Chapter 1

Generalities on ordered sets and lattices.

1.1 Ordered Sets

1.1.1 Characteristic Function

Definition 1 Let X be a reference set, and A be a subset. We define the characteristic function of A, denoted by χ_A, as the mapping defined by:

$$\chi_A : X \to \{0, 1\}.$$

$$x \to \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

1.1.2 Binary Relation

Definition 2 The binary relation T on a set X is a subset of pairs (x, y) of elements of X, i.e., $\mathrm{T} \subset \mathrm{X}^2$.

And we write x Ty is equivalent to: $(x, y) \in \mathrm{T}$ such that:

$\mathrm{T}(x, y) = 1$ if x Ty and $\mathrm{T}(x, y) = 0$ if xXy.

Definition 3 If A and B are two sets, and there is a specific property between x from A and y from B, this property can be represented by the ordered pair (x, y). The set of such pairs $(x, y), x \in \mathrm{A}$ and $y \in \mathrm{B}$ is called relation T.

1.1.3 Order Relation

A binary relation O on a set X is an order on X if it satisfies the following three properties:

1. Reflexivity : for any $x \in X, xOx$.

2. Antisymmetry : for all $x, y \in X$, $(xOy$ and $yOx)$ imply $x = y$.

3. Transitivity : for all $x, y, z \in X, (xOy$ and $yOz)$ imply xOz. The order O is called total if it is such that for all $x, y \in X, xOy$ or yOx.

1.1.4 Ordered Set

An ordered set is a pair $P = (X, O)$ where X is a set and O is an order on X. If O is a total order, $P = (X, O)$ is then called a totally ordered set (or linearly ordered set or chain).

Example

1) Let $X = \{a, b, c, d, e\}$ and $P = (X, O)$ be the ordered set where O is the following order on X:

$O = \{(a, b), (a, e), (c, b), (c, d), (c, e), (d, e), (a, a), (b, b), (c, c)\ (d, d), (e, e)\}$.

2) The set of natural numbers with division is an ordered set.

3) The set $A = D(m)$ of divisors of an integer $m > 1$, with division, is an ordered set.

Counterexample

The set of lines in the plane with:

The relation ”is parallel to” or the relation ”is orthogonal to” is not an ordered set because the relation ”is parallel to” is not antisymmetric, and the relation ”is orthogonal to” is not transitive.

1.1.5 Hasse Diagram

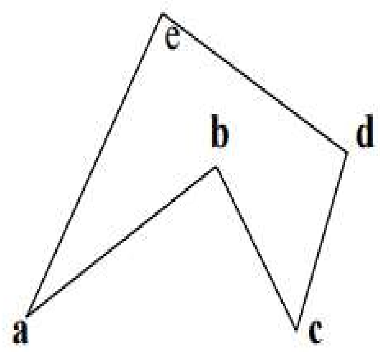
The Hasse diagram of an ordered set P = (X, $\leq$) is a representation of its covering graph in which the elements x of P are depicted as points P(x) in the plane, such that the following two rules are satisfied:

a) If $x < y$, (the horizontal line passing through) P(x) is below (the horizontal line passing through) P(y).

b) P(x) and P(y) are connected by a straight segment if and only if $x < y$.

Example

The Hasse diagram of the set X = $\{a, b, c, d, e\}$ equipped with the order relation $O = \{(a, b), (a, e), (c, b), (c, d), (c, e), (d, e), (a, a), (b, b), (c, c), (d, d)\ (e, e)\}$ is:



Remark 1

There exist infinitely many possible diagrams for the same ordered set.

1.1.6 Special Elements of a Set

Let (X, $\leq$) be an ordered set and A $\subseteq$ X.

1) We say that $x \in X$ is a lower bound of A if $x \leq a, \forall a \in A$ (the set of lower bounds of A, denoted by A^L).

2) We say that $x \in$ X is an upper bound of A if $a \leq x, \forall a \in A$ (the set of upper bounds of A, denoted by A^U).

3) We say that $m \in A$ is the minimum of A if $m \leq a, \forall a \in A$ ($m = \min$ of A).

4) We say that $M \in A$ is the maximum of A if $a \leq M, \forall a \in A$ (denoted max(A)).

5) $\text{Sup(A)} = \min\left(A^U\right)$.

6) $\text{Inf(A)} = \max\left(A^L\right)$.

7) Maximal element: We say that $x \in A$ is a maximal element in A if $\exists a \in A$ such that $x \leq a \Longrightarrow x = a$.

8) Minimal element: We say that $x \in A$ is a minimal element in A if $\exists a \in A$ such that $a \leq x \Rightarrow a = x$.

Remark The min(A) and max(A) are unique if they exist.

