Chapter 3

Generalities on fuzzy sets

This chapter reviews the concepts and notations of sets, and then introduces the concepts of fuzzy sets. The concept of fuzzy sets is a generalisation of the crisp sets.

3.1 Crisp sets

Before starting the definition of fuzzy subset, we first take care of the classical set and its properties.

The concept of a set is one of the most fundamental in mathematics. Developed at the end of the 19th century, set theory is now a ubiquitous part of mathematics, and can be used as a foundation from which nearly all of mathematics can be derived.

Etymology: The German word Menge, rendered as ”set” in English, was coined by Bernard Bolzano in his work The Paradoxes of the Infinite.

Definition 1 A set is a well-defined collection of distinct objects. The objects that make up a set (also known as the set's elements or members) can be anything: numbers, people, letters of the alphabet, other sets, and so on. Georg Cantor, one of the founders of set theory.

A set can be written:

In extension: We give the list of its elements. For example, if $a_1, a_2, a_3, \ldots, a_n$ are the elements of set A, we write:

$$A = \{a_1, a_2, a_3, \ldots, a_n\}$$

In understanding: We give the property or properties that characterize its elements. For example, if the elements of the set B satisfying the conditions $P_1, P_2, P_3, \ldots, P_n$ then the set B is defined by:

$$B = \{b/b \text{ satisfied } P_1, P_2, P_3, \ldots, P_n\}$$

In Characteristic Function: A classical subset A of X is defined by a

characteristic function χ_A

$$\chi_A : X \longrightarrow \{0, 1\}$$

$$x \longrightarrow \chi_A(x)$$

Notation

- $A = \{(x, \chi_A(x)), x \in X\}$ is crisp set
- $\mathcal{P}(X) = \{\chi_A / A \subseteq X\}$

Example (finite case)

1- The set F of the twenty smallest integers that are four less than perfect squares can be written:

$$F = \left\{n^2 - 4 : n \text{ is an integer, and } 0 \leq n \leq 19\right\}$$

2- A is the set whose members are the first four positive integers.

Definition 2 (power set) The power set of a set S is the set of all subsets of S, including S itself and the empty set.

Remark

1. The power set of a set S usually written as $\mathcal{P}(S)$.

2. The power set of a finite set with n elements has 2^n elements.

3. The power set of an infinite (either countable or uncountable) set is always uncountable.

Example

1. The power set of the set $\{1, 2, 3\}$ is $\{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \phi\}$.

2. The set $\{1, 2, 3\}$ contains three elements, and the power set shown above contains $2^3 = 8$ elements.

Definition 3 (cardinality) The cardinality $|S|$ of a set S is"the number of members of S." For example, if $B = \{$ blue, white, red $\}, |B| = 3$.

There is a unique set with no members and zero cardinality, which is called the empty set (or the null set).

The concept of the fuzzy subset was introduced by Zadeh [19] as a generalization of the notion of the classical set.

3.2 Basic concepts of fuzzy sets

3.2.1 Membership functions

Definition 4 A fuzzy set A is characterized by a generalized characteristic function $\mu_A : X \longrightarrow [0, 1]$, called the membership function of A and defined over a universe of discourse X.

Remark

$$\mu_A : X \longrightarrow [0, 1]$$

$$x \longrightarrow \mu_A(x)$$

- μ_A is called the membership function of A
- $\mu_A(x)$ is called the membership degree of x in A

Notation

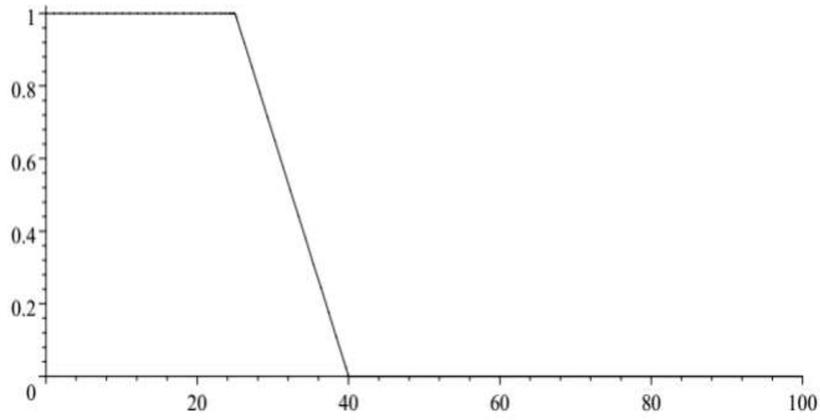

Figure 3.1: A membership function for ”Young”

- $A = \{(x, \mu_A(x)), x \in X\}$ is fuzzy set by convention

$$A = \sum_{x \in X} \frac{\mu_A(x_i)}{x_i} \text{ in the discrete case}$$

$$A = \int \frac{\mu_A(x)}{x} \text{ in the continues case}$$

- $F(X)$ is the set of all fuzzy subsets of X

Example $X = \{$ motorbike, car, train $\}$ means of transport, A: subset of X, the means of fast transport $A = \{($ motorbike, 0.7$), ($ car, 0.5$), ($ train, 1$)\}$

Example Let X the set of all possible ages of people.

$$Y(x) = \begin{cases} 1 & \text{if} \quad x < 25 \\ \frac{40-x}{15} & \text{if} \quad 25 \leq x \leq 40 \\ 0 & \text{if} \quad 40 < x \end{cases}$$

$Y(x)$ is the degree of belonging of x to the set young people

Example Let's define a fuzzy set $A = \{$ real number very near 0$\}$ can be defined and its membership function is

$$\mu_A(x) = \left(\frac{1}{1+x^2}\right)^2$$

It is easy to calculate $\mu_A(1) = 0.25, \mu_A(2) = 0.04, \mu_A(3) = 0.01$

Example Consider a universal set X which is defined on the age domain.

