Chapter 4

Tringular norms and triangular conorms

Triangular norms are essential tools for interpreting conjunctions and disjunctions in fuzzy logic. Subsequently, they play a crucial role in the intersection of fuzzy sets. However, they are also interesting mathematical objects in their own right. Triangular norms, as we use them today, also play a significant role in decision-making.

In this overview, we explore some algebraic, analytical, and logical aspects of triangular norms.

4.1 Tringular norms

4.1.1 Basic definitions and properties

Definition 1 A triangular norm (t-norm for short) is a binary operation T on the unit interval $[0,1]$, i.e., it is a function $T:[0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$: the following four axioms are satisfied:

(T1) $T(x,y) = T(y,x)$ (commutativity)

(T2) $T(x, T(y,z)) = T(T(x,y),z)$. (associativity)

(T3) $T(x,y) \leq T(x,z)$ whenever $y \leq z$ (monotonicity)

(T4) $T(x,1) = x$. (boundary condition)

Example

The following are the four basic t-norms T_M, T_P, T_L, and T_D given by, respectively:

$T_M(x,y) = \min(x,y)$	(Minimum)
$T_P(x,y) = x \cdot y$	(Product)
$T_L(x,y) = \max(x+y-1,0)$	(Lukasiewicz t-norm)
$T_D(x,y) = \{ \begin{matrix} 0 & \text{if } (x,y) \in [0,1[^2 \\ & \min(x,y) \text{ otherwise} \end{matrix}$	(Drastic product)

Example

$T(x,y) = \frac{xy}{(2-x-y+xy)}$	Einstein
$T(x,y) = \frac{xy}{(x+y-xy)}$	Hamacher
$T(x,y) = \frac{xy}{\max(x,y,\alpha)}$	Dubois and Parade (1986) $\alpha \in [0,1]$

Proposition 1 Any t-conorm T satisfies $T(0,x) = T(x,0) = 0$, for all $x \in [0,1]$.

Proof. We know that $T(x,0) \in [0,1]$, so $T(x,0) \geq 0$, and we use the axiom (S3)(monotonicity), we obtient $T(x,0) \leq T(1,0) = 0$.

Proposition 2 Let A be a set with $]0,1[\subseteq A \subseteq [0,1]$, and assume that $F : A^2 \to A$ is a binary operation on A such that for all $x,y,z \in A$ the properties (T1) - (T3) and

$F(x,y) \leq \min(x,y)$

are satisfied. Then the function $T : [0,1]^2 \to [0,1]$ defined by

$$T(x,y) = \begin{cases} F(x,y) & \text{if } (x,y) \in (A \backslash \{1\})^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

is a t-norm.

Proof The commutativity (T1) and the boundary condition (T4) are satisfied by definition. Concerning the associativity (T2), observe that for $x,y,z \in A\backslash\{0,1\}$ we have $T(T(x,y),z) = T(x,T(y,z))$ as a $\hat{A}$. consequence of the associativity of F, If $0 \in \{x,y,z\}$ then we get $T(x,T(y,z)) = 0 = T(T(x,y),z)$, and if $1 \in \{x,y,z\}$ then $\mathrm{T(T(x,y),z) = T(x,T(y,z))}$ follows from (T4). Concerning the monotonicity (T3), suppose $y \leq z$. In the cases $x,y,z \in A\backslash\{1\}$ or $x \in \{0,1\}$ or $y = 0$, the inequality

$T(x,y) \leq T(x,z)$ is inherited from the monotonicity of F and min. The only non-trivial case is when $x, y \in A\backslash\{1\}$ and $z = 1$, in which case $T(x,y) \leq T(x,z)$ follows from $(*)$.

Definition 2 A function $f : [0,1]^2 \to [0,1]$ which satisfies, for all $x, y, z \in [0,1]$, the properties (T1)- (T3) and $f(x,y) \leq \min(x,y)$ is called a t-subnorm.

Example

$$1\text{- } f(x,y) = 0.$$

$$2\text{- } f(x,y) = \frac{x \cdot y}{3}.$$

$$3\text{- } f(x,y) = x \cdot y.$$

Remark Clearly, each t-norm is a t-subnorm, but not vice versa: for example, the function $f : [0,1]^2 \to [0,1]$ given by $f(x,y) = 0$, is a t-subnorm but not a t-norm because (T4) not satisfies ($f(x,1) = 0 \neq x$).

Corollary If f is a t-subnorm then the function $T : [0,1]^2 \to [0,1]$ defined by

$$T(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

is a triangular norm.

4.1.1.1 Comparison of t-norms

Definition 3

(i) If, for two t-norms T_1 and T_2, the inequality $T_1(x,y) \leq T_2(x,y)$ holds for all $(x,y) \in [0,1]^2$, then we say that T_1 is weaker than T_2 or, equivalently, that T_2 is stronger than T_1, and we write in this case $T_1 \leq T_2$.

(ii) We shall write $T_1 < T_2$ whenever $T_1 \leq T_2$ and $T_1 \neq T_2$, i.e., if $T_1 \leq T_2$ and for some $(x_0, y_0) \in [0,1]^2$ we have $T_1(x_0, y_0) < T_2(x_0, y_0)$

Lemma

(i) The minimum T_M is the strongest t-norm ($T_M \geq T$).

(ii) The drastic product T_D is the weakest t-norm ($T_D \leq T$).

Proof

(i) For each t-norm T and for each $(x, y) \in [0, 1]^2$ we have both $T(x, y) \leq T(x, 1) = x$ and $T(x, y) \leq T(1, y) = y$, so $T(x, y) \leq \min(x, y) = T_M(x, y)$.

(ii) All t-norms coincide on the boundary of $[0, 1]^2$ and for all $(x, y) \in]\,0, 1\,[^2$ we trivially have $T(x, y) \geq 0 = T_D(x, y)$.

Example

- $T_0(x, y) = \begin{cases} 0 \text{ if } (x, y) \in [0, 1\,[^2, \\ \min(x, y) \text{ otherwise.} \end{cases}$ (Drastic product of weber).
- $T_1(x, y) = \max(x + y - 1, 0)$ (Eukasiewicz).
- $T_{1.5}(x, y) = \frac{xy}{2-x-y+xy}$ (Einstein).
- $T_2(x, y) = xy$ (Algebraic or probaliste).
- $T_{2.5}(x, y) = \frac{xy}{x+y-xy}$ (Hamacher) .
- $T_3(x, y) = \min(x, y)$ (Zadeh).

We have: $T_0 \leq T_1 \leq T_{1.5} \leq T_2 \leq T_{2.5} \leq T_3$.

Definition 4 (Domination of t-norm)Let T_1 and T_2 be two t-norms. Then we say that T_1 dominates T_2 (in symbols $T_1 \gg T_2$) if for all $x, y, u, v \in [0, 1]$

$$T_1\left(T_2(x, y), T_2(u, v)\right) \geq T_2\left(T_1(x, u), T_1(y, v)\right)$$

Lemma

(i) For each t-norm T we have $T_M \gg T$ and $T \gg T_D$.

(ii) If for two t-norms T_1 and T_2 we have T_1 dominates $T_2\,(T_1 \gg T_2)$ then, T_1, is stronger than $T_2\,(T_1 \geq T_2)$.

(iii) The relation $\gg$ on the set of all t-norms is reflexive and antisymmetric.

Proof

(i) Trivially, (par separation des cas)

(ii) If for two t-norms T_1 and T_2 we have $T_1 \gg T_2$ then, putting $y = u = 1$ in (Equ 1), we immediately see that also $T_1 \geq T_2$ holds.

(iii) from the commutativity (T_1) and the associativity (T_2) we obtain for each t-norm T and all $x, y, u, v \in [0, 1]$ the equality $T(T(x, y), T(u, v)) = T(T(x, u), T(y, v))$, $(T(T(x, y), T(u, v)) = T(x, T(y, T(u, v)) = T(x, T(T(y, u), v)) = T(x, T(T(u, y), v) = T(x, T(u, T(y, v)) = T(T(x, u), T(y, v)))$. i.e., $T \gg T$, and the assumptions $T_1 \gg T_2$ and $T_2 \gg T_1$ imply, as a consequence of (ii), $T_1 = T_2$

Remark The converse is false: $T_1 \geq T_2$ does not imply $T_1 \gg T_2$.

consider the t-norm T_P and the t norm T given by:

$$T(x, y) = \begin{cases} \frac{xy}{2} & \text{if } (x, y) \in [0, 1[^2, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

we have $T_P \geq T$ but $T_P \gg T$ is false.

let $(x, y) \in [0, 1]^2$ if $(x, y) \in [0, 1[^2$ hence $T_P = xy > \frac{xy}{2} = T(x, y)$

if $\max(x, y) = 1 T_P(x, y) = \min(x, y) = T(x, y)$.

So $\forall (x, y) \in [0, 1]^2$ we have $T_P(x, y) \geq T(x, y)$ i,e,. $T_P \geq T$

but $T_P(T(x, y), T(u, v)) \ngeq T\left(T_P(x, u), T_P(y, v)\right)$,

because if $(x, y) \in [0, 1[^2$ and $(u, v) \in [0, 1[^2$ we get $T_P(Tx, y), T(u, v)) = \frac{xyuv}{4}$ and $T\left(T_P(x, u), T_P(y, v)\right) = \frac{xyuv}{2}$.

Proposition 3

(i) The only t-norm T satisfying $T(x, x) = x$ for all $x \in [0, 1]$ is the minimum T_M.

(ii) The only t-norm T satisfying $T(x, x) = 0$ for all $x \in [0, 1[$ is the drastic product T_D.

Proof

(i) If for a t-norm T we have $T(x, x) = x$ for each $x \in [0, 1]$, then for all $(x, y) \in [0, 1]^2$ with $y \leq x$ the monotonicity (T3) implies $y = T(y, y) \leq T(x, y) \leq T_M(x, y) = y$,

which, together with (T1), means $T = T_M$.

(ii) Assume $T(x, x) = 0$ for each $x \in [0, 1[$. Then for all $(x, y) \in [0, 1[^2$ with $y \leq x$ we have $0 \leq T(x, y) \leq T(x, x) = 0$, hence, together with (T1) and (T4), yielding $T = T_D$.

4.2 Triangular conorms

4.2.1 Basic definitions and properties

Definition 5 A triangular conorm (t-conorm for short) is a binary operation S on the unit interval $[0, 1]$, i.e., it is a function $S : [0, 1]^2 \to [0, 1]$ such that for all $x, y, z \in [0, 1]$: the following four axioms are satisfied:

(S1) $S(x, y) = S(y, x)$ (commutativity)

(S2) $S(x, S(y, z)) = S(S(x, y), z)$ (associativity)

(S3) $S(x, y) \leq S(x, z)$ whenever $y \leq z$ (monotonicity)

(S4) $S(x, 0) = x$ (boundary condition)

Example

The following are the four basic t-norms S_M, S_P, S_L, and S_D given by, respectively:

$S_M(x, y) = \max(x, y)$	$(maximum)$
$S_P(x, y) = x + y - x \cdot y$	$(probabilisticsum)$
$S_L(x, y) = \min(x + y, 1)$	$(Lukasiewiczt - conorm, boundedsum)$
$S_D(x, y) = \begin{cases} 1 & if(x, y) \in]0, 1]^2 \\ \max(x, y) otherwise \end{cases}$	$(drasticsum)$

Example

$T(x,y) = \frac{x+y}{(1+xy)}$	Einstein
$T(x,y) = \frac{x+y-2xy}{(1-xy)}$	Hamacher
$T(x,y) = \frac{x+y+xy-\min(x,y,1-\alpha)}{\max(1-\alpha,1-y,\alpha)}$	Dubois and Parade (1986) $\alpha \in [0,1]$

Proposition 4 Any t-conorm S satisfies $S(1,x) = S(x,1) = 1$, for all $x \in [0,1]$.

Proof We know that $S(x,1) \in [0,1]$, so $S(x,1) \leq 1$, and we use the axiom (S3)(monotonicity), we obtient $S(x,1) \geq S(0,1) = 1$.

Proposition 5 A function $S : [0,1]^2 \to [0,1]$ is a t-conorm if and only if there exists a t-norm T such that for all $(x,y) \in [0,1]^2$

$$S(x,y) = 1 - T(1-x, 1-y)$$

Proof. If T is a t-norm then obviously the operation S defined by (*) satisfies (S1)- (S3) and (S4)

(S_1) $S(x,y) = 1 - T(1-x,1-y) = 1-(1-y,1-x) = S(y,x)$,

(S_2) $S(x,S(y,z)) = 1-T(1-x,1-S(y,z)) = 1-T(1-x,1-(1-T(1-y,1-z))) = 1-T(1-x,T(1-y,1-z))$,

$S(S(x,y),z) == 1-T(1-s(x,y),1-z) = 1-T(1-(1-T(1-x,1-y)),1-z) = 1-T(T(1-x,1-y),1-z) = 1-T(1-x,T(1-y,1-z))$,

(S_3) $S(x,y) = 1-T(1-x,1-y) \leq 1-T(1-x,1-z) = S(x,z)$ whenever $y \leq z$

(S_4) $S(x,0) = 1-T(1-x,1) = 1-(1-x) = x$,

and is, therefore, a t-conorm. On the other hand, if S is a t-conorm, then define the function $T : [0,1]^2 \to [0,1]$ by

$$T(x,y) = 1 - S(1-x, 1-y)$$

Again, it is trivial to T is a t-norm and that $(*)$ holds.

Remark

(i) The t-conorm given by (*) is called the dual t-conorm of T and, analogously, the t-norm given by (**) is said to be the dual t-norm of S.

(ii) The proof of ($Proposition$ 5) makes it clear that also each t-norm is the dual operation of some t-conorm. Note that (T_M, S_M), (T_P, S_P), (T_L, S_L), and (T_D, S_D) are pairs of t-norms and t-conorms which are mutually dual to each other.

Definition 6 Let T be a t-norm and S be a t-conorm. Then we say that T is distributive over S if for all $x, y, z \in [0, 1]$

$$T(x, S(y, z)) = S(T(x, y), T(x, z))$$

and that S is distributive over T if for all $x, y, z \in [0, 1]$

$$S(x, T(y, z)) = T(S(x, y), S(x, z))$$

Remark If T is distributive over S and S is distributive over T, then (T, S) is called a distributive pair (of t-norms and t-conorms).

Proposition 6 Let T be a t-norm and S a t-conorm. Then we have:

(i) S is distributive over T if and only if $T = T_M$.

(ii) T is distributive over S if and only if $S = S_M$.

(iii) (T, S) is a distributive pair if and only if $T = T_M$ and $S = S_M$.

Proof Obviously, each t-conorm is distributive over T_M because of the monotonicity (S3) of the t-conorm.

($\subseteq$) we have

$$S(x, T_M(y, z)) \leq S(x, y) .... (a)$$

$$S(x, T_M(y, z)) \leq S(x, z) .... (b)$$

(a) and (b) given that $S(x, T(y, z)) \leq T_M(S(x, y), S(x, z))$.

($\supseteq$) Conversely, if S is distributive over T then for all $x \in [0, 1]$ we have $x = S(x, T(0, 0)) = T(S(x, 0), S(x, 0)) = T(x, x)$, and from Proposition (..) we obtain $T = T_M$. An analogaus argument proves (ii), and (iii) is just the combination of (i) and (ii).

Remark

(i) The duality changes the order: if, for some t-norms T_1 and T_2 we have $T_1 \leq T_2$, and if S_1 and S_2 are the dual t-conorms of T_1 and T_2, respectively, then we get $S_1 \geq S_2$. Consequently, for each t-conorm S we have

$$S_M \leq S \leq S_D$$

i.e., the maximum S_M is the weakest and the drastic sum S_D is the strongest t-conorm.

(ii) For the t-conorms in example 18 we get this ordering:

$$S_M < S_P < S_L < S_D$$

The continuity of t-conorm S is equivalente to the continuity of the t-norm duale T.

Definition 7

A T-conorm $S : [0,1]^2 \to [0,1]$ is continue if for all the sequences convergentes $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ we have :

$$S\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} S\left(x_n, y_n\right)$$

Example

- the t-conorms S_M, S_P, S_L are continues, and the drastic sum S_D is not continue.

4.2.2 Elementary algebraic properties

Definition 8

(i) An element $a \in [0,1]$ is called an idempotent element of S if $S(a,a) = a$. The numbers 0 and 1 (which are idempotent elements for each t-conorm S) are called trivial idempotent elements of S, each idempotent element in $]0,1[$ will be called a non-trivial idempotent element of S.

(ii) An element $a \in]0,1[$ is called a nilpotent element of S if there exists some $n \in N$ such that $a_S^{(n)} = 0$.

(iii) An element $a \in]0,1[$ is called a zero divisor of S if there exists some $b \in]0,1[$ such that $S(a,b) = 0$.

Chapter 5

Representation of Lukasiewicz trivalent algebras by fuzzy sets.

5.1 Generalities on Fuzzy Sets

Let E be a non-empty set, and $P(E)$ be the set of subsets of E. $P(E)$ equipped with the usual operations of intersection ($\cap$), union ($\cup$), and complement (C) forms a Boolean algebra.

If we denote by U the two-element set $U = \{0,1\}$, we know that there is a correspondence (bijection) between $P(E)$ and U^E (the set of functions from E to $\{0,1\}$) as follows:

To each subset A of E, we associate its characteristic function

$$f_{\mathrm{A}} : P(E) \to U^E \text{ defined by:}$$

$$\mathrm{A} \to f_A$$

$$f_{\mathrm{A}}(x) = \begin{cases} 1 & \text{if } x \in \mathrm{A} \\ 0 & \text{if } x \notin \mathrm{A} \end{cases}$$

To each function $\delta : E \to \mathrm{U}$, we associate the subset $\mathrm{A} = \delta^{-1}(1)$.

Throughout the following, we will agree to identify each subset A with its function f_{A}.

Thus, we can write

$$x \in \mathrm{A} \text{ or } \mathrm{A}(x) = 1$$

$$x \notin \mathrm{A} \text{ or } \mathrm{A}(x) = 0$$

Furthermore, the ordered set U (naturally ordered as $0 < 1$) is a Boolean algebra (as well as a chain) with the operations

$$\alpha \wedge \beta = \min(\alpha, \beta)$$

$$\alpha \vee \beta = \max(\alpha, \beta)$$

$$\rceil \alpha = 1 - \alpha.$$

With the previous identifications, set operations can be translated as follows:

A ∩ B is defined by: $(\mathrm{A} \cap \mathrm{B})(x) = \mathrm{A}(x) \wedge \mathrm{B}(x)$, for all $x \in E$.

A ∪ B is defined by: $(\mathrm{A} \cup \mathrm{B})(x) = \mathrm{A}(x) \vee \mathrm{B}(x)$, for all $x \in E$.

$\overline{\mathrm{A}}$ is defined by: $(\overline{\mathrm{A}})(x) = 1 - \mathrm{A}(x)$, for all $x \in E$.

∅ is defined by: $\emptyset(x) = 0$, for all $x \in E$.

5.1.1 Fuzzy Structure

A fuzzy structure is defined as any pair $(E,\ \mathrm{J})$ where:

- E is any non-empty set (whose elements will be denoted $x, y, z, \ldots$).

- J is a closed chain (that is, with a smallest element 0 and a largest element 1, with $0 \neq 1$). The elements of J will be denoted $\alpha, \beta, \ldots$

5.1.2 Fuzzy Set

In a fuzzy structure, a fuzzy set of E is any function from E to J.

Fuzzy sets will generally be denoted $\widetilde{A}, \widetilde{B}, \ldots$

The set of fuzzy sets of E will be denoted $\tilde{P}(E)$.

5.1.3 Crisp Sets

If $\mathrm{J} = \mathrm{U} = \{0, 1\}$ then $\tilde{P}(E) = P(E)$, which is the set of subsets of E, also known as the crisp sets of E.

5.1.4 Order Relation on Fuzzy Sets

We will now define an order relation on $\tilde{P}(E)$:

$\tilde{A} \subseteq \tilde{B}$ if and only if for all $x \in E : \tilde{A}(x) \leq \tilde{B}(x)$.

This inclusion relation is obviously an order relation.

5.1.5 Union and Intersection of Fuzzy Sets

The union and intersection in $\tilde{P}(E)$ naturally extend the union and intersection in $P(E)$.

The union of two fuzzy sets $\tilde{A}, \tilde{B}$ is defined by:

$(\tilde{A} \cup \tilde{B})(x) = \max(\tilde{A}(x), \tilde{B}(x))$.

The intersection of two fuzzy sets $\tilde{A}, \tilde{B}$ is defined by:

$(\tilde{A} \cap \tilde{B})(x) = \min(\tilde{A}(x), \tilde{B}(x))$.

And we have: $\overline{\widetilde{A}}(x) = 1 - \widetilde{A}(x)$.

- $(\tilde{P}(E), \cup, \cap, C, \emptyset, E)$ is a Boolean algebra.

5.1.6 Fuzziness Levels

Let's define $J^0 = \mathrm{J} - \{0\}$, $J^1 = \mathrm{J} - \{1\}$, $J^{01} = \mathrm{J} - \{0, 1\}$.

Let $\tilde{A}$ be a fuzzy set in a fuzzy structure $(E,\ \mathrm{J})$.

For any $\alpha \in \mathrm{J}$, we define the fuzziness level of degree α as the function $N_\alpha : \tilde{P}(E) \rightarrow P(E)$ defined by:

$$N_\alpha(\tilde{A}) = \{x \in E / \tilde{A}(x) \geq \alpha\}$$

For any $\alpha \in J^1$, we define the strict fuzziness level of degree α as the function $N'_\alpha : \tilde{P}(E) \rightarrow P(E)$ defined by:

$$N'_\alpha(\tilde{A}) = \{x \in E / \tilde{A}(x) > \alpha\}$$

$\left(\tilde{P}(E), \cup, \cap, C, \emptyset, E, N_{1/2}, N_1\right)$ is an L_3 algebra.

Properties 1

- $N_\alpha(\tilde{A} \cup \tilde{B}) = N_\alpha(\tilde{A}) \cup N_\alpha(\tilde{B})$.

  Proof

$$N_\alpha(\tilde{A} \cup \tilde{B}) = \{x \in E/(\tilde{A} \cup \tilde{B})(x) \geq \alpha\}.$$
$$= \{x \in E/ \max(\tilde{A}(x), \tilde{B}(x)) \geq \alpha\}.$$
$$= \{x \in E/\tilde{A}(x) \geq \alpha \text{ or } \tilde{B}(x) \geq \alpha\}.$$
$$= \{x \in E/\tilde{A}(x) \geq \alpha\} \cup \{x \in E/\tilde{B}(x) \geq \alpha\}.$$
$$= N_\alpha(\tilde{A}) \cup N_\alpha(\tilde{B}).$$

- $N_\alpha(\tilde{A} \cap \tilde{B}) = N_\alpha(\tilde{A}) \cap N_\alpha(\tilde{B})$.

  Proof

$$N_\alpha(\tilde{A} \cap \tilde{B}) = \{x \in E/(\tilde{A} \cap \tilde{B})(x) \geq \alpha\}.$$
$$= \{x \in E/ \min(\tilde{A}(x), \tilde{B}(x)) \geq \alpha\}.$$
$$= \{x \in E/\tilde{A}(x) \geq \alpha \text{ and } \tilde{B}(x) \geq \alpha\}.$$
$$= \{x \in E/\tilde{A}(x) \geq \alpha\} \cap \{x \in E/\tilde{B}(x) \geq \alpha\}.$$
$$N_\alpha(\tilde{A} \cap \tilde{B}) = N_\alpha(\tilde{A}) \cap N_\alpha(\tilde{B}).$$

- $N_\alpha(\varnothing) = \varnothing$

  Proof

$$N_\alpha(\emptyset) = \{x \in E/\emptyset(x) \geq \alpha\}$$
$$= \{x \in E/0 \geq \alpha\} \text{ such that } \alpha \in \left\{\frac{1}{2}, 1\right\}$$
$$= \varnothing$$

- $N_\alpha(E) = E$.

  Proof

$$N_\alpha(E) = \{x \in E/E(x) \geq \alpha\}$$
$$N_\alpha(E) = \{\text{x} \in E/1 \geq \alpha\} = E$$

- If $\alpha \leq \beta \Rightarrow N_\beta \leq N_\alpha$, i.e., $N_\beta(\tilde{A}) \subseteq N_\alpha(\tilde{A})$ for all $\tilde{A}$ in $\tilde{P}(E)$.
- If A is a crisp set: $N_\alpha(A) = A$ for all $\alpha \in J^0$.
- For any α, β in $J^0 : N_\alpha N_\beta = N_\beta$.
- If for all $\alpha \in J^0 : N_\alpha(\tilde{A}) = N_\alpha(\tilde{B})$ then $\tilde{A} = \tilde{B}$ (Moisil's determination principle).