1.2 Lattices

1.2.1 Definition of a Lattice

Definition 4 An ordered set X is an inf-semilattice if every pair $\{x, y\}$ of its elements has an infimum denoted by $x \wedge y$. It is a sup-semilattice if every pair of its elements has a supremum denoted by $x \vee y$. It is a lattice if every pair of its elements has both a supremum and an infimum, making it both an inf-semilattice and a sup-semilattice.

A lattice is often denoted as $T = (X, \leq, \wedge, \vee)$.

Definition 5 A lattice is an ordered set $(T, \leq)$ such that for every pair of elements $\{x, y\}$, there exists an upper bound denoted by $x \vee y$ and a lower bound denoted by $x \wedge y$.

Examples

1) Any chain C is a lattice, where for all $x, y \in$ C:

$x \vee y = \max(x, y)$

$x \wedge y = \min(x, y)$.

2) $D(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$, equipped with division, is a lattice with:

$x \vee y = \text{ppcm}(x, y)$.

$x \wedge y = \text{gcd}(x, y)$.

3) The set $(2^X, \subseteq)$, equipped with union and intersection, is a lattice. Its laws are defined as follows: $\forall X, Y \in 2^X : X \vee Y = X \cup Y$. $X \wedge Y = X \cap Y$

Properties

In any lattice, the following hold:

- $x \leq y \Leftrightarrow x = x \wedge y$.

$$\Leftrightarrow y = x \vee y.$$

- Idempotence: $x \wedge x = x \vee x = x$.
- Commutativity: $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$.
- Associativity: $x \vee (y \vee z) = (x \vee y) \vee z$.

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z$$

- Absorption laws: $x \wedge (y \vee x) = x$.

$$x \vee (y \wedge x) = x$$

Theorem

Let T be a set equipped with two internal laws $\wedge, \vee$ that are idempotent, commutative, associative, and satisfy the absorption laws. Then, there exists a unique order relation $(\leq)$ on T such that:

T is a lattice with: $x \wedge y = \inf(x, y)$.

$$x \vee y = \sup(x, y).$$

1.2.2 Closed Lattices

Definition 6 A lattice T is called closed if it has a smallest element denoted as "0" and a largest element denoted as "1".

Examples

1. The set of subsets of a set E, $P(\mathrm{E})$, equipped with the inclusion relation, is a closed lattice. The smallest element is $\varnothing$, and the largest element is E.
2. The set of divisors of 6 : $\mathrm{D}(6) = \{1, 2, 3, 6\}$ is a closed lattice with the minimum being 1, and the maximum being 6.
3. $(\mathbb{N}^*, |)$ is not a closed lattice as it does not have a largest element.

1.2.3 Filter in a Lattice

Definition 7 Let T be a lattice. A filter in the lattice T is any non-empty subset F of T satisfying the following two conditions:

$$\checkmark \quad x \in F, y \geq x \Rightarrow y \in F.$$

$$\checkmark \quad x \in F, \text{ and } y \in F \Longrightarrow x \wedge y \in F.$$

Example

In the set $\mathrm{D}(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$, we have:

$F_1 = \{2, 6, 10, 30\}$, $F_2 = \{30\}$, $F_3 = \{6, 30\}$ are filters.

Remark

1) A filter F is called proper if $F \neq \mathrm{T}$.

2) F is proper if and only if $0 \notin F$.

3) Let "1" be the largest element of T; then $\{1\}$ is the smallest filter in T.

4) Any intersection of filters is a filter.

Definition 8

Let G be a non-empty subset of T. The filter generated by G, denoted F_{G}, is the intersection of all filters containing G, i.e., it is the smallest filter containing G.

F_{G} is defined as follows:

F_{G} is the set of $x \in \mathrm{T}$ such that there exist a finite number $a_1, a_2, a_3, \ldots,$ a_n of elements from G with: $x \geq a_1 \wedge a_2 \wedge a_3 \wedge \ldots \wedge a_n$, and we write:

$F_{\mathrm{G}} = \{x \in \mathrm{T}/x \geq a_1 \wedge a_2 \wedge a_3 \wedge \ldots \wedge a_n, a_i \in \mathrm{G}\}$.

If $\mathrm{G} = \emptyset$, then $F_{\mathrm{G}} = \{1\}$.

Example

Consider the lattice $\mathrm{D}(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and take $\mathrm{G} = \{10\} \neq \emptyset$.

In this case, the filter generated by G is:

$$F_{\mathrm{G}} = \{10, 30\}.$$

Definition 8 A principal filter is the filter generated by a single element, defined as: $F_a = \{x \in \mathrm{T}/x \geq a\}$.

Example

In the lattice $\mathrm{D}(6) = \{1, 2, 3, 6\}$.

