Chapter 2

Logic and Lukasiewicz Trivalent Algebras

2.1 Lukasiewicz Trivalent Logic

2.1.1 Semantics

Lukasiewicz's main work in mathematical logic was the creation of so-called "multivalent" logics. He had the idea of assigning a status to the proposition "it will rain tomorrow." In 1917, he outlined the first version of a trivalent logic, associating the third logical value, different from true and false, with the notion of possibility. His first publications on trivalent logic date back to 1920.

He used the following notations: 1 for true, 0 for false, and 1/2 for the third logical value, which can be interpreted as problematic or possible. Lukasiewicz defined his trivalent logic semantically using the following connectors:

2.1.1.1 Negation

Denoted by N, defined by the table:

x	$N(x)$
0	1
1/2	1/2
1	0

Note that it is a decreasing involution in the set $\{0, 1/2, 1\}$ naturally ordered. In particular, we have the principle of double negation $NNP = P$.

2.1.1.2 Implication

Implication, denoted $x \to y$: defined by the table

$x \backslash y$	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

We observe that in the set $T = \{0, 1/2, 1\}$ naturally ordered, $x \to y = 1$ if and only if $x \leq y$.

2.1.1.3 Disjunction

Lukasiewicz defined disjunction as:

$x \vee y = (x \to y) \to y$.

And the truth table is:

$x \backslash y$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

$x \vee y$

In a Boolean algebra, we have:

$$
\begin{aligned}
(x \to y) \to y &= 1(7x \vee y) \vee y, \\
&= (1(7x) \wedge 1y) \vee y, \\
&= (x \wedge 7y) \vee y, \\
&= (x \vee y) \wedge (1y \vee y), \\
&= (x \vee y) \wedge 1,
\end{aligned}
$$

$(x \to y) \to y = x \vee y$.

2.1.1.4 Conjunction

This connector defined in Lukasiewicz's trivalent logic as follows:

$x \wedge y = N(Nx \vee Ny)$.

Its truth table is:

$_x\backslash^y$	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

In a Boolean algebra, we have:

$$\begin{aligned} N(Nx \vee Ny) &= N(Nx) \wedge N(Ny) \\ &= x \wedge y. \end{aligned},$$

2.1.1.5 Equivalence

Lukasiewicz's equivalence in trivalent algebra is defined by:

$x \Leftrightarrow y = (x \Rightarrow y) \wedge (y \Rightarrow x)$.

Its truth table is:

$_x\backslash^y$	0	1/2	1
0	1	1/2	0
1/2	1/2	1	1/2
1	0	1/2	1

2.1.1.6 Possibility

Lukasiewicz then tried to give a definition of the concept of possibility by attempting to solve certain problems in modal logic. It was one of his students, Tarski, who in 1921 gave a unary possibility connector, which he denoted by μ. The definition is as follows:

$\mu x = Nx \rightarrow x$. And its truth table is:

x	μx
0	0
1/2	1
1	1
μx	

Proof

x	Nx	$Nx \to x$	μx
0	1	0	0
1/2	1/2	1	1
1	0	1	1

2.1.1.7 Necessity denoted ϑ

It is a unary connector defined by: ϑx = $N\mu N$x.

x	ϑx
0	0
1/2	0
1	1
ϑx	

Proof

x	Nx	μNx	$N\mu Nx$	ϑx
0	1	1	0	0
1/2	1/2	1	0	0
1	0	0	1	1

2.1.1.8 Impossibility and Contingency

In Lukasiewicz's trivalent calculus, we can define the impossibility (η) and contingency (γ) connectors:

$\eta x = N\mu x.$

$\gamma x = \mu N x.$

Which gives the tables:

x	ηx
0	1
1/2	0
1	0

x	γx
0	1
1/2	1
1	0

2.1.1.9 Weak Implication

Monteiro introduced the weak implication, denoted by:

$x \underset{M}{\rightarrow} y$

$x \backslash y$	0	1/2	1
0	1	1	1
1/2	1	1	1
1	0	1/2	1
$x \rightarrow y$			

Note that: $x \rightarrow y = \gamma(x) \vee y.$

Lukasiewicz's implication can be derived from weak implication:

$x \rightarrow y = (x \underset{M}{\rightarrow} y) \wedge (Ny \underset{M}{\rightarrow} Nx).$

2.1.2 Wajsberg's Axiomatization (1931)

Wajsberg was the first in 1931 to provide an axiomatization of Lukasiewicz's trivalent logic using the following four axioms:

W1: $x \rightarrow (y \rightarrow x).$

W2: $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)).$

W3 : $((x \rightarrow Nx) \rightarrow x) \rightarrow x.$

W4 : $(Nx \rightarrow Ny) \rightarrow (y \rightarrow x)$.

