Chapter 3

Binary Relations on a Set

3.1 Basic Definitions

Definition 3.1 (Binary Relation) Let E be a set. A binary relation $\mathcal{R}$ on E is a property that applies to pairs of elements from E. We denote $x\mathcal{R}y$ to indicate that the property is true for the pair $(x, y) \in E \times E$.

Example

1. The inequality $\leq$ is a relation on $\mathbb{N}, \mathbb{Z}$, and $\mathbb{R}$.
2. The inclusion relation in the power set of E: $A\mathcal{R}B \Leftrightarrow A \subset B$.
3. The divisibility relation on the integers: $m\mathcal{R}n \Leftrightarrow m$ divides n.

Definition 3.2 Let $\mathcal{R}$ be a relation on a set E.

1. $\mathcal{R}$ is reflexive if for every $x \in E$, $x\mathcal{R}x$ holds.
2. $\mathcal{R}$ is symmetric if for all $x, y \in E$, $x\mathcal{R}y \Rightarrow y\mathcal{R}x$.
3. $\mathcal{R}$ is antisymmetric if for all $x, y \in E$, $(x\mathcal{R}y \wedge y\mathcal{R}x) \Rightarrow x = y$.
4. $\mathcal{R}$ is transitive if for all $x, y, z \in E$, $(x\mathcal{R}y \wedge y\mathcal{R}z) \Rightarrow x\mathcal{R}z$.

3.2 Equivalence Relations

Definition 3.3 (Equivalence Relation) A binary relation $\mathcal{R}$ on E is an equivalence relation if and only if it is reflexive, symmetric, and transitive.

Example 1 The relation $\mathcal{R}$ of "being parallel" is an equivalence relation for the set E of affine lines in the plane:

1. Reflexivity: A line is parallel to itself.

2. Symmetry: If line D is parallel to D', then D' is parallel to D.

3. Transitivity: If line D is parallel to D' and D' is parallel to D'', then D is parallel to D''.

Example 2 Consider the following relation on $\mathbb{Z}$:

$$x\mathcal{R}y \Leftrightarrow \exists k \in \mathbb{Z} \mid x - y = 2k$$

1. $\mathcal{R}$ is reflexive because $\exists k = 0 \mid x - x = 2k = 0$, thus $x\mathcal{R}x$.

2. Suppose $x\mathcal{R}y$, then $\exists k \in \mathbb{Z} \mid x - y = 2k \Rightarrow y - x = 2k'$ with $k' = -k \in \mathbb{Z}$. Therefore, $y\mathcal{R}x$. Hence, $\mathcal{R}$ is symmetric.

3. Suppose $x\mathcal{R}y$ and $y\mathcal{R}z$. Then, $(\exists k \in \mathbb{Z} \mid x - y = 2k)$ and $(\exists k' \in \mathbb{Z} \mid y - z = 2k')$ by adding these equations, we obtain $x - z = 2k''$ with $k'' = (k + k') \in \mathbb{Z}$. Thus, $x\mathcal{R}z$. Therefore, $\mathcal{R}$ is transitive. Consequently, $\mathcal{R}$ is an equivalence relation.

Definition 3.4 Let $\mathcal{R}$ be an equivalence relation on a set E. The equivalence class of an element $x \in E$ is the set of elements in E that are related to x by $\mathcal{R}$, denoted by $\mathcal{C}(x)$ or $\bar{x}$:

$$\bar{x} = \{y \in E \mid y\mathcal{R}x\}$$

Definition 3.5 Let $\mathcal{R}$ be an equivalence relation on a set E. The quotient set of E by $\mathcal{R}$ is the set of equivalence classes of $\mathcal{R}$, denoted by $E/\mathcal{R}$:

$$E/\mathcal{R} = \{\bar{x} \mid x \in E\}$$

Example In the previous example, we have

$$\bar{x} = \{y \in E \mid y\mathcal{R}x\}$$
$$= \{y \in E \mid x - y = 2k\}$$
$$= \{x - 2k : k \in \mathbb{Z}\}$$
$$= \{\dots, x-4, x-2, x, x+2, x+4, \dots\}.$$