$X = \{5, 15, 25, 35, 45, 55, 65, 75, 85\}$, and $\mu : X \to [0, 1]$ the membership function given by

Age	Infant	Young	Adult	Senior
5	0.00	0.00	0.00	0.00
15	0.00	0.20	0.10	0.00
25	0.00	1.00	0.90	0.00
35	0.00	0.80	1.00	0.00
45	0.00	0.40	1.00	0.10
55	0.00	0.10	1.00	0.20
65	0.00	0.00	1.00	0.60
75	0.00	0.00	1.00	1.00
85	0.00	0.00	1.00	1.00

3.3 Fuzzy sets operations

3.3.1 Standard Operations

Let $F(x)$ denote the collection of all fuzzy sets on a given universe of discourse X.

The basic connectives in fuzzy set theory are inclusion, union, intersection, and complementation. When Zadeh introduced these operations, he based union and intersection connectives on the max and min operations.

- **Inclusion:** Let $A, B \in F(X)$. We say that the set A is included in B if

$$A(x) \leq B(x), \forall x \in X$$

The empty (fuzzy) set $\emptyset$ is defined as $\emptyset(x) = 0, \forall x \in X$, and the total set x is $X(x) = 1, \forall x \in X$.

- **Intersection:** Let $A, B \in F(X)$. The intersection of A and B is the fuzzy set C with

$$C(x) = \min\{A(x), B(x)\} = A(x) \wedge B(x), \forall x \in X$$

We denote $C = A \wedge B$.

- **Union:** Let $A, B \in F(X)$. The union of A and B is the fuzzy set D with

$$D(x) = \max\{A(x), B(x)\} = A(x) \vee B(x), \forall x \in X$$

We denote $D = A \vee B$.

- **Complementation:** Let $A \in F(X)$ be a fuzzy set. The complement of A is the fuzzy set B given by

$$B(x) = 1 - A(x), \forall x \in X.$$

We denote $B = \bar{A}$.

Example If we consider the fuzzy sets

$$A_1(x) = \begin{cases} 1 & \text{if} \quad 40 \le x < 50 \\ 1 - \frac{x-50}{10} & \text{if} \quad 50 \le x < 60 \\ 0 & \text{if} \quad 60 \le x \le 100 \end{cases}$$

$$A_2(x) = \begin{cases} 0 & \text{if} \quad 40 \le x < 50 \\ \frac{x-50}{10} & \text{if} \quad 50 \le x < 60 \\ 1 - \frac{x-60}{10} & \text{if} \quad 60 \le x < 70 \\ 0 & \text{if} \quad 70 \le x \le 100 \end{cases}$$

then their union is

$$(A_1 \vee A_2)(x) = \begin{cases} 1 & \text{if} \quad 40 \le x < 50 \\ 1 - \frac{x-50}{10} & \text{if} \quad 50 \le x < 55 \\ \frac{x-50}{10} & \text{if} \quad 55 \le x \le 60 \\ 1 - \frac{x-60}{10} & \text{if} \quad 60 \le x \le 70 \\ 0 & \text{if} \quad 70 \le x \le 100 \end{cases}$$

The intersection can be expressed as

$$(A_1 \wedge A_2)(x) = \begin{cases} 0 & \text{if} \quad 40 \le x < 50 \\ \frac{x-50}{10} & \text{if} \quad 50 \le x < 55 \\ 1 - \frac{x-50}{10} & \text{if} \quad 55 \le x < 60 \\ 0 & \text{if} \quad 60 < x \le 100 \end{cases}$$

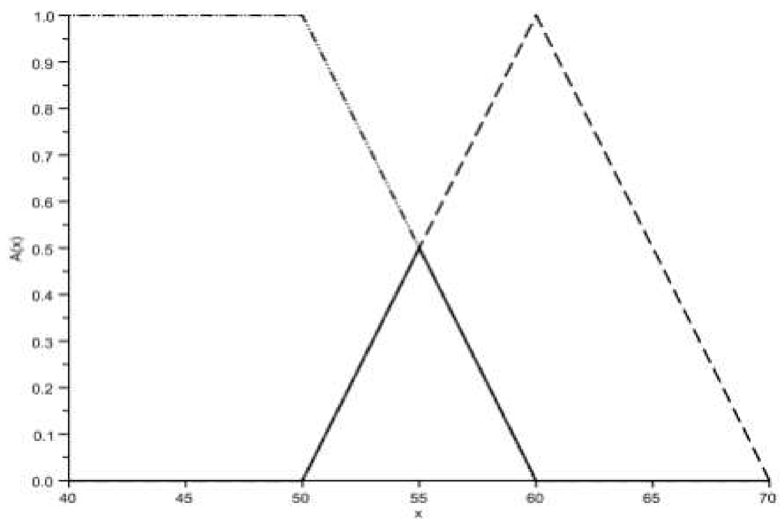

Figure 3.2: Fuzzy Intersection

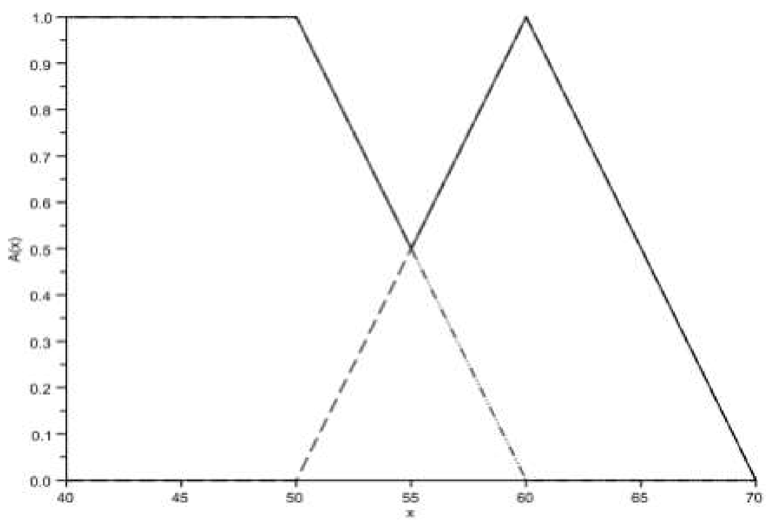

Figure 3.3: Fuzzy Union

The complement of A_1 can be written

$$\bar{A}_1(x) = \begin{cases} 0 & \text{if} \quad 40 \leq x < 50 \\ \frac{x-50}{10} & \text{if} \quad 50 \leq x < 60 \\ 1 & \text{if} \quad 60 \leq x \leq 100 \end{cases}$$