Note that all four axiom schemes are theses in classical propositional calculus denoted by L_2.

2.2 Algebrization

2.2.1 Lukasiewicz Trivalent Algebra

2.2.1.1 Definition

In 1940, Moisil introduced the concept of Lukasiewicz trivalent algebra by giving a rather complex axiomatization that can be stated as follows:

A system $(L, \vee, \wedge, 0, 1, N, \mu)$ formed by a non-empty set L, two elements 1 and 0 of L, two binary operations $\wedge$ and $\vee$ defined on L, and two unary operations N and μ defined on L, is a Lukasiewicz trivalent algebra if:

$\mathbf{L}_1$: $(L, \vee, \wedge, 0, 1)$ is a closed distributive lattice.

$\mathbf{L}_2$: The unary operation N is a decreasing involution, i.e.:

- $X \leq y \Rightarrow Ny \leq Nx$.
- $NNx = x$.
- $N1 = 0$.
- $N0 = 1$.

$\mathbf{L}_3$: The unary operation μ is an endomorphism on L, idempotent, and extensive.

- Endomorphism: $\mu(x \vee y) = \mu(x) \vee \mu(y)$.

$$\mu(x \wedge y) = \mu(x) \wedge \mu(y)$$

- Idempotent: $\mu\mu(x) = \mu(x)$.
- Extensive: $\mu(x) \geq x$.

$\mathbf{L}_4: \ N\mu N\mu x = \mu x.$

$\mathbf{L}_5: \ Nx \vee \mu x = 1.$

$\mathbf{L}_6: \ x \wedge Nx = \mu x \wedge Nx.$

This algebra is denoted as t_3-algebra.

2.2.1.2 Examples

1. $\mathrm{T} = \{0, 1/2, 1\}$ is the smallest £$_3$-algebra.

2. Let B be a Boolean algebra, $\mathrm{B}^2 = \{(x, y)/x \in \mathrm{B} \text{ and } y \in \mathrm{B}\}$ is a Boolean algebra.

The set $\mathrm{L(B)} = \{(x, y)/x \leq y\}$ is a sublattice of B^2, equipped with the laws:

(1) $(x, y) \vee (x', y') = (x \vee x', y \vee y').$

(2) $(x, y) \wedge (x', y') = (x \wedge x', y \wedge y').$

(3) $N(x, y) = (7y, 7x).$

(4) $\mu(x, y) = (y, y).$

It is an L_3 algebra because:

1. $(0, 0) \in \mathrm{L(B)}$ is the smallest element of L(B), and $(1, 1)$ is the largest element of L(B), so L(B) is closed.

2. L(B) is distributive because:

Let $(x, y), (x', y'), (x'', y'') \in \mathrm{L(B)}$

$$\begin{aligned}
(x, y) \vee [(x', y') \wedge (x'', y'')] &= (x, y) \vee [(x' \wedge x'', y' \wedge y'')] \\
&= (x \vee (x' \wedge x''), y \vee (y' \wedge y'')) \\
&= ((x \vee x') \wedge (x \vee x''), (y \vee y') \wedge (y \vee y'')) \\
&= (x \vee x', y \vee y') \wedge (x \vee x'', y \vee y'') \\
&= [(x, y) \vee (x', y')] \wedge [(x, y) \vee (x'', y'')]
\end{aligned}$$

Therefore

$(x, y) \vee [(x', y') \wedge (x'', y'')] = [(x, y) \vee (x', y')] \wedge [(x, y) \vee (x'', y'')]$.

3. N is a decreasing involution because:

Assume that $(x, y), (x', y') \in \mathrm{L(B)}$

Suppose $(x, y) \leq (x', y')$, and we need to show that $N\,(x', y') \leq N(x, y)$

We have $(x, y) \leq (x', y') \Longrightarrow x \leq x'$

$$y \leq y'$$

and as $x, y \in \mathrm{B}$, a Boolean algebra, so: $7x' \leq 7x$

$$7y' \leq 7y$$

Therefore $(7y', 7x') \leq (7y, 7x)$, which means: $N\,(x', y') \leq N(x, y)$

So if $(x, y) \leq (x', y') \Longrightarrow N\,(x', y') \leq N(x, y)$.