$\overline{0} = \{y \in E \mid 0\mathcal{R}y\} = \{\dots, -4, -2, 0, 2, 4, \dots\}$, $\overline{1} = \{y \in E \mid 1\mathcal{R}y\} = \{\dots, -3, -1, 1, 3, \dots\}$

and $\overline{2} = \overline{0}$.**Therefore,**$\mathbb{Z}/\mathcal{R} = \{\bar{x} \mid x \in E\} = \{\overline{0}, \overline{1}\}$

Proposition 3.1 Let $\mathcal{R}$ be an equivalence relation on E. Then

1. An equivalence class is a subset of the set E, i.e., for all $x \in E$, $\bar{x} \subset E$.

2. An equivalence class is never empty, i.e., for all $x \in E$, $\bar{x} \neq \phi$.

3. The intersection of two distinct equivalence classes is empty, i.e., for all $x, y \in E$, $\bar{x} \cap \bar{y} = \phi$.

4. For all $x, y \in E$, $x\mathcal{R}y \Leftrightarrow \bar{x} = \bar{y}$.

Theorem 3.1 Let $\mathcal{R}$ be an equivalence relation on E. The equivalence classes $(\bar{x})_{x \in E}$ form a partition of E:

$$E = \cup_{x \in E} \bar{x}$$

3.3 Order Relation

Definition 3.6 (Order Relation) A binary relation $\mathcal{R}$ on E is an order relation if and only if it is reflexive, antisymmetric, and transitive. We then say that $(E, \mathcal{R})$ is an ordered set.

Example.

1. The inequality $\leq$ is an order relation on $\mathbb{N}, \mathbb{Z}$, and $\mathbb{R}$.

2. The inclusion relation in the power set of E is an order relation: $A\mathcal{R}B \Leftrightarrow A \subset B$.

Definition 3.7 Let $\mathcal{R}$ be an order relation on E. Two elements x and y of E are said to be comparable if $x\mathcal{R}y$ or $y\mathcal{R}x$.

Definition 3.8 (Total Order and Partial Order) Let $\mathcal{R}$ be an order relation on E. If any two elements x and y are always comparable, we say that $\mathcal{R}$ is a total order relation and the set E is called totally ordered. Otherwise (i.e., if there exist at least two non-comparable elements x and y), we say that $\mathcal{R}$ is a partial order relation and the set E is called partially ordered.

Example.

1. $\leq$ is a total order on $\mathbb{N}, \mathbb{Z}$, and $\mathbb{R}$.
2. The divisibility relation in $\mathbb{N}^*$ is a partial order.

Definition 3.9 Let $\mathcal{R}$ be an order relation on E, and let M, m be two elements of E.

1. M is an upper bound of a subset A of E if $x\mathcal{R}M$ for every $x \in A$.
2. m is a lower bound of a subset A of E if $m\mathcal{R}x$ for every $x \in A$.

Example.

1. The set $\{8, 10, 12\}$ is bounded above by 120 and bounded below by 2 for the divisibility relation "/" on $\mathbb{N}$.
2. $\mathcal{P}(E)$ is bounded below by $\emptyset$ and bounded above by E for the inclusion relation $\subset$.

3.4 Exercises with Solutions

Exercise 1. In $\mathbb{R}$, the binary relation $\mathcal{R}$ is defined as follows:

$$\forall x, y \in \mathbb{R} : x\mathcal{R}y \Longleftrightarrow x^2 - 1 = y^2 - 1$$

1. Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{R}$.
2. Determine the quotient set $\mathbb{R}/\mathcal{R}$.

Exercise 2. For every $n \in \mathbb{N}^*$, a binary relation on $\mathbb{Z}$ is defined by

$$\forall x, y \in \mathbb{Z} : x\mathcal{R}y \Longleftrightarrow \exists k \in \mathbb{Z} \mid x - y = kn$$

1. Show that $\mathcal{R}$ is an equivalence relation on $\mathbb{Z}$.

2. Assume that $n = 3$:

(a) Determine the equivalence class of $x \in \mathbb{Z}$. Deduce the classes $\overline{0}, \overline{1}, \overline{2}$.