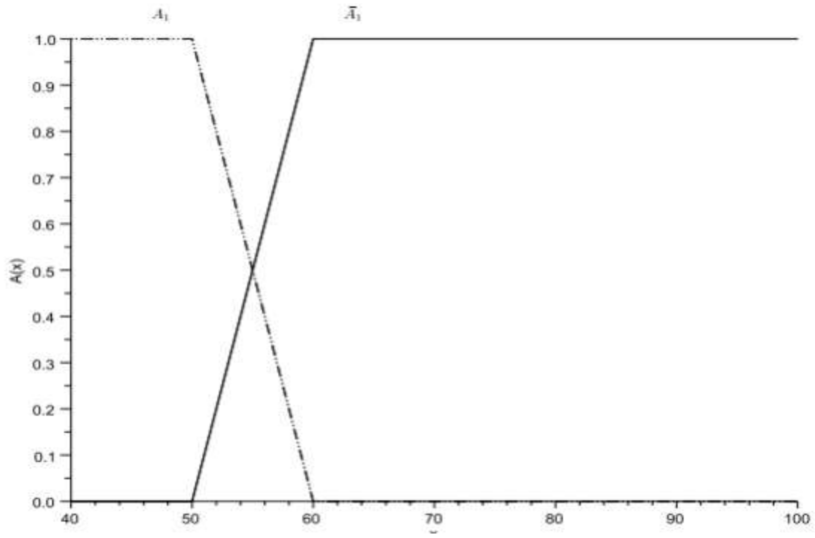

Figure 3.4: The complement of a fuzzy set

3.3.2 Fuzzy complement

Complement set $\bar{A}$ of set A carries the sense of negation. Complement set may be defined by the following function C.

$$C : [0, 1] \longrightarrow [0, 1]$$

Definition 5 The complement function C is designed to map membership function $\mu_A(x)$ of fuzzy set A to $[0, 1]$ and the mapped value is written as $C(\mu_A(x))$. To be a fuzzy complement function, four axioms should be satisfied.

(Axiom C1) $C(0) = 1, C(1) = 0$ (boundary condition)

(Axiom C2) (monotonic nonicreasing), $a, b \in [0, 1]$

$$\text{if } a < b, \text{ then } C(a) \geqslant C(b)$$

(Axiom C3) C is a continuous function.

(Axiom C4) C is involutive.

$$C(C(a)) = a \text{ for all } a \in [0, 1]$$

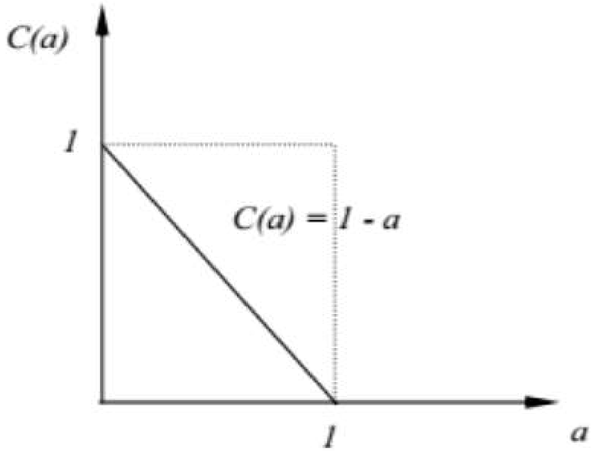

Figure 3.5: Standard complement set function

Remark $C1$ and $C2$ are fundamental requisites to be a complement function. These two axioms are called ”axiomatic skeleton”.

Example of Complement Function

Above four axioms hold in standard complement operator

$$C\left(\mu_A(x)\right) = 1 - \mu_A(x) \quad \text{or} \quad \mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

this standard function is shown in (Figure (1.5))

Proposition 1 The function defined by

$$C_w(a) = (1 - a^w)^{\frac{1}{w}}$$

is a negation, called Yager’s function.

Proof

1. $C_w(0) = 1, C_w(1) = 0.$ (boundary condition)

2. $a, b \in [0, 1]$ if $a < b$, then

$$\begin{aligned} a^w < b^w &\Rightarrow 1 - a^w \geq 1 - b^w \\ &\Rightarrow \left((1 - a^w)^{\frac{1}{w}}\right) \geq \left((1 - b^w)^{\frac{1}{w}}\right) \\ &\Rightarrow C_w(a) \geq C_w(b) \end{aligned}$$