$N(0, 0) = (10, 10) = (1, 1)$.

$N(1, 1) = (11, 71) = (0, 0)$.

$NN(x, y) = N(7y, 7x) = (77x, 77y) = (x, y)$.

In the end, we get that N is a decreasing involution.

4. μ is an endomorphism:

$$\begin{aligned}\mu\left((x, y) \vee (x', y')\right) &= \mu\left(x \vee x', y \vee y'\right)\\ &= (y \vee y', y \vee y')\\ &= (y, y) \vee (y', y')\\ &= \mu(x, y) \vee \mu\,(x', y')\end{aligned}$$

So: $\mu\left((x, y) \vee (x', y')\right) = \mu(x, y) \vee \mu\,(x', y')$.

$\mu\left((x, y) \wedge (x', y')\right) = \mu\left(x \wedge x', y \wedge y'\right)$

$$\begin{aligned}\mu\left((x, y) \wedge (x', y')\right) &= (y \wedge y', y \wedge y')\\ &= (y, y) \wedge (y', y')\\ &= \mu(x, y) \wedge \mu\,(x', y')\end{aligned}$$

So: $\mu\left((x, y) \wedge (x', y')\right) = \mu(x, y) \wedge \mu\left(x', y'\right)$.

5. μ is idempotent:

We have: $\mu(\mu(x, y)) = \mu(y, y) = (y, y) = \mu(x, y)$.

So μ is idempotent.

6. μ is extensive:

$\mu(x, y) = (y, y)$

We have: $(x, y) \in$ L(B), so $y \geq x$ and $y \geq y$, then $(y, y) \geq (x, y)$

That is: $\mu(x, y) \geq (x, y)$.

So μ is extensive.

7.

$$\begin{aligned} N\mu N\mu(x, y) &= N\mu N(y, y) \\ &= N\mu(7y, 7y) \\ &= N(7y, 7y) \\ &= (y, y) \\ &= \mu(x, y). \end{aligned}$$

So $N\mu N\mu(x, y) = \mu(x, y)$.

8.

$$\begin{aligned} N(x, y) \vee \mu(x, y) &= (7y, 7x) \vee (y, y) \\ &= (7y \vee y, 7x \vee y) \\ &= (1, 7x \vee y) \\ &= (1, x \to y) \end{aligned}$$

And since $(x, y) \in$ L(B), i.e., $x \leq y$

And according to the property of implication: $x \to y = 1$ if and only if $x \leq y$.

Therefore: $x \to y = 1 (7x \vee y = 1)$

So: $N(x,y) \vee \mu(x,y) = (1, 7x \vee y)$

$$= (1, x \rightarrow y)$$

$$= (1,1).$$

Or $7x \leq 7x$

$$x \leq y \Rightarrow 7x \leq 7y$$

$$\Rightarrow 1 \leq 7x \vee y \leq 1$$

$$\Rightarrow 7x \vee y = 1$$

So $N(x,y) \vee \mu(x,y) = (1,1)$.

9.

$$\begin{aligned} (x,y) \wedge N(x,y) &= (x,y) \wedge (7y, 7x) \\ &= (x \wedge 7y, y \wedge 7x) \end{aligned}$$

We have: $x \leq y \Rightarrow 7y \leq 7x$ and $x \leq x$

So: $x \wedge 7y \leq 0$ and $x \wedge 7y \geq 0 \Rightarrow 0 \leq x \wedge 7y \leq 0$ then $x \wedge 7y = 0$.

Therefore $(x,y) \wedge N(x,y) = (x \wedge 7y, y \wedge \neg x)$

$$= (0, y \wedge \rceil x)$$

$\mu(x,y) \wedge N(x,y) = (y,y) \wedge (7y, 7x)$

$$= (y \wedge \rceil y, y \wedge \rceil x)$$

$$= (0, y \wedge \rceil x).$$

Finally, we get: $(x,y) \wedge N(x,y) = \mu(x,y) \wedge N(x,y)$.

2.2.1.3 Properties

1. The negation N satisfies De Morgan's laws:

$N(x \vee y) = N(x) \wedge N(y)$.