(b) Show that $\forall m \in \mathbb{Z} : \overline{0} = \overline{3m}, \overline{1} = \overline{3m+1}, \overline{2} = \overline{3m+2}$.

(c) Show that $\overline{0} \cap \overline{1} = \emptyset, \overline{1} \cap \overline{2} = \emptyset, \overline{0} \cap \overline{2} = \emptyset$. Deduce the quotient set $\mathbb{Z}/\mathcal{R}$.

Exercise 3. Let E be a set and let A be a subset of E. A binary relation $\mathcal{R}$ is defined on $\mathcal{P}(E)$ as follows:

$$\forall X, Y \in \mathcal{P}(E) : X\mathcal{R}Y \Longleftrightarrow A \cap X = A \cap Y$$

1. Show that $\mathcal{R}$ is an equivalence relation on $\mathcal{P}(E)$.

2. Determine the equivalence classes of $\emptyset$ and E. Deduce $\bar{A}$ and $\overline{C_E(A)}$.

Exercise 4. Let $\mathcal{R}$ be a binary relation on $\mathbb{R}^3$ defined by

$$(x, y, z)\mathcal{R}(a, b, c) \Longleftrightarrow (|x - a| \leq b - y \text{ and } z = c).$$

1. Show that $\mathcal{R}$ is a partial order relation on $\mathbb{R}^3$.

2. Is the order total on $\mathbb{R}^3$?

Exercise 5. A binary relation $\mathcal{R}$ is defined on $\mathbb{R}^2$ as follows:

$$\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 : (x_1, y_1) \mathcal{R} (x_2, y_2) \Longleftrightarrow x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

1. Show that $\mathcal{R}$ is an order relation on $\mathbb{R}^2$.

2. Are the elements $(2, 4), (3, 1)$ of $\mathbb{R}^2$ comparable by $\mathcal{R}$?

3. Is the order total on $\mathbb{R}^2$?

4. Determine the set of upper bounds of $A = \{(1, 2), (3, 1)\} \subset \mathbb{R}^2$.

Exercise 6. Determine whether the following relations $\mathcal{R}$ are order relations:

1. $\forall x, y \in \mathbb{R} : x\mathcal{R}y \iff e^x \leq e^y$;

2. $\forall x, y \in \mathbb{R} : x\mathcal{R}y \iff |x| \leq |y|$;

3. $\forall x, y \in \mathbb{N} : x\mathcal{R}y \iff \exists p, q \geq 1 \mid y = px^q$ (where p and q are integers);

4. $\forall x, y \in \mathbb{N}^* : x\mathcal{R}y \iff \exists m \in \mathbb{N}^* \mid y = mx$;

5. $\forall x, y \in]1, +\infty[: x\mathcal{R}y \iff \frac{x}{1+x^2} \geq \frac{y}{1+y^2}$.

3.4.1 Solution

Exercise 1.

1. $\forall x, y \in \mathbb{R} : x\mathcal{R}y \iff x^2 - 1 = y^2 - 1$

(i) Reflexivity: $\forall x \in \mathbb{R}, x^2 - 1 = x^2 - 1 \Rightarrow xRx$.

(ii) Symmetry: $x\mathcal{R}y \Leftrightarrow x^2 - 1 = y^2 - 1 \Rightarrow y^2 - 1 = x^2 - 1 \Rightarrow y\mathcal{R}x$.

(iii) Transitivity:

$$\begin{cases} x\mathcal{R}y \\ y\mathcal{R}z \end{cases} \Leftrightarrow \begin{cases} x^2 - 1 = y^2 - 1 \\ y^2 - 1 = z^2 - 1 \end{cases} \Rightarrow x^2 - 1 = z^2 - 1 \Rightarrow x\mathcal{R}z.$$

Therefore, $\mathcal{R}$ is an equivalence relation.