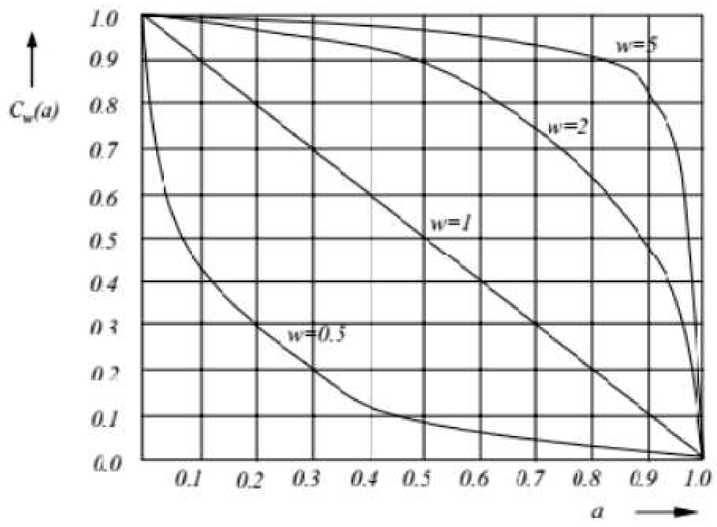

Figure 3.6: Yager complement function

3. C involutive

$$\begin{aligned}
C_w\left(C_w(a)\right) &= C\left(\left(1-a^w\right)^{\frac{1}{w}}\right) \\
&= \left(1-\left[\left(1-a^w\right)^{\frac{1}{w}}\right]^w\right)^{\frac{1}{w}} \\
&= \left(1-\left(1-a^w\right)\right)^{\frac{1}{w}} \\
&= \left(a^w\right)^{\frac{1}{w}} \qquad \text{(monotonic nonicreasing)}
\end{aligned}$$

4. C is a continuous function.

The shape of the function is dependent on the parameter (Figure(1.6))

Remark

(i) When $w = 1$, the Yager's function becomes the standard complement function $c(a) = 1 - a$.

(ii) The fuzzy complement function C is not unique see Figure(1.6)

Proposition 2 (Fundamental properties of fuzzy sets operations) Let $A, B, C \in F(X)$, we have the following propriety:

Involution	$\overline{\overline{A}} = A$
Commutativity	$A \cup B = B \cup A, A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$\{ \begin{array}{l} A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \end{array}$
Absorption	$A \cup (A \cap B) = A, A \cap (A \cup B) = A$
Idempotence	$A \cup A = A, A \cap A = A$
Absorption by X and $\emptyset$	$A \times X = X, A \cap \emptyset = \emptyset$
Identity	$A \cup \emptyset = A$
Law of contradiction	$A \cap \bar{A} = \emptyset$
Law of excluded middle	$A \cup \bar{A} = X$
De Morgan's laws	$\overline{A \cap B} = \bar{A} \cup \bar{B}$ and $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Remark The two principles of classical logic (the non contradiction and the excluded teirs) no longer remains valid in the theory of fuzzy sets i.e. $A \cap \bar{A} \neq \emptyset, A \cup \bar{A} \neq X$.

Example let $X = \{$ smal, medium, large $\}$ with $\mu_A = (x, \mu_A(x)) = \{($ smal, $0.3), ($ medium, $1), ($ large, $0.6)\}$. $\mu_{\bar{A}}(x) = 1 - \mu_A(x) = \{($ smal, $0.7), ($ medium, $0), ($ small, $0.4)\}$. Hence, $\mu_A \cap \mu_{\bar{A}} = \{($ smal, $0.3), ($ medium, $0), ($ large, $0.4)\}$.

then, $A \cap \bar{A} \neq \emptyset$, and $A \cup \bar{A} \neq X$. So, min and max is not checked.

Fuzzy partition

Let A be a crisp set in universal set X and $\bar{A}$ be a complement set of A. The conditions $A \neq \emptyset$ and $A \neq X$ result in couple the $(A, \bar{A})$ which decomposes X into 2 subsets.

Definition 6 (Fuzzy partition) In the same manner, consider a fuzzy set satisfying $A \neq \emptyset$ and $A \neq X$. The pair $(A, \bar{A})$ is defined as fuzzy partition. Usually, if m subsets are defined in X, m-tuple $(A_1, A_2, A_3, \ldots, A_n)$ holding the following conditions is called a fuzzy partition.

(i) $\forall i, \quad A_i \neq \emptyset$

(ii) $A_i \cap A_j = \emptyset$ for $i \neq j$,

(iii) $\forall x \in X, \quad \sum_{i=0}^{m} \mu_{A_i}(x) = 1.$

3.3.3 Characteristics of fuzzy subsets

In this section, we will give definitions for characteristics of fuzzy sets : support, kernel, height and cardinality of a fuzzy subset, and we will give an example and proposition.

Definition 7 (Support of fuzzy subset) Let A be a fuzzy set on a set X. The support of A is the crisp subset on X given by

$$\text{Supp}(A) = \{x \in X / \mu_A(x) > 0\}$$

Definition 8 (Kernel of a fuzzy subset)Let A be a fuzzy set on a set X. The kernel of A is the crisp subset on X given by

$$\text{Ker}(A) = \{x \in X / \mu_A(x) = 1\}$$

Definition 9 (Height of fuzzy subset) Let A be a fuzzy set on a set X. The height of A is the highest value taken by its membership function

given by

$$H(A) = \sup \{\mu_A(x) / x \in X\}$$

Definition 10 A fuzzy subset A is said to be normal whenever $Ht(A) = 1$.

Definition 11 (Cardinality of a fuzzy subset)The cardinality of a finite fuzzy subset A denoted $|A|$ is defined by

$$|A| = \sum_{x \in X} \mu_A(x)$$

Example Let $X = [0,1]$ with $\alpha, \beta \in \mathbb{R}$ and let $a, b \in \mathbb{R}$. We define the fuzzy set A on X by

$$\mu_A(x) = \begin{cases} 0, \text{ if } x < a - \alpha \text{ or } b + \beta < x \\ 1, \text{ if } a < x < b \\ 1 + \left(\frac{x-a}{\alpha}\right), \quad \text{if} \quad a - \alpha < x < a \\ 1 - \left(\frac{b-x}{\beta}\right), \quad \text{if} \quad b < x < b + \beta \end{cases}$$

Then $\mathrm{Ker}(A) = [0,1], \mathrm{Supp}(A) = [a - \alpha, b + \beta]$ and $H(A) = 1$.

