6} < \sqrt{6 + U_n} < 4, \quad (0.5Pt)$$

   then

$$0 < \sqrt{6} < U_{n+1} < 4 < 10, \quad \text{hence :} \quad U_{n+1} \in]0, 10[. \quad (0.5Pt)$$

2. By induction we must prove that (U_n) is increasing $\forall n \in \mathbb{N} : U_n \leq U_{n+1}$

   For $n = 0,\ U_0 = 1,\ U_1 = \sqrt{6 + 1} = \sqrt{7},$ so $U_0 \leq U_1$. (01 Pt)

   We suppose that

$$U_n \leq U_{n+1} \Rightarrow 6 + U_n \leq 6 + U_{n+1} \Rightarrow \sqrt{6 + U_n} \leq \sqrt{6 + U_{n+1}} \Rightarrow U_{n+1} \leq U_{n+2}$$

   then (U_n) is increasing. (01 Pt)

3. (U_n) is increasing and majorante (bounded from above) then (U_n) is convergent. (0.5 Pt) let l its limit, l verify the equation

$$l = \sqrt{6 + l} \Leftrightarrow l^2 - l - 6 = 0 \Leftrightarrow l = -2 \text{ or } l = 3\ (0.5 + 0.5Pt)$$

   $l > 0,\ \forall n \in \mathbb{N} : U_n > 0,$ then $l = 3$. $(0.5Pt)$

Exercise 2 : The domain of definition is $D = \mathbb{R}$ (0.5 Pt)

1. **The continuity of** f on $\mathbb{R}$
   - On $]-\infty, 1[$, f is continuous (product two continuous functions.) (0.25 Pt)
   - On $]1, +\infty[$, f is continuous (sum and fractions two continuous functions). (0.25 Pt)

— <u>The continuity at $x_0 = 1$</u>

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \cos^2(\pi x) = (-1)^2 = 1 = f(1); (0.5Pt)$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1 + \frac{\ln x}{x}) = 1 = f(1); (0.5Pt)$$

then f is continuous at 1 because

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1);$$

from which f is continuous on $\mathbb{R}$. (0.5 Pt)

2. **The differentiability of** f on $\mathbb{R}$
   — On $]-\infty, 1[$, f is differentiable (product two differentiable functions.) (0.25 Pt)
   — On $]1, +\infty[$, f is differentiable (sum and fractions two differentiable functions).(0.25 Pt)
   — <u>The differentiability of</u> f at $x_0 = 1$

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{\cos^2(\pi x) - 1}{x - 1} = \lim_{x\to 1^-} \frac{-2\pi \sin(\pi x)\cos(\pi x)}{1} = 0 \ (0.75Pt)$$

$$\lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{1 + \frac{\ln x}{x} - 1}{x - 1} = \lim_{x\to 1^+} \frac{\ln x}{x(x-1)} = \lim_{x\to 1^+} \frac{1}{x(2x-1)} = 1 \ (0.75Pt)$$

Then f is not differentiable on $\mathbb{R}$

Exercise 4 :

Note that $f'(x) = -1 + \frac{1}{x+2} = \frac{-1-x}{x+2}$. then $f'(x) = 0 \Leftrightarrow x = -1$ (0.5 Pt)
So, on $]-2, -1]$, the function f is strictly increasing (0.5 Pt)
on $[-1, +\infty[$, the function f is strictly decreasing (0.5 Pt)
on the other hand, we have

$$\lim_{x\to -2} f(x) = -\infty \text{ and } \lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} -x[\frac{\ln(x+2)}{-x} + 1] = -\infty \quad (0.25 + 0.25Pt)$$

Thus

1. On $]-2, -1]$, we have
   — f is continuous (0,25)
   — $(\lim_{x\to -2} f(x)).f(-1) = -\infty$ (0.5 Pt)
   — f is strictly increasing.
   Then, according to the intermediate value theorem $\exists! c_1 \in]-2, -1]$ (0.5 Pt)
2. on $[-1, 0]$, since $f(-1) = 1$ and $f(0) = \ln(2) > 0$ then the equation $f(x) = 0$ has not solution (0.5 Pt)
3. On $[0, +\infty[$, we have
   — f is continuous (0,25)
   — $f(0).(\lim_{x\to =+\infty} f(x)) = -\infty < 0$ (0.5)
   — f is strictly decreasing
   Then, according to the intermediate value theorem $\exists! c_2 \in [0, +\infty[$ (0.5 Pt)