2. $\mathbb{R}/\mathbb{R} = \{\bar{x} : x \in \mathbb{R}\}$.

We have $\bar{x} = \{y \in \mathbb{R} \mid y\mathbb{R}x\} = \{y \in \mathbb{R} \mid y^2 - 1 = x^2 - 1\} = \{x, -x \mid x \in \mathbb{R}\}$

Thus, $\mathbb{R}/\mathbb{R} = \{\{x_1 - x\}, x \in \mathbb{R}\}$.

Exercise 2.

1. $\forall x, y \in \mathbb{Z} : x\mathcal{R}y \Leftrightarrow \exists k \in \mathbb{Z} \mid x - y = kn$.

- Reflexivity: We know that $\forall x \in \mathbb{Z} : x - x = 0 = 0 \cdot n$ with $k = 0 \in \mathbb{Z}$, so $x\mathcal{R}x$.
- Symmetry: $x\mathcal{R}y \Leftrightarrow x - y = kn \Rightarrow y - x = (-k) \cdot n = k' \cdot n$ with $k' = -k \in \mathbb{Z}$. Thus, $y\mathcal{R}x$.

- Transitivity:

$$\begin{cases} x\mathcal{R}y \\ y\mathcal{R}z \end{cases} \Leftrightarrow \begin{cases} x-y=k_1\cdot n/k_1\in\mathbb{Z} \\ y-z=k_2\cdot n/k_2\in\mathbb{Z} \text{ gives:} \end{cases} \text{; Summing both sides:}$$

$$x-z=(k_1+k_2)n=k_3\cdot n \text{ with } k_3=k_1+k_2\in\mathbb{Z}$$

Therefore, $x\mathcal{R}z$

2. For $n=3:\forall x,y\in\mathbb{Z}:x\mathcal{R}y\Leftrightarrow\exists k\in\mathbb{Z}:x-y=3k$.

(a) For any

$$x\in\mathbb{Z}:\bar{x}=\{y\in\mathbb{Z}:y\mathcal{R}x\}=\{y\in\mathbb{Z}:y=x+3k\}$$
$$=\{x+3k\mid k\in\mathbb{Z}\}.$$

In particular:

$$\bar{0}=\{y\in\mathbb{Z}:y\mathcal{R}0\}=\{3k\mid k\in\mathbb{Z}\}=3\mathbb{Z}$$

$$\bar{1}=\{y\in\mathbb{Z}:y\mathcal{R}1\}=\{3k+1\mid k\in\mathbb{Z}\}=3\mathbb{Z}+1$$

$$\bar{2}=\{y\in\mathbb{Z}:y\mathcal{R}2\}=\{3k+2\mid k\in\mathbb{Z}\}=3\mathbb{Z}+2.$$

(b)

For all $m\in\mathbb{Z}$:

$$\begin{cases} \bar{0}=\bar{3m} \\ \bar{1}=\overline{3m+1} \\ \bar{2}=\overline{3m+2} \end{cases} \quad \text{because} \quad \forall m\in\mathbb{Z}:\begin{cases} 0\mathcal{R}(3m) \\ 1\mathcal{R}(3m+1) \\ 2\mathcal{R}(3m+2) \end{cases}.$$

Indeed, for all $m\in\mathbb{Z}$:

$$\begin{cases} 0-(3m)=3(-m) \\ 1-(3m+1)=3(-m) \\ 2-(3m+2)=3(-m) \end{cases}, \quad -m\in\mathbb{Z}.$$

(C)

We have:

$$\begin{cases} \bar{0} \cap \bar{1} = \emptyset \\ \bar{1} \cap \bar{2} = \emptyset \\ \bar{0} \cap \bar{2} = \emptyset \end{cases}, \text{because} \begin{cases} 0 \not{\mathcal{R}} 1 \\ 1 \not{\mathcal{R}} 2 \\ 0 \not{\mathcal{R}} 2 \end{cases}. \text{Indeed,} \begin{cases} 0-1=-1 \neq 3k_1 \\ 1-2=-1 \neq 3k_2 \\ 0-2=-2 \neq 3k_3 \end{cases}, \quad k_1, k_2, k_3 \in \mathbb{Z}\,.$$