Take $a = 2$, so the principal filter generated by 2 is:

$$F_2 = \{2, 6\}.$$

Definition 9 By definition, ultrafilters are the maximal proper filters in the order (proper filters that are not contained in any other proper filter).

Example

In $\mathrm{D}(30)$, the ultrafilters are:

$F_2 = \{x \in \mathrm{D}(30)/x \geq 2\} = \{2, 6, 10, 30\}$.

$F_3 = \{x \in \mathrm{D}(30)/x \geq 3\} = \{3, 6, 15, 30\}$.

$F_5 = \{x \in \mathrm{D}(30)/x \geq 5\} = \{5, 10, 15, 30\}$.

Proposition 1

Let F be a proper filter; the following two assertions are equivalent :

- F is an ultrafilter.
- $\forall x \in \mathrm{T}, x \notin F \Longrightarrow$ there exists $y \in F$ such that: $x \wedge y = 0$.

1.2.4 Ideal in a Lattice

Definition 10

An ideal is any non-empty subset I of E such that:

✓ $x \in \mathrm{I}, y \leq x \Rightarrow y \in \mathrm{I}$.

✓ $x, y \in \mathrm{I} \Rightarrow x \vee y \in \mathrm{I}$.

We say that I is proper if and only if $1 \notin \mathrm{I}$.

$\{0\}$ is an ideal, and it is the smallest ideal.

Definition 11

Let G be a non-empty subset of E. The ideal generated by G, denoted I_G, is defined by:

$$\mathrm{I_G} = \{x \in \mathrm{E}/x \leq a_1 \vee a_2 \vee \ldots \vee a_\mathrm{n}, a_\mathrm{i} \in \mathrm{G}\}$$

Definition 12

G is a $\vee$-compatible set if $\mathrm{I_G} \neq \mathrm{E}$.

G is a $\vee$-incompatible set $\Leftrightarrow \exists a_1, a_2, \ldots, a_\mathrm{n}$ in G such that : $a_1 \vee a_2 \vee \ldots \vee a_\mathrm{n} = 1$.

The set of proper ideals is inductive, therefore it has maximal elements (maximal ideal).

1.2.5 Distributive Lattice

Definition 13

Let T be a lattice; we say that T is a distributive lattice if every triplet (x, y, z) in T satisfies one of two conditions:

1. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

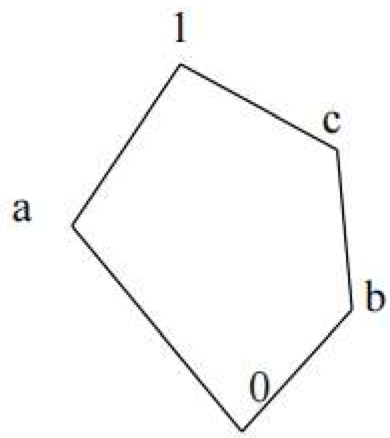
Examples

- Any chain is a distributive lattice:

  $\text{Min}(x, \max(y, z)) = \max(\min(x, y), \min(x, z))$.

- $(P(\text{E}), \subseteq)$ is a distributive lattice.

- The set defined by:



Is not a distributive lattice because:

We have : $c \wedge (a \vee b) = c \wedge l = c$.

$(c \wedge a) \vee (c \wedge b) = 0 \vee b = b$.

Theorem

T is a distributive lattice if and only if it satisfies one of the following three conditions:

a) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

b) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

c) $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$

d) $[x \wedge y = x \wedge z$ and $x \vee y = x \vee z] \Rightarrow (y = z)$.

1.2.6 Complemented Lattice

Definition 14 Let T be a lattice.

We say that the lattice T is complemented if every $x \in \mathrm{T}$ has at least one complement, i.e., an element $x' \in \mathrm{T}$ satisfying $x \wedge x' = 0$ and $x \vee x' = 1$.

Example

1) $(P(E), \subseteq)$ is complemented.

$\mathrm{A} \cap \mathrm{C}A = \emptyset, \quad \mathrm{A} \cup \mathrm{C}A = E$

2) $\mathrm{D}(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$ is complemented:

$2' = 15, 15' = 2, 3' = 10, 10' = 3, 5' = 6, 6' = 5, 1' = 30, 30' = 1.$

Note

The distributivity of a lattice preserves the uniqueness of complements if they exist.

1.2.7 Boolean Lattice

Definition 15 By definition, a Boolean lattice is a closed and complemented distributive lattice.

Examples

- The chain $\mathrm{u} = \{0, 1\}$ is a Boolean lattice.
- $(P(\mathrm{E}), \subseteq)$ is a Boolean lattice.
- $\mathrm{D}(6) = \{1, 2, 3, 6\}$ is a Boolean lattice.

- The set of positive integers with division is not a Boolean lattice (does not have a greatest element).