We have: $\begin{cases} x \leq x \vee y \\ y \leq x \vee y \end{cases} \Rightarrow \begin{cases} N(x \vee y) \leq N(x). \\ N(x \vee y) \leq N(y). \end{cases}$

$\Rightarrow N(x \vee y) \leq N(x) \wedge N(y)$............... (1)

Let z: $\begin{cases} z \leq N(x) \\ z \leq N(y) \end{cases} \Rightarrow \begin{cases} x \leq N(z) \\ y \leq N(z) \end{cases} \Rightarrow x \vee y \leq N(z)$.

$\Rightarrow z \leq N(x \vee y)$.

And $\mathrm{l}(x) \wedge N(y) \leq N(x)$.

$$N(x) \wedge N(y) \leq N(y)$$

Therefore $N(x) \wedge N(y) \leq N(x \vee y)$................. (2)

From (1) and (2) we obtain $N(x \vee y) = N(x) \wedge N(y)$.

Also, $N(x \wedge y) = N(x) \vee N(y)$.

2. $N(0) = 1, N(1) = 0$.

We have $\forall x \in L : N(x) \geq 0$.

So $N(1) \geq 0$............ (1)

And $\forall x \in L : N(x) \leq 1$.

$x \geq N(1)$ and $x \geq 0$. So: $N(1) \leq 0$.............. (2)

From (1) and (2): $0 \leq N(1) \leq 0$, so: $N(1) = 0$.

3. $\mu(1) = 1$.

We have $\forall x \in L : \mu(x) \geq x$, so: $\mu(1) \geq 1$.

And on the other hand: $\forall x \in L : \mu(x) \leq 1$ then $\mu(1) \leq 1$.

$\mu(1) \geq 1$ and $\mu(1) \leq 1$ so $\mu(1) = 1$.

4. $\mu(0) = 0$.

$\forall x \in L : X \wedge N(x) = \mu(x) \wedge N(x)$.

So: $0 \wedge N(0) = \mu(0) \wedge N(0)$.

$0 \wedge 1 = \mu(0) \wedge 1$.

$0 = \mu(0)$.

Therefore: $\mu(0) = 0$.

5. $N\mu(x) \leq Nx \leq \mu N(x)$.

We have: $\forall x \in L : \mu(x) \geq x$.

$\Longrightarrow Nx \geq N\mu(x)$.............. (1)

And we have: $\forall x \in L : x \leq \mu(x)$.

So for the element $N(x) : Nx \leq \mu N(x)$................. (2)

From (1) and (2) we obtain: $N\mu(x) \leq Nx \leq \mu N(x)$.

6. $N\mu N(x) \leq x \leq \mu(x)$.

Using property (5) we get:

We have:

$$\begin{aligned} N\mu(x) \leq Nx \quad &\Rightarrow N(N(x)) \leq N(N\mu(x)) \\ &\Rightarrow x \leq \mu(x) \text{.............. (1)} \end{aligned}$$

On the other hand: we have

$$\begin{aligned} N(x) \leq \mu N(x) \quad &\Longrightarrow N\mu(N(x)) \leq N(N(x)) \\ &\Longrightarrow N\mu(N(x)) \leq x \text{................ (2)} \end{aligned}.$$

From (1) and (2) we find: $N\mu N(x) \leq x \leq \mu(x)$.

7. ϑ and η are retracting, idempotent endomorphisms.

$\vartheta = N\mu N$

a. $\boldsymbol{\vartheta}$ endomorphism:

$$\begin{aligned} \vartheta(x \vee y) \quad &= N\mu N(x \vee y). \\ &= N\mu(Nx \wedge Ny). \\ &= N(\mu Nx \wedge \mu Ny)(\text{car } \mu \text{ endomorphism }). \\ &= N\mu N(x) \vee N\mu N(y). \\ \vartheta(x \vee y) \quad &= \vartheta(x) \vee \vartheta(y) \end{aligned}$$

$$\begin{aligned}\vartheta(x \wedge y) &= N\mu N(x \wedge y).\\ &= N\mu(Nx \vee Ny).\\ &= N(\mu Nx \vee \mu Ny).\\ &= N\mu Nx \wedge N\mu Ny.\\ &= \vartheta x \wedge \vartheta y\end{aligned}$$

Therefore ϑ endomorphism.

b. $\boldsymbol{\vartheta}$ is a retraction:

$\vartheta x = N\mu Nx \leq x$ (according to property 6).

$\vartheta x \leq x$, therefore ϑ is a retraction.

c. $\boldsymbol{\vartheta}$ is idempotent:

We have:

$$\begin{aligned}\vartheta\vartheta x &= N\mu N(N\mu Nx).\\ &= N\mu NN(\mu Nx)\\ &= N\mu\mu(Nx)\\ &= N\mu Nx\\ &= \vartheta x\end{aligned}$$

Therefore ϑ is idempotent.

8. η and γ satisfying De Morgan's laws, called dualities.

$\eta(x \vee y) = \eta x \wedge \eta y$ and $\eta(x \wedge y) = \eta x \vee \eta y$.

$\gamma(x \vee y) = \gamma x \wedge \gamma y$ and $\gamma(x \wedge y) = \gamma x \vee \gamma y$.

9. $\eta x \leq Nx \leq \gamma x$.

$\vartheta x \leq x \leq \mu x$.

The six operators I, $N, \vartheta, \eta, \gamma$ form a monoid with the following table:

$\circ$	I	N	μ	ϑ	η	γ
I	I	N	μ	ϑ	η	γ
N	N	I	η	γ	μ	ϑ
μ	μ	γ	μ	ϑ	η	γ
ϑ	ϑ	η	μ	ϑ	η	γ
η	η	ϑ	η	γ	μ	ϑ
γ	γ	μ	η	γ	μ	ϑ

This monoid is generated by N and any of $\mu, \vartheta, \eta, \gamma$.

10. L_5 is equivalent to:

a. $x \vee \gamma x = 1$.

b. $\eta x \vee \mu x = 1$.

c. $\vartheta x \vee \gamma x = 1$.

d. $\mu x \vee \gamma x = 1$.

Proof Recall that $L_5 : Nx \vee \mu x = 1$.

$(L_5) \Rightarrow (1)$.

We assume that: $Nx \vee \mu x = 1$. i.e., L_5

Replace x with Nx, we get:

$Nx \vee \mu x = NNx \vee \mu Nx = 1$.

$\Longrightarrow x \vee \gamma x = 1$

So $(L_5) \Longrightarrow (1)$.

$(1) \Rightarrow (L_5)$

We assume that: $x \vee \gamma x = 1$.

Replacing x with Nx:

$Nx \vee \gamma Nx = 1 \Longrightarrow Nx \vee \mu x = 1$.

So $(L_5) \Leftrightarrow (1)$.

$(1) \Longrightarrow (2)$.

We assume that: $x \vee \gamma x = 1$.

Replacing x with μx:

$x \vee \gamma x = 1$.

$\Rightarrow \mu x \vee \gamma\mu x = 1$ (according to property (9), $\gamma\mu = \eta$)

$\Longrightarrow \mu x \vee \eta x = 1$.

So $(1) \Rightarrow (2)$.

$(2) \Longrightarrow (3)$.

Suppose:

$\eta x \vee \mu x = 1$.

Replacing x with Nx

$$\begin{aligned}\eta x \vee \mu x &= \eta Nx \vee \mu Nx.\\ &= \vartheta x \vee \gamma x\\ &= 1.\end{aligned}$$

So $(2) \Rightarrow (3)$.

$(3) \Rightarrow (1)$.

Suppose: $\vartheta x \vee \gamma x = 1$.

We have $\forall x \in L : \vartheta x \leq x$.

So $\vartheta x \vee \gamma x \leq x \vee \gamma x$.

That is $1 \leq x \vee \gamma x$.

So $x \vee \gamma x = 1$.

Then $(3) \Rightarrow (1)$

$(L_5) \Longrightarrow (4)$.

We assume that: $Nx \vee \mu x = 1$.

We have $\forall x \in L : Nx \leq \mu Nx$ (according to property (5)).

$$Nx \leq \gamma x$$

So $Nx \vee \mu x \leq \mu x \vee \gamma x$.

$\Rightarrow 1 \leq \mu x \vee \gamma x$.

$\Rightarrow \mu x \vee \gamma x = 1$.

So $(L_5) \Rightarrow (4)$.

$(4) \Rightarrow (2)$.

We assume that: $\mu x \vee \gamma x = 1$.

Replaced x with μx.

$$\begin{aligned} \mu x \vee \gamma x &= \mu\mu x \vee \gamma\mu x \\ &= \mu x \vee \eta x \\ &= 1 \end{aligned}$$

So $(4) \Rightarrow (2)$.

11. L_5 is equivalent to:

a. $\eta x \wedge \mathrm{x} = 0$.

b. $Nx \wedge \vartheta x = 0$

c. $Nx \wedge \eta x = 0$

d. $\gamma x \wedge \vartheta x = 0$

e. $x \wedge \vartheta x = 0$

12. L_6 is equivalent to:

a. $x \wedge Nx = x \wedge \gamma x$.

b. $x \vee Nx = x \vee \eta x$.

c. $x \vee Nx = \mu x \vee Nx$.

13. $\mu x \wedge Nx = x \wedge \gamma x$.

$x \vee \eta x = Nx \vee \vartheta x$.

2.2.1.4 Fundamental Properties

$$Nx = \eta x \vee (x \wedge \gamma x)$$

Proof: Let's assume that:

$x \leq \mu x$ hence $x \wedge \mu x = x$.

$$\begin{aligned} x \vee 0 &= x \vee (\mu x \wedge \eta x) \\ &= (x \wedge \mu x) \vee (\mu x \wedge \eta x) \\ &= \mu x \wedge (x \vee \eta x) \end{aligned}$$

And we have: $x = \mu x \wedge (Nx \vee \vartheta x)$................. $(*)$

So:

$$\begin{aligned} Nx &= N\mu x \vee (NNx \wedge N\vartheta x) \\ &= \eta x \vee (x \wedge \gamma x) \end{aligned}.$$

Moisil's Determination Principle:

$$\begin{cases} \mu x = \mu y \\ \vartheta x = \vartheta y \end{cases} \Rightarrow x = y$$

Proof: Let's assume that:

$\mu x = \mu y$ and $\vartheta x = \vartheta y$.

From $(*)$:

$$x \vee y = \mu(x \vee y) \wedge (N(x \vee y) \vee \vartheta(x \vee y))$$

$$= (\mu x \vee \mu y) \wedge (Nx \wedge Ny \vee \vartheta x \vee \vartheta y).$$

$$x \vee y = \mu x \wedge (Nx \vee \vartheta y) \wedge \mu x \wedge (Ny \vee \vartheta y).$$

So : $x \vee y = x \wedge y$.

$x \wedge y \leq x \leq x \vee y$.

$x \vee y \leq x \leq x \vee y$.

$\Longrightarrow x \vee y = x$. But $x \vee y = y$.

$\Longrightarrow x = y$.

2.2.1.5 Second Axiomatization of $\mathfrak{f}_3$ - Algebras

The 1st axiomatization of Moisil, consisting of the set of axioms L_1 to L_6, is equivalent to the following:

A $\mathfrak{f}_3$-algebra is a structure $(L, \wedge, \vee, 0, 1, \mu, \vartheta)$ such that:

L'_1 : $(L, \wedge, \vee, 0, 1)$ is a closed distributive lattice.

L'_2 : μ, ϑ are endomorphisms conserving 0 and 1.

L'_3 : $\mu\vartheta = \vartheta, \vartheta\mu = \mu$.

L'_4 : $\vartheta \leq \mu$.

L'_5 : $\mu(x) = \mu(y), \vartheta(x) = \vartheta(y) \Rightarrow x = y$.

L'_6 : μ, and ϑ are chrysipians: $\mu, \vartheta : L \to CL$, where $C(L) = \{$ the complements of $L\}$.

2.2.1.6 Equivalence of the Two Axiomatizations of $\mathfrak{f}_3$ Algebras

The proof of the equivalence between the two axiomatizations is left to the reader as an exercise.

2.3 Tutorial 1 : Algebraic Logic

The purpose of this problem is to obtain representation theorems for L_3-algebras (Lukasiewicz's trivalent algebras).

I - Preliminary on Lattices

Consider a distributive and closed lattice L (with 0 and 1). Let $C(L)$ denote the Boolean sublattice of complemented (or chrysippean) elements of L. If $x \in C(L)$, we denote its complement by x'.

1. Let a be a fixed element of L. Define the binary relation in L, denoted $x \equiv y(a)$, by:

$a \wedge x = a \wedge y$.

Show that this is an equivalence relation compatible with the laws $\wedge$ and $\vee$. We can define the quotient lattice, denoted by L/a.

2. Now, fix $a \in C(L)$. For any element x in L, denote by $\bar{x}$ its class in L/a and $\tilde{x}$ its class in $L/\grave{a}$.

Show that the mapping θ defined by $\theta(x) = (\bar{x}, \tilde{x})$ is an isomorphism from the lattice L to the product lattice $L/a \times L/\grave{a}$.

II - Supplementary on L_3 Algebras

Let $(L, \wedge, \vee, 1, 0, N, \mu)$ be an E_3-algebra (in the sense of Moisil's first definition). As usual, define the other modalities η, γ, υ. Also, define two new operators σ and τ by: $\sigma x = \upsilon x \vee \eta x$, $\tau x = \mu x \wedge \gamma x$. Finally, introduce the following definitions:

An element x of L is said to be possible if $\mu x = 1$.

An element x of L is said to be contingent if $\gamma x = 1$.

An element x of L is said to be a center if it is both possible and contingent.

1. a) Show that for all $x \in L$, $x \vee Nx$ is possible.

b) Show that for all $x \in L$, $x \wedge Nx$ is contingent.

c) Show that if there exists a center, it is unique (in this case, it is called a centered E_{3-} algebra).

d) Show that x is a center if and only if $x = Nx$.

2. Let $a \in C(L)$. Show that the equivalence relation defined in I- 1° is compatible with N and μ. This allows us to define a structure of Ł$_3$-algebra on L/a.

3. Using the classical component-wise definition, show that the product of two L_3-algebras is also an L_3-algebra (a result that can be considered evident). Show that the mapping θ defined in $I - 2^\circ$ is then an ι_3-isomorphism.

4. a) Show that an element x of L is chrysippean if and only if $\tau x = 0$.

b) Show that an element x of L is the center if and only if $\sigma x = 0$.

c) Show that for any element a of L, L is isomorphic (in terms of L_3-algebras) to $L/\sigma a \times L/\tau a$.

5. Let L be an E_3-algebra with a center ω, and all other elements being chrysippean. Show that L is isomorphic to the chain $T = \{0, 1/2, 1\}$.

Hint: If $x \in L$ and $x \neq \omega$, distinguish the two cases: $x \wedge \omega = \omega$ or $x \wedge \omega$ is chrysippean.

III - Representation Theorems for Finite L_3-Algebras

Show that any finite E_3-algebra L is isomorphic to a finite product $U^p \times T^q$ $(p \geq 0, q \geq 0)$, where $U = \{0, 1\}, T = \{0, 1/2, 1\}$.

Hint: If possible, choose an element a in L that is neither chrysippean nor the potential center, and use II- 3°.

Note: It follows that $\text{card}(L) = 2^p \times 3^q$.

IV - Boolean Representation

1. Let B be a Boolean algebra, and construct the Boolean product algebra B^2. If $x \in B$, denote its Boolean complement by x'. Define: $L(B) = \{(x, y) \in B^2 / x \leq y\}$.

a) Show that $L(B)$ is a sublattice of the lattice B^2.

b) Show that $L(B)$ is an L_3-algebra (in the sense of Moisil's first definition) by defining the operators:

$$N(x, y) = (y', x')$$
$$\mu(x, y) = (y, y)$$

2. a) Conversely, let L be an arbitrary E_3-algebra. Show that the mapping

1. ϕ defined by:

if $x \in L \quad \phi(x) = (vx, \mu x)$

is an L_3 - monomorphism from L into $L(C(L))$.

b) Show that ϕ is an isomorphism if and only if L is an L_3-algebra.

Chapter 3

Generalities on fuzzy sets

This chapter reviews the concepts and notations of sets, and then introduces the concepts of fuzzy sets. The concept of fuzzy sets is a generalisation of the crisp sets.

3.1 Crisp sets

Before starting the definition of fuzzy subset, we first take care of the classical set and its properties.

The concept of a set is one of the most fundamental in mathematics. Developed at the end of the 19th century, set theory is now a ubiquitous part of mathematics, and can be used as a foundation from which nearly all of mathematics can be derived.

Etymology: The German word Menge, rendered as ”set” in English, was coined by Bernard Bolzano in his work The Paradoxes of the Infinite.

Definition 1 A set is a well-defined collection of distinct objects. The objects that make up a set (also known as the set's elements or members) can be anything: numbers, people, letters of the alphabet, other sets, and so on. Georg Cantor, one of the founders of set theory.

A set can be written:

In extension: We give the list of its elements. For example, if $a_1, a_2, a_3, \dots, a_n$ are the elements of set A, we write:

$$A = \{a_1, a_2, a_3, \ldots, a_n\}$$

In understanding: We give the property or properties that characterize its elements.

For example, if the elements of the set B satisfying the conditions $P_1, P_2, P_3, \ldots, P_n$ then the set B is defined by:

$$B = \{b/b \text{ satisfied } P_1, P_2, P_3, \ldots, P_n\}$$

In Characteristic Function: A classical subset A of X is defined by a

characteristic function χ_A

$$\chi_A : X \longrightarrow \{0, 1\}$$

$$x \longrightarrow \chi_A(x)$$

Notation

- $A = \{(x, \chi_A(x)), x \in X\}$ is crisp set
- $\mathcal{P}(X) = \{\chi_A / A \subseteq X\}$

Example (finite case)

1- The set F of the twenty smallest integers that are four less than perfect squares can be written:

$$F = \left\{n^2 - 4 : n \text{ is an integer, and } 0 \leq n \leq 19\right\}$$

2- A is the set whose members are the first four positive integers.

Definition 2 (power set) The power set of a set S is the set of all subsets of S, including S itself and the empty set.

Remark

1. The power set of a set S usually written as $\mathcal{P}(S)$.
2. The power set of a finite set with n elements has 2^n elements.

3. The power set of an infinite (either countable or uncountable) set is always uncountable.

Example

1. The power set of the set $\{1, 2, 3\}$ is $\{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \phi\}$.

2. The set $\{1, 2, 3\}$ contains three elements, and the power set shown above contains $2^3 = 8$ elements.

Definition 3 (cardinality) The cardinality $|S|$ of a set S is"the number of members of S." For example, if $B = \{$ blue, white, red $\}, |B| = 3$.

There is a unique set with no members and zero cardinality, which is called the empty set (or the null set).

The concept of the fuzzy subset was introduced by Zadeh [19] as a generalization of the notion of the classical set.

3.2 Basic concepts of fuzzy sets

3.2.1 Membership functions

Definition 4 A fuzzy set A is characterized by a generalized characteristic function $\mu_A : X \longrightarrow [0, 1]$, called the membership function of A and defined over a universe of discourse X.

Remark

$$\mu_A : X \longrightarrow [0, 1]$$

$$x \longrightarrow \mu_A(x)$$

- μ_A is called the membership function of A
- $\mu_A(x)$ is called the membership degree of x in A

Notation

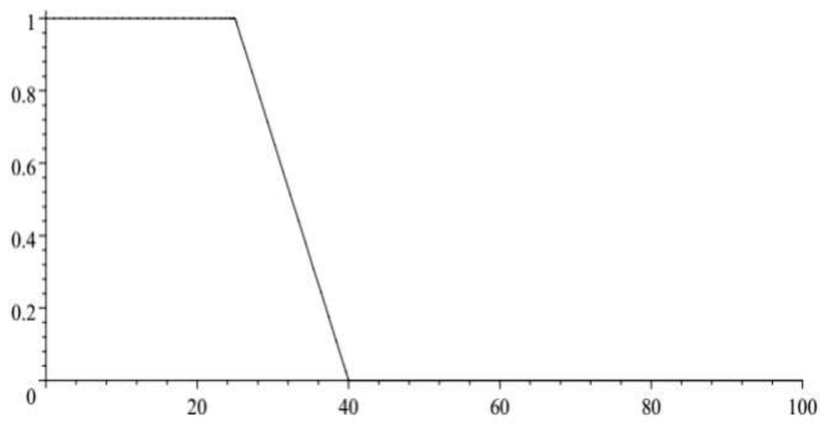

Figure 3.1: A membership function for ”Young”

- $A = \{(x, \mu_A(x)), x \in X\}$ is fuzzy set by convention

$$A = \sum_{x \in X} \frac{\mu_A(x_i)}{x_i} \text{ in the discrete case}$$

$$A = \int \frac{\mu_A(x)}{x} \text{ in the continues case}$$

- $F(X)$ is the set of all fuzzy subsets of X

Example $X = \{$ motorbike, car, train $\}$ means of transport, A: subset of X, the means of fast transport $A = \{($ motorbike, 0.7$), ($ car, 0.5$), ($ train, 1$)\}$

Example Let X the set of all possible ages of people.

$$Y(x) = \begin{cases} 1 & \text{if} \quad x < 25 \\ \frac{40-x}{15} & \text{if} \quad 25 \leq x \leq 40 \\ 0 & \text{if} \quad 40 < x \end{cases}$$

$Y(x)$ is the degree of belonging of x to the set young people

Example Let's define a fuzzy set $A = \{$ real number very near 0$\}$ can be defined and its membership function is

$$\mu_A(x) = \left(\frac{1}{1+x^2}\right)^2$$

It is easy to calculate $\mu_A(1) = 0.25, \mu_A(2) = 0.04, \mu_A(3) = 0.01$