We know that:

$$\begin{aligned} \mathbb{Z}/R &= \{\bar{x} : x \in \mathbb{Z}\} \\ &= \{\bar{x} : x = 3m\} \cup \{\bar{x} : x = 3m+1\} \cup \{\bar{x} : x = 3x+2\}. \\ &= \{\bar{0}, \bar{1}, \bar{2}\}. \end{aligned}$$

Exercise 4. $(x,y,z)R(a,b,c) \Leftrightarrow (|x-a| \leqslant b-y \text{ and } z=c)$

(1)

(i) Reflexivity: $(x,y,z)R(x,y,z) \Leftrightarrow (|x-x| = 0 \leqslant y-y = 0 \text{ and } z = z)$, hence R is reflexive.

(ii) Anti-symmetry: Suppose $(v,y,z)R(a,b,c)$ and $(a,b,c)R(x,y,z)$

This implies $[(|x-a| \leqslant b-y \quad (*) \text{ and } |a-x| \leqslant y-b \quad (**)) \text{ and } z=c]$

Then, $(*)+(**)$ gives: $x=a$, replacing $x=a$ in $(*)$ and $(**)$ we find $y=b$. Thus, $(x,y,z)=(a,b,c)$. Therefore, R is anti-symmetric.

(iii) Transitivity: Suppose $(v,y,z)R(a,b,c)$ and $(a,b,c)R(\alpha,\beta,\gamma)$

This implies $[(|x-a| \leqslant b-y \ (*) \text{ and } |a-\alpha| \leqslant \beta-b \ (**)) \text{ and } z=c=\gamma]$

Thus, $(*)+(**)$ gives $(|x-a|+|a-\alpha| \leqslant b-y+\beta-b \text{ and } z=c=\gamma)$.

And since $(|x-\alpha| = |x-a+a-\alpha| \leqslant |x-a|+|a-\alpha| \leqslant y+\beta \text{ and } z=\gamma)$ implies $(x,y,z)R(\alpha,\beta,\gamma)$. Hence, R is transitive.

Therefore, R is a partial order relation on $\mathbb{R}^3$.

(2) R is not total because $\exists (x,y,z) = (0,0,2) \in \mathbb{R}^3$ and $(a,b,c) = (0,0,3) \in \mathbb{R}^3$ such that $(0,0,2)\not{R}(0,0,3)$ and $(0,0,3)\not{R}(0,0,2)$.

Exercise 5. $\forall (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2 : x_1 \leqslant x_2$ and $y_1 \leqslant y_2$.

(1)

(i) Reflexivity: We know that

$$\forall(x,y)\in\mathbb{R}^2:\begin{cases} x\leqslant x \\ y\leqslant y\end{cases} \Rightarrow (x,y)R(x,y)\Rightarrow R \text{ is reflexive.}$$

(ii) Anti-symmetry: Suppose $(x_1,y_1)\,R\,(x_2,y_2)$ and $(x_1,y_2)\,R\,(x_1,y_1)$

$$\Rightarrow\begin{cases} x_1\leqslant x_2\wedge y_1\leqslant y_2 \\ \wedge \\ x_2\leqslant x_1\wedge y_2\leqslant y_1\end{cases}\Rightarrow\begin{cases} x_1=x_2 \\ \wedge \\ y_1=y_2\end{cases}\Rightarrow (x_1,y_1)=(x_2,y_2)\ .\ \text{Thus, } R \text{ is anti-symmetric.}$$

(iii) Transitivity: Let $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in\mathbb{R}^2$

$$\begin{cases} (x_1,y_1)\,R\,(x_2,y_2) \\ \wedge \\ (x_2,y_2)\,R\,(x_3,y_3)\end{cases}\Rightarrow\begin{cases} x_1\leqslant x_2\wedge y_1\leqslant y_2 \\ \wedge \\ x_2\leqslant x_3\wedge y_2\leqslant y_3\end{cases}\Rightarrow\begin{cases} x_1\leqslant x_3 \\ \wedge \\ y_1\leqslant y_3\end{cases}\Rightarrow (x_1,y_1)\,R\,(x_3,y_3)$$

Therefore, R is transitive. Hence, R is a partial order relation on $\mathbb{R}^2$.

(2) $(2,4)$ and $(3,1)$ are not comparable because $(1,4)$ and $(3,1)$ do not satisfy the relation. In fact, $\begin{cases} 2\leqslant 3 \\ \wedge \\ 4\not\leqslant 1\end{cases}$ and $\begin{cases} 3\not\leqslant 2 \\ \wedge \\ 1\leqslant 2\end{cases}\Rightarrow\begin{cases} (2,4)\not R(3,1) \\ \wedge \\ (3,1)\not R(2,4)\end{cases}$

(3) The order is partial because $\exists a=(2,4)$ and $b=(3,1)$ where $a\not R b$ and $b\not R a$.

(4) $t=(x,y)\in\mathbb{R}^2$ is an upper bound of A if $\forall a\in A: aRt$.

$$\Rightarrow\begin{cases} (1,2)R(x,y) \\ \wedge \\ (3,1)R(x,y)\end{cases}\Rightarrow\begin{cases} 1\leqslant x\wedge 2\leqslant y. \\ \wedge \\ 3\leqslant x\wedge 1\leqslant y.\end{cases}\Rightarrow\begin{cases} x\geqslant 3 \\ \wedge \\ y\geqslant 2\end{cases}$$

$$\Rightarrow \underset{\mathbb{R}^2}{\mathrm{Maj}}(A)=\{(x,y): x\geqslant 3\wedge y\geqslant 2\}.$$

Chapter 4

Algebraic Structures

4.1 Internal Composition Laws and Their Properties

4.1.1 Internal Composition Laws

Definition 4.1 Let E be a set. An internal composition law $*$ on E is a mapping from $E \times E$ to E:

$$\begin{aligned} * : E \times E &\longrightarrow E \\ (x, y) &\mapsto x * y \end{aligned}$$

Notations

1. Instead of ”internal composition law,” we also say ”operation of internal composition” or simply ”internal operation.”
2. $(E, *)$ is often used to denote a set E equipped with an internal operation $*$.

Example.

1. The laws $\cup$ (union), $\cap$ (intersection), and $\triangle$ (symmetric difference) on $\mathcal{P}(E)$ (the power set of E).
2. The law (composition) on $\mathcal{F}(E)$ (the set of functions from E to E).

3. The laws + and × on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$.

4. Let * be defined on $\mathbb{R}$ by $x * y = \frac{1}{x+y}$. Then * is not an internal operation since $(-1, 1)$ does not have an image.

Definition 4.2 (Stable Subset for an Operation) Let E be a set equipped with an internal composition law * and F be a subset of E. We say that F is stable under the law * if

$$\forall (x, y) \in F \times F : x * y \in F$$

Example.

1. $\mathbb{R}^+$ and $\mathbb{R}^-$ are two stable subsets of $\mathbb{R}$ under the operation +.

2. For the operation ×, $\mathbb{R}^+$ is still a stable subset, but $\mathbb{R}^-$ is not.

4.1.2 Properties of internal composition laws

Definition 4.3 (Commutativity and Associativity) Let E be a set equipped with an internal composition law $*$.

We say that $*$ is commutative if $\forall (x, y) \in E^2 : x * y = y * x$.

We say that $*$ is associative if $\forall (x, y, z) \in E^3 : (x * y) * z = x * (y * z)$.

Example.

1. The addition and multiplication laws on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ are commutative and associative.

2. Also, the union ($\cup$), intersection ($\cap$), and symmetric difference ($\triangle$) laws on $\mathcal{P}(E)$ are commutative and associative.

3. The composition law ($\circ$) on $\mathcal{F}(E)$ is associative but not commutative, because $f \circ g \neq g \circ f$ in general.

4. Let $*$ be the law defined on $\mathbb{Q}$ by: $x * y = \frac{x+y}{2}$. Then $*$ is commutative, because $x * y = \frac{x+y}{2} = \frac{y+x}{2} = y * x$, but it is not associative,

because $(-1 * 0) * 1 = \frac{1}{4} \neq -1 * (0 * 1) = \frac{-1}{4}$.

Definition 4.4 (Neutral Element) Let E be a set equipped with an internal composition law *. Let e be an element of E. We say that e is the neutral element for the law *, if

$$\forall x \in E : x * e = e * x = x$$

Remark 4.1 If the law $*$ is commutative, the equality $x * e = e * x$ is automatically satisfied.

Example.

1. In $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$, 0 is neutral for the addition law, and 1 is neutral for the multiplication law.
2. In $\mathcal{P}(E)$, the empty set $(\emptyset)$ is neutral for the union law $(\cup)$, and E is neutral for the intersection law $(\cap)$.
3. Let $*$ be the law defined on $\mathbb{R}$ by: $x * y = x + y - 1$. Then $e = 1$ is a neutral element, because $x * e = x \Rightarrow x + e - 1 = x$. Thus, $e = 1$.

Proposition 4.1 (Uniqueness of the Neutral Element) The neutral element of E for the law * if it exists, is unique.

Proof. Indeed, let e' be another neutral element for $*$, then $e' = e' * e = e * e' = e$. Thus, the neutral element is unique.

Definition 4.5 (Inverse Element) Let E be a set equipped with an internal composition law * and let e be a neutral element. We say that the element x of E has an inverse element x' of E, if $\forall x \in E : x * x' = x' * x = e$.

Example.

1. In $\mathbb{R}$, the invertible elements for the multiplication law $(\times)$ are the non-zero elements.
2. Let $*$ be the law defined on $\mathbb{R}$ by: $x * y = x + y - 1$. Then $x \in \mathbb{R}$ has an inverse element $x' = 2 - x$, because $x * x' = 1 \Rightarrow x + x' - 1 = 1$. Thus, $x' = 2 - x$.

4.1.3 Properties of internal composition laws

Definition 4.3 (Commutativity and Associativity) Let E be a set equipped with an internal composition law $*$.

We say that $*$ is commutative if $\forall(x, y) \in E^2 : x * y = y * x$.

We say that $*$ is associative if $\forall(x, y, z) \in E^3 : (x * y) * z = x * (y * z)$.

Example.

1. The addition and multiplication laws on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are commutative and associative.

2. The union ($\cup$), intersection ($\cap$), and symmetric difference ($\triangle$) laws on $\mathcal{P}(E)$ are commutative and associative.

3. The composition law ($\circ$) on $\mathcal{F}(E)$ is associative but not commutative, because $f \circ g \neq g \circ f$ in general.

4. Let $*$ be the law defined on $\mathbb{Q}$ by: $x * y = \frac{x+y}{2}$. Then $*$ is commutative, because $x * y = \frac{x+y}{2} = \frac{y+x}{2} = y * x$, but it is not associative, because $(-1 * 0) * 1 = \frac{1}{4} \neq -1 * (0 * 1) = \frac{-1}{4}$.

Definition 4.4 (Neutral Element) Let E be a set equipped with an internal composition law $*$. Let e be an element of E. We say that e is the neutral element for the law $*$ if $\forall x \in E : x * e = e * x = x$.

Remark 4.1 If the law $*$ is commutative, the equality $x * e = e * x$ is automatically satisfied.

Example.

1. In $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, 0 is the neutral element for the addition law, and 1 is the neutral element for the multiplication law.

2. In $\mathcal{P}(E)$, the empty set $\emptyset$ is the neutral element for the union law $\cup$, and E is the neutral element for the intersection law $\cap$.