Example Let $X = \{1, 2, \ldots, 6\}$, and A be a fuzzy set of X given by:

$$A = \{\langle x, \mu_A(x)\rangle\} = \{\langle 1, 0.2\rangle, \langle 2, 0.0\rangle, \langle 3, 0.8\rangle, \langle 4, 1.0\rangle, \langle 5, 0.5\rangle, \langle 6, 1.0\rangle\}.$$

Then $\mathrm{supp}(A) = \{1, 3, 4, 5, 6\}, \mathrm{Ker}(A) = \{4, 6\}, H(A) = \{1\}, |A| = 3.5$.

Proposition 4 Let A a fuzzy subset of X. The kernel and support of a fuzzy subset verify the following properties:

$$\mathrm{supp}\,(A^c) = (k\,\mathrm{er}(A))^c$$

$$\ker\,(A^c) = (\mathrm{supp}(A))^c$$

3.3.4 Other fuzzy subset operations

Disjunctive sum

The disjunctive sum is the name of operation corresponding ”exclusive OR” logic. And it is expressed as the following((Figure (1.7))

$$A \oplus B = (A \cap \bar{B}) \cup (\bar{A} \cap B)$$

Definition 12 (Simple disjunctive sum)By means of fuzzy union and fuzzy intersection, the definition of the disjunctive sum in a fuzzy set is allowed just like in the crisp set.

$A \oplus B = (A \cap \bar{B}) \cup (\bar{A} \cap B)$, then

$\mu_{A \oplus B}(x) = \mathrm{Max}\left\{\mathrm{Min}\left[\mu_A(x), 1 - \mu_B(x)\right], \mathrm{Min}\left[1 - \mu_A(x), \mu_B(x)\right]\right\}$

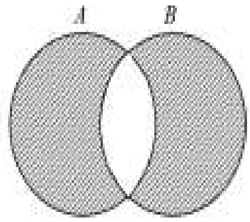

Figure 3.7: Disjunctive sum of two sets

Example Here goes procedures obtaining disjunctive sum of A and B.

$A = \{(x_1, 0.2), (x_2, 0.7), (x_3, 1), (x_4, 0)\}$

$B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1), (x_4, 0.1)\}$

consequence,

$A \oplus B = (A \cap \bar{B}) \cup (\bar{A} \cap B) = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0), (x_4, 0.1)\}$

Definition 13 (Disjoint sum) We can define an operator Δ for the exclusive OR disjoint sum as follows.

$$\mu_{A\Delta B}(x) = |\mu_A(x) - \mu_B(x)|$$

Difference in Fuzzy Set

The difference in crisp set is defined by

$$A - B = A \cap \bar{B}$$

In a fuzzy set, there are two means of obtaining the difference

(1) Simple difference

Example By using standard complement and intersection operations, the difference operation would be simple. If we reconsider the previews example, $A - B$ would be,(Figure(1.9))

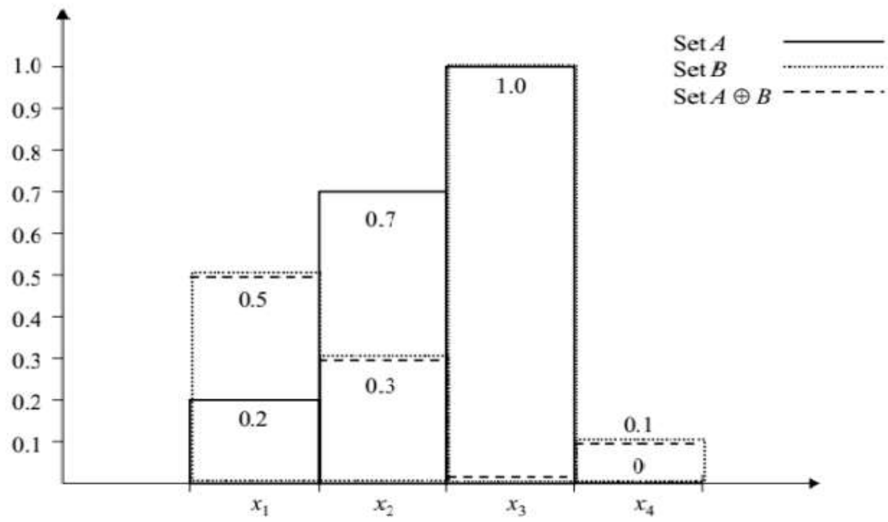

Figure 3.8: Example of simple disjunctive sum

$$A = \{(x_1, 0.2), (x_2, 0.7), (x_3, 1), (x_4, 0)\}$$

$$B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1), (x_4, 0.1)\}$$

$$\bar{B} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0), (x_4, 0.9)\}$$

$$A - B = A \cap \bar{B} = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0), (x_4, 0)\}$$

(2) Bounded difference

Definition 14 (Bounded difference) For novice-operator θ, we define the membership function as,

$$\mu_{A\theta B}(x) = \text{Max}\,[0, \mu_A(x) - \mu_B(x)]$$

Distance in Fuzzy Set

The concept 'distance' is designated to describe the difference. Measures for distance are defined in the following.

(1) Hamming distance

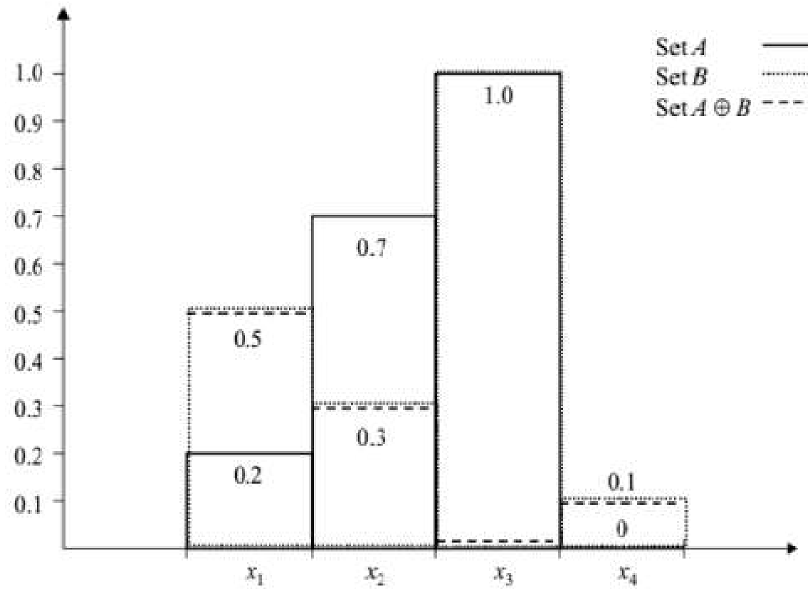

Figure 3.9: Simple difference $A - B$

This concept is marked as,

$$d(A, B) = \sum_{i=0}^{n} |\mu_A(x_i) - \mu_B(x_i)|$$

Example Following A and B for instance,

$$A = \{(x_1, 0.4), (x_2, 0.8), (x_3, 1), (x_4; 0)\}$$

$$B = \{(x_1, 0.4), (x_2, 0.3), (x_3, 0), (x_4; 0)\}$$

$$d(A, B) = |0| + |0.5| + |1| + |0| = 1.5$$

Remark Hamming distance contains the usual mathematical senses of 'distance'

(2) Euclidean distance

$$e(A, B) = \sqrt{\sum_{i=0}^{n} (\mu_A(x_i) - \mu_B(x_i))^2}$$

Example Euclidean distance between sets A and B used for the previous Hamming distance is

$$e(A, B) = \sqrt{0^2 + 0.5^2 + 1^2 + 0^2}$$

(3)Minkowski distance

$$d_w(A,B) = \left(\sum_{x \in X} \left| \mu_A(x) - \mu_B(x) \right) \right|^{w} \right)^{\frac{1}{w}}, \quad w \in [1,\infty]$$

Generalizing Hamming distance and Euclidean distance results in Minkowski distance. It becomes the Hamming distance for $w = 1$ while the Euclidean distance for $w = 2$.

3.3.4.1 Cartesian product and Projection of fuzzy subsets

Definition 15 (Cartesian product) The Cartesian product applied to n fuzzy sets can be defined as follows: Let $\mu_{A_1}(x), \mu_{A_2}(x), \mu_{A_3}(x), \ldots, \mu_{A_n}(x)$ as membership function of $A_1, A_2, A_3, \ldots, A_n$. Then, the membership degree of $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$ on the fuzzy sets $A_1 \times \ldots \times A_n$ is,

$$\mu_{A_1 \times A_2 \times \ldots \times A_n} = \min\left[\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)\right]$$

Example Lets $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2,\}$ and lets A_1, A_2 are two fuzzy subsets respectively defined on X and Y given by : $A_1 = \{\langle x_1, 0.1\rangle; \langle x_2, 0.4\rangle; \langle x_3, 0.75\rangle\}$, and $A_2 = \{\langle y_1, 0.2\rangle; \langle y_2, 0.6\rangle\}$. So, we find:

$\mu_{A_1 \times A_2} = \{\langle (x_1, y_1), 0.1\rangle; \langle (x_1, y_2), 0.1\rangle; \langle (x_2, y_1), 0.2\rangle; \langle (x_2, y_2), 0.4\rangle;$

$\langle (x_3, y_1), 0.2\rangle; \langle (x_3, y_2), 0.6\rangle\}$

Definition 16 (Power of fuzzy sets)The second power of fuzzy set A is defined by:

$$\mu_{A^2}(x) = [\mu_A(x)]^2, \quad \forall x \in X$$

Similarly, m^{th} power of fuzzy set A^m may be computed as,

$$\mu_{A^m}(x) = [\mu_A(x)]^m, \quad \forall x \in X.$$

Let A be a fuzzy subset defined on a universe $X_1 \times X_2$ cartesian product of two reference sets X_1 and X_2.

Definition 17 (Projection of fuzzy subsets) The projection on X_1 of the fuzzy set A of $X_1 \times X_2$ is the fuzzy set $\text{Proj}_{X_1}(A)$ of X_1, whose the membership function is defined by

$$\forall x_1 \in X_1, \mu_{\mathrm{Proj}_{X_1}(A)}(x_1) = \sup_{x_2 \in X_2} \mu_A(x_1, x_2).$$

We define analogously the projection of A on X_2.

3.3.5 Representation of fuzzy subset from classical subsets

3.3.5.1 Alpha-cuts of a Fuzzy sets

One of the most important concepts of fuzzy sets is the concept of an α-cuts and it's variant.

Definition 18 For a given fuzzy set A on a universe X, The α-cuts of A, written A_α is defined as

$$A_\alpha = \{x \in X, \mu_A(x) \geq \alpha\}, \quad \text{for} \quad \alpha \in [0,1]$$

particular cases:

(1) if $\alpha = 0$, then $A_0 = X$

(2) if $\alpha = 1$, then $A_1 = \ker(A)$

Remark if A is a crisp set then $\mathrm{supp}(A) = \ker(A) = A = A_\alpha$

Example let $X = \{1, 2, 3, \ldots, 10\}$, and A be a fuzzy subset of X given by

$A = \{< 1; 0.2 >, < 2; 0.5 >, < 3; 0.8 >, < 4; 1 >, < 5; 0.7 >, < 6; 0.3 >, < 7; 0 >, < 8; 0 >, < 9; 0 >, < 10; 0 >\}$

the α-cuts of A :

$A_0 = X$

$A_{0.2} = \{x \in X, A(x) \geq 0.2\} = \{1, 2, 3, 4, 5, 6\}$

$A_{0.3} = \{x \in X, A(x) \geq 0.3\} = \{2, 3, 4, 5, 6\}$

$A_{0.5} = \{x \in X, A(x) \geq 0.5\} = \{2, 3, 4, 5\}$

$A_{0.7} = \{x \in X, A(x) \geq 0.2\} = \{3, 4, 5\}$

$A_{0.8} = \{x \in X, A(x) \geq 0.2\} = \{3, 4\}$

$A_1 = \{x \in X, A(x) \geq 1\} = \{4\}$

Properties (Basic properties of α-cuts)Let A, B are two a fuzzy subset on a universe X and $\alpha, \beta \in [0, 1]$

(1) if $\alpha \leq \beta$, then $A_\beta \subseteq A_\alpha$

(2) $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$

(3) $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$

Definition 19 (The strong α-cuts)For any α of $[0, 1]$, we define the

strong α-cut of the fuzzy subset A as the subset

$$A^\alpha = \{x \in X, \mu_A(x) > \alpha\}$$

Remark The strong α-cuts have the same properties as the α-cuts.

3.3.5.2 Representation of a fuzzy set by means of its α-cuts

Theorem (Decomposition theorem)Any fuzzy subset A of the reference set X is defined from its α-cuts by:

$$\forall x \in X \mu_A(x) = \sup_{0<\alpha\leq 1} \alpha \cdot \chi_{A_\alpha}(x)$$

χ_{A^α} is the characteristic function of A^α.

Proof. Let $x \in X$ and put $\mu(x) = \alpha, \alpha \in [0, 1]$ we have,

$$\begin{cases} \mu_\alpha(x) = 1 & \text{if} \quad \mu_\alpha(x) \geq \alpha \\ \mu_\alpha(x) = 0 & \text{if} \quad \mu_\alpha(x) < \alpha \end{cases}$$

So, $\alpha\mu_\alpha(x) = \alpha = \mu(x)$;

From where,

$$\sup_{\alpha\in[0,1]} (\alpha\mu_\alpha(x)) \geq \mu(x)$$

On the other hand we have:

$$\text{for all } \alpha \in [0,1], \begin{cases} \mu_\alpha(x) = 1 & \text{if} \quad \mu_\alpha(x) \geq \alpha \\ \mu_\alpha(x) = 0 & \text{if} \quad \mu_\alpha(x) < \alpha \end{cases}$$

we have two cases: $\alpha\mu_\alpha(x) \leq \alpha \quad \forall\alpha \in [0,1]$

Hence,

$$\sup_{\alpha\in[0,1]} (\alpha\mu_\alpha(x)) \leq \mu(x)$$

According to $(*)$ and $(**)$ then $\forall x \in X \quad \mu(x) = \sup_{\alpha\in[0,1]} (\alpha\mu_\alpha(x))$

Example Let X be the set of some countries

$X = \{$ Germany, Belgium, Spain, France, G-Brittany, Italy $\}$. We can take the fuzzy subset associated with the ”southern” property:

$$A = \{< G, 0 >, < B, 0 >, < S, 1 >, < F, 0.8 >, < GB, 0 >, < I, 1 >\}$$

and build it 1-cut $A_1 = \{S, I\}$ identical to its core, as well as it 0.8-cut $A_{0.8} = \{S, F, I\}$, which is identical to all α-cuts, for all $0 < \alpha < 0.8$. It 0 -cut $A_0 = X$ himself.

So we get

$$\begin{aligned}
\mu_A(G) &= \max(1 \times 0, \ldots, 0.1 \times 0, 0 \times 1) = 0, \\
\mu_A(B) &= \max(1 \times 0, \ldots, 0, 1 \times 0, 0 \times 1) = 0, \\
\mu_A(S) &= \max(1 \times 1, \ldots, 0 \times 1) = 1.0, \\
\mu_A(F) &= \max(1 \times 0, 0.9 \times 0, 0.8 \times 1, \ldots, 0 \times 1) = 0.8, \\
\mu_A(GB) &= \max(1 \times 0, \ldots, 0.1 \times 0, 0 \times 1) = 0, \\
\mu_A(I) &= \max(1 \times 1, \ldots, 0 \times 1) = 1.
\end{aligned}$$

Which provides the definition of A.

Chapter 4

Tringular norms and triangular conorms

Triangular norms are essential tools for interpreting conjunctions and disjunctions in fuzzy logic. Subsequently, they play a crucial role in the intersection of fuzzy sets. However, they are also interesting mathematical objects in their own right. Triangular norms, as we use them today, also play a significant role in decision-making.

In this overview, we explore some algebraic, analytical, and logical aspects of triangular norms.

4.1 Tringular norms

4.1.1 Basic definitions and properties

Definition 1 A triangular norm (t-norm for short) is a binary operation T on the unit interval $[0,1]$, i.e., it is a function $T:[0,1]^2 \to [0,1]$ such that for all $x,y,z \in [0,1]$: the following four axioms are satisfied:

(T1) $T(x,y) = T(y,x)$ (commutativity)

(T2) $T(x,T(y,z)) = T(T(x,y),z)$. (associativity)

(T3) $T(x,y) \leq T(x,z)$ whenever $y \leq z$ (monotonicity)

(T4) $T(x,1) = x$. (boundary condition)

Example

The following are the four basic t-norms T_M, T_P, T_L, and T_D given by, respectively:

$T_M(x, y) = \min(x, y)$	(Minimum)
$T_P(x, y) = x \cdot y$	(Product)
$T_L(x, y) = \max(x + y - 1, 0)$	(Lukasiewicz t-norm)
$T_D(x, y) = \{ \begin{matrix} 0 \quad \text{if } (x, y) \in [0, 1[^2 \\ \min(x, y) \text{ otherwise} \end{matrix}$	(Drastic product)

Example

$T(x, y) = \frac{xy}{(2-x-y+xy)}$	Einstein
$T(x, y) = \frac{xy}{(x+y-xy)}$	Hamacher
$T(x, y) = \frac{xy}{\max(x,y,\alpha)}$	Dubois and Parade (1986) $\alpha \in [0, 1]$

Proposition 1 Any t-conorm T satisfies $T(0, x) = T(x, 0) = 0$, for all $x \in [0, 1]$.

Proof. We know that $T(x, 0) \in [0, 1]$, so $T(x, 0) \geq 0$, and we use the axiom (S3)(monotonicity), we obtient $T(x, 0) \leq T(1, 0) = 0$.

Proposition 2 Let A be a set with $]0, 1[\subseteq A \subseteq [0, 1]$, and assume that $F : A^2 \to A$ is a binary operation on A such that for all $x, y, z \in A$ the properties (T1) - (T3) and

$F(x, y) \leq \min(x, y)$

are satisfied. Then the function $T : [0, 1]^2 \to [0, 1]$ defined by

$$T(x, y) = \begin{cases} F(x, y) & \text{if } (x, y) \in (A \backslash \{1\})^2, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

is a t-norm.

Proof The commutativity (T1) and the boundary condition (T4) are satisfied by definition. Concerning the associativity (T2), observe that for $x, y, z \in A \backslash \{0, 1\}$ we have $T(T(x, y), z) = T(x, T(y, z))$ as a $\hat{A}$. consequence of the associativity of F, If $0 \in \{x, y, z\}$ then we get $T(x, T(y, z)) = 0 = T(T(x, y), z)$, and if $1 \in \{x, y, z\}$ then $\mathrm{T(T(x, y), z) = T(x, T(y, z))}$ follows from (T4). Concerning the monotonicity (T3), suppose $y \leq z$. In the cases $x, y, z \in A \backslash \{1\}$ or $x \in \{0, 1\}$ or $y = 0$, the inequality

$T(x,y) \leq T(x,z)$ is inherited from the monotonicity of F and min. The only non-trivial case is when $x, y \in A\backslash\{1\}$ and $z = 1$, in which case $T(x,y) \leq T(x,z)$ follows from $(*)$.

Definition 2 A function $f : [0,1]^2 \to [0,1]$ which satisfies, for all $x, y, z \in [0,1]$, the properties (T1)- (T3) and $f(x,y) \leq \min(x,y)$ is called a t-subnorm.

Example

1- $f(x,y) = 0$.

2- $f(x,y) = \dfrac{x \cdot y}{3}$.

3- $f(x,y) = x \cdot y$.

Remark Clearly, each t-norm is a t-subnorm, but not vice versa: for example, the function $f : [0,1]^2 \to [0,1]$ given by $f(x,y) = 0$, is a t-subnorm but not a t-norm because (T4) not satisfies ($f(x,1) = 0 \neq x$).

Corollary If f is a t-subnorm then the function $T : [0,1]^2 \to [0,1]$ defined by

$$T(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

is a triangular norm.

4.1.1.1 Comparison of t-norms

Definition 3

(i) If, for two t-norms T_1 and T_2, the inequality $T_1(x,y) \leq T_2(x,y)$ holds for all $(x,y) \in [0,1]^2$, then we say that T_1 is weaker than T_2 or, equivalently, that T_2 is stronger than T_1, and we write in this case $T_1 \leq T_2$.

(ii) We shall write $T_1 < T_2$ whenever $T_1 \leq T_2$ and $T_1 \neq T_2$, i.e., if $T_1 \leq T_2$ and for some $(x_0, y_0) \in [0,1]^2$ we have $T_1(x_0, y_0) < T_2(x_0, y_0)$

Lemma

(i) The minimum T_M is the strongest t-norm ($T_M \geq T$).

(ii) The drastic product T_D is the weakest t-norm ($T_D \leq T$).

Proof

(i) For each t-norm T and for each $(x, y) \in [0, 1]^2$ we have both $T(x, y) \leq T(x, 1) = x$ and $T(x, y) \leq T(1, y) = y$, so $T(x, y) \leq \min(x, y) = T_M(x, y)$.

(ii) All t-norms coincide on the boundary of $[0, 1]^2$ and for all $(x, y) \in]\,0, 1\,[^2$ we trivially have $T(x, y) \geq 0 = T_D(x, y)$.

Example

- $T_0(x, y) = \begin{cases} 0 \text{ if } (x, y) \in [0, 1\,[^2, \\ \min(x, y) \text{ otherwise.} \end{cases}$ (Drastic product of weber).
- $T_1(x, y) = \max(x + y - 1, 0)$ (Eukasiewicz).
- $T_{1.5}(x, y) = \frac{xy}{2-x-y+xy}$ (Einstein).
- $T_2(x, y) = xy$ (Algebraic or probaliste).
- $T_{2.5}(x, y) = \frac{xy}{x+y-xy}$ (Hamacher) .
- $T_3(x, y) = \min(x, y)$ (Zadeh).

We have: $T_0 \leq T_1 \leq T_{1.5} \leq T_2 \leq T_{2.5} \leq T_3$.

Definition 4 (Domination of t-norm)Let T_1 and T_2 be two t-norms. Then we say that T_1 dominates T_2 (in symbols $T_1 \gg T_2$) if for all $x, y, u, v \in [0, 1]$

$$T_1\left(T_2(x, y), T_2(u, v)\right) \geq T_2\left(T_1(x, u), T_1(y, v)\right)$$

Lemma

(i) For each t-norm T we have $T_M \gg T$ and $T \gg T_D$.

(ii) If for two t-norms T_1 and T_2 we have T_1 dominates $T_2\,(T_1 \gg T_2)$ then, T_1, is stronger than $T_2\,(T_1 \geq T_2)$.

(iii) The relation $\gg$ on the set of all t-norms is reflexive and antisymmetric.

Proof