Theorem 1 Let B be a Boolean algebra and F a proper filter, there is an equivalence between:

- F is an ultrafilter.
- $\forall x \in \mathrm{B}, x \in F$ or $\rceil x \in F$.

Theorem 2

Let F be a non-empty subset of a Boolean algebra.

For F to be an ultrafilter, it is necessary and sufficient that it satisfies:

$$\checkmark x \in F \Leftrightarrow \rceil x \notin F$$

$$\checkmark \quad x, y \in F \Leftrightarrow x \wedge y \in F.$$

Theorem 3

Let F be a non-empty subset of B (Boolean algebra), for F to be an ultrafilter, it is necessary and sufficient that its characteristic function χ is a Boolean morphism from B to $\mathrm{u} = \{0, 1\}$.

$$\chi : \mathrm{B} \to \mathrm{u} \begin{cases} \chi(x) = 1 \text{ if x} \in \mathrm{F}; \\ \chi(x) = 0 \text{ if x} \notin F. \end{cases}$$

Theorem 4 (Stone's Representation Theorem)

Every Boolean algebra B is isomorphic to a subalgebra of the form $P(X)$ such that: X is the set of ultrafilters of B.

Chapter 2

Logic and Lukasiewicz Trivalent Algebras

2.1 Lukasiewicz Trivalent Logic

2.1.1 Semantics

Lukasiewicz's main work in mathematical logic was the creation of so-called ”multivalent” logics. He had the idea of assigning a status to the proposition ”it will rain tomorrow.” In 1917, he outlined the first version of a trivalent logic, associating the third logical value, different from true and false, with the notion of possibility. His first publications on trivalent logic date back to 1920.

He used the following notations: 1 for true, 0 for false, and 1/2 for the third logical value, which can be interpreted as problematic or possible. Lukasiewicz defined his trivalent logic semantically using the following connectors:

2.1.1.1 Negation

Denoted by N, defined by the table:

x	$N(x)$
0	1
1/2	1/2
1	0

Note that it is a decreasing involution in the set $\{0, 1/2, 1\}$ naturally ordered. In particular, we have the principle of double negation $NNP = P$.

2.1.1.2 Implication

Implication, denoted $x \to y$: defined by the table

$x \backslash y$	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

We observe that in the set $T = \{0, 1/2, 1\}$ naturally ordered, $x \to y = 1$ if and only if $x \leq y$.

2.1.1.3 Disjunction

Lukasiewicz defined disjunction as:

$x \vee y = (x \to y) \to y.$

And the truth table is:

$x \backslash y$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

$x \vee y$

In a Boolean algebra, we have:

$$(x \to y) \to y = 1(7x \vee y) \vee y,$$

$$= (1(7x) \wedge 1y) \vee y,$$

$$= (x \wedge 7y) \vee y,$$

$$= (x \vee y) \wedge (1y \vee y),$$

$$= (x \vee y) \wedge 1,$$

$(x \to y) \to y = x \vee y.$

2.1.1.4 Conjunction

This connector defined in Lukasiewicz's trivalent logic as follows:

$x \wedge y = N(Nx \vee Ny)$.

Its truth table is:

$_x\backslash^y$	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

In a Boolean algebra, we have:

$$\begin{aligned} N(Nx \vee Ny) &= N(Nx) \wedge N(Ny) \\ &= x \wedge y. \end{aligned},$$

2.1.1.5 Equivalence

Lukasiewicz's equivalence in trivalent algebra is defined by:

$x \Leftrightarrow y = (x \Rightarrow y) \wedge (y \Rightarrow x)$.

Its truth table is:

$_x\backslash^y$	0	1/2	1
0	1	1/2	0
1/2	1/2	1	1/2
1	0	1/2	1

2.1.1.6 Possibility

Lukasiewicz then tried to give a definition of the concept of possibility by attempting to solve certain problems in modal logic. It was one of his students, Tarski, who in 1921 gave a unary possibility connector, which he denoted by μ. The definition is as follows:

$\mu x = Nx \rightarrow x$. And its truth table is:

x	μx
0	0
1/2	1
1	1
μx	

Proof

x	Nx	$Nx \to x$	μx
0	1	0	0
1/2	1/2	1	1
1	0	1	1

2.1.1.7 Necessity denoted ϑ

It is a unary connector defined by: ϑx = $N\mu N$x.

x	ϑx
0	0
1/2	0
1	1
ϑx	

Proof

x	Nx	μNx	$N\mu Nx$	ϑx
0	1	1	0	0
1/2	1/2	1	0	0
1	0	0	1	1

2.1.1.8 Impossibility and Contingency

In Lukasiewicz's trivalent calculus, we can define the impossibility (η) and contingency (γ) connectors: