Chapter 5

Representation of Lukasiewicz trivalent algebras by fuzzy sets.

5.1 Generalities on Fuzzy Sets

Let E be a non-empty set, and $P(E)$ be the set of subsets of E. $P(E)$ equipped with the usual operations of intersection ($\cap$), union ($\cup$), and complement (C) forms a Boolean algebra.

If we denote by U the two-element set $U = \{0, 1\}$, we know that there is a correspondence (bijection) between $P(E)$ and U^E (the set of functions from E to $\{0, 1\}$) as follows:

To each subset A of E, we associate its characteristic function

$$f_{\mathrm{A}} : P(E) \to U^E \text{ defined by:}$$

$$\mathrm{A} \to f_A$$

$$f_{\mathrm{A}}(x) = \begin{cases} 1 & \text{if } x \in \mathrm{A} \\ 0 & \text{if } x \notin \mathrm{A} \end{cases}$$

To each function $\delta : E \to \mathrm{U}$, we associate the subset $\mathrm{A} = \delta^{-1}(1)$.

Throughout the following, we will agree to identify each subset A with its function f_{A}.

Thus, we can write

$$x \in \mathrm{A} \text{ or } \mathrm{A}(x) = 1$$

$$x \notin \mathrm{A} \text{ or } \mathrm{A}(x) = 0$$

Furthermore, the ordered set U (naturally ordered as $0 < 1$) is a Boolean algebra (as well as a chain) with the operations

$$\alpha \wedge \beta = \min(\alpha, \beta)$$

$$\alpha \vee \beta = \max(\alpha, \beta)$$

$$\rceil\alpha = 1 - \alpha.$$

With the previous identifications, set operations can be translated as follows:

$\mathrm{A} \cap \mathrm{B}$ is defined by: $(\mathrm{A} \cap \mathrm{B})(x) = \mathrm{A}(x) \wedge \mathrm{B}(x)$, for all $x \in E$.

$\mathrm{A} \cup \mathrm{B}$ is defined by: $(\mathrm{A} \cup \mathrm{B})(x) = \mathrm{A}(x) \vee \mathrm{B}(x)$, for all $x \in E$.

$\overline{\mathrm{A}}$ is defined by: $(\overline{\mathrm{A}})(x) = 1 - \mathrm{A}(x)$, for all $x \in E$.

$\varnothing$ is defined by: $\emptyset(x) = 0$, for all $x \in E$.

5.1.1 Fuzzy Structure

A fuzzy structure is defined as any pair $(E,\ \mathrm{J})$ where:

- E is any non-empty set (whose elements will be denoted $x, y, z, \ldots$).
- J is a closed chain (that is, with a smallest element 0 and a largest element 1, with $0 \neq 1$). The elements of J will be denoted $\alpha, \beta, \ldots$

5.1.2 Fuzzy Set

In a fuzzy structure, a fuzzy set of E is any function from E to J.

Fuzzy sets will generally be denoted $\widetilde{A}, \widetilde{B}, \ldots$

The set of fuzzy sets of E will be denoted $\tilde{P}(E)$.

5.1.3 Crisp Sets

If $\mathrm{J} = \mathrm{U} = \{0, 1\}$ then $\tilde{P}(E) = P(E)$, which is the set of subsets of E, also known as the crisp sets of E.

5.1.4 Order Relation on Fuzzy Sets

We will now define an order relation on $\tilde{P}(E)$:

$\tilde{A} \subseteq \tilde{B}$ if and only if for all $x \in E : \tilde{A}(x) \leq \tilde{B}(x)$.

This inclusion relation is obviously an order relation.

5.1.5 Union and Intersection of Fuzzy Sets

The union and intersection in $\tilde{P}(E)$ naturally extend the union and intersection in $P(E)$.

The union of two fuzzy sets $\tilde{A}, \tilde{B}$ is defined by:

$(\tilde{A} \cup \tilde{B})(x) = \max(\tilde{A}(x), \tilde{B}(x))$.

The intersection of two fuzzy sets $\tilde{A}, \tilde{B}$ is defined by:

$(\tilde{A} \cap \tilde{B})(x) = \min(\tilde{A}(x), \tilde{B}(x))$.

And we have: $\overline{\widetilde{A}}(x) = 1 - \widetilde{A}(x)$.

- $(\tilde{P}(E), \cup, \cap, C, \emptyset, E)$ is a Boolean algebra.

5.1.6 Fuzziness Levels

Let's define $J^0 = \mathrm{J} - \{0\}$, $J^1 = \mathrm{J} - \{1\}$, $J^{01} = \mathrm{J} - \{0, 1\}$.

Let $\tilde{A}$ be a fuzzy set in a fuzzy structure $(E,\ \mathrm{J})$.

For any $\alpha \in \mathrm{J}$, we define the fuzziness level of degree α as the function $N_\alpha : \tilde{P}(E) \to P(E)$ defined by:

$$N_\alpha(\tilde{A}) = \{x \in E / \tilde{A}(x) \geq \alpha\}$$

For any $\alpha \in J^1$, we define the strict fuzziness level of degree α as the function $N'_\alpha : \tilde{P}(E) \to P(E)$ defined by:

$$N'_\alpha(\tilde{A}) = \{x \in E / \tilde{A}(x) > \alpha\}$$

$\left(\tilde{P}(E), \cup, \cap, C, \emptyset, E, N_{1/2}, N_1\right)$ is an L_3 algebra.

Properties 1

- $N_\alpha(\tilde{A} \cup \tilde{B}) = N_\alpha(\tilde{A}) \cup N_\alpha(\tilde{B})$.

  Proof

$$\begin{aligned} N_\alpha(\tilde{A} \cup \tilde{B}) &= \{x \in E/(\tilde{A} \cup \tilde{B})(x) \geq \alpha\}. \\ &= \{x \in E/\max(\tilde{A}(x), \tilde{B}(x)) \geq \alpha\}. \\ &= \{x \in E/\tilde{A}(x) \geq \alpha \text{ or } \tilde{B}(x) \geq \alpha\}. \\ &= \{x \in E/\tilde{A}(x) \geq \alpha\} \cup \{x \in E/\tilde{B}(x) \geq \alpha\}. \\ &= N_\alpha(\tilde{A}) \cup N_\alpha(\tilde{B}). \end{aligned}$$

- $N_\alpha(\tilde{A} \cap \tilde{B}) = N_\alpha(\tilde{A}) \cap N_\alpha(\tilde{B})$.

  Proof

$$\begin{aligned} N_\alpha(\tilde{A} \cap \tilde{B}) &= \{x \in E/(\tilde{A} \cap \tilde{B})(x) \geq \alpha\}. \\ &= \{x \in E/\min(\tilde{A}(x), \tilde{B}(x)) \geq \alpha\}. \\ &= \{x \in E/\tilde{A}(x) \geq \alpha \text{ and } \tilde{B}(x) \geq \alpha\}. \\ &= \{x \in E/\tilde{A}(x) \geq \alpha\} \cap \{x \in E/\tilde{B}(x) \geq \alpha\}. \\ N_\alpha(\tilde{A} \cap \tilde{B}) &= N_\alpha(\tilde{A}) \cap N_\alpha(\tilde{B}). \end{aligned}$$

- $N_\alpha(\varnothing) = \varnothing$

  Proof

$$\begin{aligned} N_\alpha(\emptyset) &= \{x \in E/\emptyset(x) \geq \alpha\} \\ &= \{x \in E/0 \geq \alpha\} \text{ such that } \alpha \in \left\{\frac{1}{2}, 1\right\} \\ &= \varnothing \end{aligned}$$

- $N_\alpha(E) = E$.

  Proof

$$N_\alpha(E) = \{x \in E/E(x) \geq \alpha\}$$

$$N_\alpha(E) = \{\text{x} \in E/1 \geq \alpha\} = E$$

- If $\alpha \leq \beta \Rightarrow N_\beta \leq N_\alpha$, i.e., $N_\beta(\tilde{A}) \subseteq N_\alpha(\tilde{A})$ for all $\tilde{A}$ in $\tilde{P}(E)$.

- If A is a crisp set: $N_\alpha(A) = A$ for all $\alpha \in J^0$.

- For any α, β in $J^0 : N_\alpha N_\beta = N_\beta$.

- If for all $\alpha \in J^0 : N_\alpha(\tilde{A}) = N_\alpha(\tilde{B})$ then $\tilde{A} = \tilde{B}$ (Moisil's determination principle).

These properties remain valid for strict fuzziness levels.

Definition 1 For any $x \in L$, we define:

$\varphi_\alpha : L_3 \to C(L)$ (the set of complemented elements of L).

Such that:

1. φ_α is an endomorphism of L, preserves 0 and 1;

2. If $\alpha \leq \beta$ then $\varphi_\beta \leq \varphi_\alpha$;

3. For any α, β in $J^0 : \varphi_\alpha \varphi_\beta = \varphi_\beta$;

4. If $\varphi_\alpha(x) = \varphi_\alpha(y)$ for all α in J^0 then $x = y$ (Moisil's determination principle).

5.2 Representation Theorem for L_3 Algebras

Consider an L_3 algebra.

X: dual space of L (the set of ultrafilters on L). We consider the fuzzy structure $\tilde{P}(X) = (X, J)$

$$f : L \to \widetilde{P}(X)$$

$x \to f(x)$ the function defined by:

$f(x)(U) = \sup \{\alpha \in J^0 / \varphi_\alpha(x) \in U\}$.

Proof

1) $\boldsymbol{f}$ is a morphism of closed lattice

- $f(x \vee y)(U) = \sup \{\alpha \in J^0 / \varphi_\alpha(x \vee y) \in U\}$

  $= \sup \left\{\alpha \in J^0 / \left(\varphi_\alpha(x) \vee \varphi_\alpha(y)\right) \in U\right\}$ because φ_α is an endomorphism.

  $= \sup \left\{\alpha \in J^0 / \varphi_\alpha(x) \in U \text{ or } \varphi_\alpha(y) \in U\right\}$

  $= \sup \left\{\alpha \in J^0 / \varphi_\alpha(x) \in U\right\} \cup \sup \left\{\alpha \in J^0 / \varphi_\alpha(y) \in U\right\}$

  $= f(x)(U) \cup f(y)(U)$.

  Hence $f(x \vee y) = f(x) \cup f(y)$.

- $f(x \wedge y)(U) = \sup\{\alpha \in J^0/\varphi_\alpha(x \wedge y) \in U\}$

  is an endomorphism.

$$= \sup\left\{\alpha \in J^0/\left(\varphi_\alpha(x) \wedge \varphi_\alpha(y)\right) \in U\right\} \text{ because } \varphi_\alpha$$

$$= \sup\left\{\alpha \in J^0/\varphi_\alpha(x) \in U \text{ and } \varphi_\alpha(y) \in U\right\}$$

$$= \sup\left\{\alpha \in J^0/\varphi_\alpha(x) \in U\right\} \cap \sup\left\{\alpha \in J^0/\varphi_\alpha(y) \in U\right\}$$

$$= f(x)(U) \cap f(y)(U).$$

  Hence $f(x \wedge y) = f(x) \cap f(y)$.

- $f(0)(U) = \sup\{\alpha \in J^0/\varphi_\alpha(0) \in U\}$

$$= \sup\{\alpha \in J^0/0 \in U\} \text{ (since } \varphi_\alpha \text{ preserves } 0).$$

$$= \sup \emptyset$$

$$= 0.$$

  Hence $f(0) = \emptyset$.

- $f(1)(U) = \sup\{\alpha \in J^0/\varphi_\alpha(1) \in U\}$

$$= \sup\{\alpha \in J^0/1 \in U\} \text{ (since } \varphi_\alpha \text{ preserves } 1).$$

$$= \sup J^0$$

$$= 1.$$

  Hence $f(1) = \text{X}$.

Thus f is a morphism of closed lattice.

2) $f(\varphi_\beta(x)) = N_\beta(f(x))$.

$$\begin{aligned} f(\varphi_\beta(x)(U) &= \sup\left\{\alpha \in J^0/\varphi_\alpha(\varphi_\beta(x)) \in U\right\} \\ &= \sup\left\{\alpha \in J^0/\varphi_\beta(x) \in U\right\} \\ &= \begin{cases} 0 & \text{if } \varphi_\beta(x) \notin U; \\ 1 & \text{if } \varphi_\beta(x) \in U \end{cases} \\ &= \sigma(\varphi_\beta(x)). \end{aligned}$$

Where σ is the Stone monomorphism relative to X.

$$\begin{aligned} N_\beta(f(x)) &= \{\mathrm{U} \in X / f(x)(U) \geq \beta\} \\ &= \{\mathrm{U} \in X / f(x)(U) \in [\beta, 1]\} \\ &= \left\{\mathrm{U} \in X / \sup \left\{\alpha \in J^0 / \varphi_\alpha(x) \in U\right\} \in [\beta, 1]\right\} \\ &= \left\{\mathrm{U} \in X / \sup \left\{\alpha \in J^0 / \varphi_\alpha(x) \in U\right\} \geq \beta\right\} \\ &= \sigma\left(\varphi_\beta(x)\right). \end{aligned}$$

So $f\left(\varphi_\beta(x)\right) = N_\beta(f(x))$.

3) $\boldsymbol{f}$ is injective

Finally, let's show that f is injective.

f is injective $\Leftrightarrow$ Ker $f = \emptyset$.

Ker $f = \{x \in L / f(x) = \emptyset\}$.

If $x \in$ Ker $f \Leftrightarrow f(x) = \emptyset$.

$$\begin{aligned} &\Leftrightarrow f(x)(U) = 0. \\ &\Leftrightarrow \sup \left\{\alpha \in J^0 / \varphi_\alpha(x) \in U\right\} = 0. \\ &\Leftrightarrow \varphi_\alpha(x) \notin U, \forall U \in X, \forall \alpha \in J^0. \\ &\Leftrightarrow \rceil \varphi_\alpha(x) \in U, \forall U \in X. \\ &\Leftrightarrow \rceil \varphi_\alpha(x) \in \cap U. \\ &\Leftrightarrow \rceil \varphi_\alpha(x) = 1. \\ &\Leftrightarrow \varphi_\alpha(x) = 0. \\ &\Leftrightarrow \varphi_\alpha(x) = \varphi_\alpha(0), \forall \alpha \in J^0. \end{aligned}$$

If $x \in$ Ker $f \Leftrightarrow x = 0$ (Moisil's Determination Principle).

Thus f is injective.

Therefore, f is a Lukasiewicz algebra monomorphism.

Chapter 6

Tutorial Sessions

6.1 Tutorial 1

Exercise 1 Let $\left(L, I^0, (\varphi_\alpha)_{\alpha\in I^0}, (\psi_\alpha)_{\alpha\in I^1}, n, N\right)$ be a Lukasiewicz multivalent algebra with involution L:

Show that the following conditions are equivalent for an element x of L:

(i) $x \in C(L)$;

(ii) $\exists y \in L, \exists i \in I^0$ such that $x = \varphi_i(y)$;

(iii) $\exists i \in I^0$ such that $x = \varphi_i(x)$;

(iv) $\forall i \in I^0, x = \varphi_i(x)$;

(iv) $\forall i, j \in I^0, \varphi_i(x) = \varphi_j(x)$.

Exercise 2 Show that any involutive multivalent Lukasiewicz algebra is a Kleene algebra, i.e.,

$$x \wedge Nx \leq y \vee Ny, (\forall x, y \in L)$$

Exercise 3 Let $\wp(E)$ be the set of fuzzy subsets of a finite set $E = \{x, y\}$ with $J = \left\{0, \frac{1}{2}, 1\right\}$;

1. Provide $\widetilde{\wp(E)}$.

2. Draw the Hasse diagram of $(\widetilde{\wp(E)}, \subset)$.

3. Show that $\left(\widetilde{\wp(E)}, \subset, J^0, (N_\alpha)_{\alpha\in J^0}, (N'_\alpha)_{\alpha\in J^1}, n, C\right)$ is an involutive trivalent Lukasiewicz algebra.

4. Let $\left(L, I^0, (\varphi_\alpha)_{\alpha\in I^0}, (\psi_\alpha)_{\alpha\in I^1}, n, N\right)$ be an involutive multivalent Lukasiewicz algebra. Show that it can be embedded in an algebra of fuzzy subsets.

6.2 Tutorial 2

Exercise

A- We know that an algebra $(L, \rightarrow, N, 1)$ of type $(2, 1, 0)$ is equivalent to a Lukasiewicz trivalent algebra $(L, \wedge, \vee, N, 0, 1, \mu)$ via the transformations

$(\mathbf{LW}) x \rightarrow y = (\mu N x \vee y) \wedge (\mu N y \vee x) = [(\mu N x \wedge y) \vee N x \vee y]$ and

(WL1) $x \vee y = (x \rightarrow y) \rightarrow y$,

(WL2) $x \wedge y = N(Nx \vee Ny)$,

(WL3) $\mu x = Nx \rightarrow x$,

(WL4) $N0 = 1$,

if and only if

(W1) $x \rightarrow (y \rightarrow x) = 1$,

(W2) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,

(W3) $((x \rightarrow Nx) \rightarrow x) \rightarrow x = 1$,

(W4) $(Nx \rightarrow Ny) \rightarrow (y \rightarrow x) = 1$,

(W5) $1 \rightarrow x = 1 \Rightarrow x = 1$,

(W6) $x \rightarrow y = 1$ and $y \rightarrow x = 1 \Rightarrow x = y$.

1. Show that the relation defined on L by $x \leq y$ if and only if $x \rightarrow y = 1$ is a partial order on L.

2. Show that if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$.

3. Show that $(x \to Nx) \to x = x$, and $Nx \leq x \to y$.

4. Show that $NNx = x$, and $Ny \to Nx = x \to y$

5. Show that $x \leq y \Rightarrow Ny \leq Nx$ and $1 \to x = x$

6. Show that $x \leq x \vee y$ and $x \leq y \Leftrightarrow x \vee y = y$.

7. Show that $x \to (y \to z) = y \to (x \to z)$.

B- Let $\left(L, I^0, (\varphi_\alpha)_{\alpha \in I^0}, (\psi_\alpha)_{\alpha \in I^1}, n, N\right)$ be an involutive multivalent Lukasiewicz algebra.

1. Show that any involutive multivalent Lukasiewicz algebra is a Kleene algebra, i.e., $x \wedge Nx \leq y \vee Ny, (\forall x, y \in L)$

2. Now, suppose that $L = \{1, 2, \ldots, p-1\}$.

Show that $N(i) = p - i$, for all $i \in L$.

6.3 Tutorial 3

Exercise 1

Definition: Let G be a group. A fuzzy subset A of the group G is called a fuzzy subgroup of G if:

i. $\mu_A(xy) = \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in G$;

ii. $\mu_A(x^{-1}) = \mu_A(x)$ for all $x \in G$.

Definition: Let G be a group, e denote the identity element of the group G. A fuzzy subset A of the group G is called a fuzzy subgroup of G if:

i. $\mu_A(xy^{-1}) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in G$;

ii. $\mu_A(e) = 1$.

1. Show that a fuzzy subset A of the group G is a fuzzy subgroup of G if and only if: $\mu_A(xy^{-1}) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in G$.

2. Let A be a fuzzy subgroup of the group G and x an element of G then:

$\mu_A(xy) = \mu_A(y)$ for all $y \in G$ if and only if $\mu_A(x) = \mu_A(e)$.

Exercise 2

Let $(G, .)$ be a group, i.e., a set equipped with a binary operation denoted by dot, which is associative, has an identity element denoted by 1, and such that for every x in G, there exists an inverse x' satisfying $x.x' = x'.x = 1$. A subgroup is a subset of G stable under the inverse and the binary operation. Show that if A is a fuzzy subset of G, then the following are equivalent:

$\forall x \in G \mu A(x') \geq \mu A(x)$

$\Leftrightarrow \forall x, y \in G, \mu_A(x.y') \geq \min(\mu_A(x), \mu_A(y))$

$\Leftrightarrow \forall \alpha \in [0,1], A_\alpha$ is a subgroup of G.

Then we say that A is a fuzzy group in G.

6.4 Tutorial 4

Exercise 1

Let $X = [0,1]$ with $\alpha, \beta \in R$ and let $a, b \in R$. Define the fuzzy set A on X as follows:

$$\mu A(x) = \begin{cases} 0, \text{ if } x < a - \alpha \text{ or } b + \beta < x \\ 1, \text{ if } a < x < b \\ 1 + x - \alpha a, \text{ if } a - \alpha < x < a \\ 1 - b - \beta x, \text{ if } b < x < b + \beta \end{cases}$$

Determine $\text{Ker}(A), \text{Supp}(A)$ and $H(A)$.

Exercise 2

1. Determine their union and intersection.

2. Give the complement of A_1

3. Draw the diagrams of the union, intersection, and complement of A_1.

Exercise 3