CLASSICAL SETS AND FUZZY SETS

CLASSICAL SETS

Define a universe of discourse, X, as a collection of objects all having the same characteristics. The individual elements in the universe X will be denoted as x. The features of the elements in X can be discrete, countable integers, or continuous valued quantities on the real line.

For crisp sets A and B consisting of collections of some elements in X, the following notation is defined:

$x \in X$	$\rightarrow$	x belongs to X
$x \in A$	$\rightarrow$	x belongs to A
$x \notin A$	$\rightarrow$	x does not belong to A

For sets A and B on X, we also have

$A \subset B$	$\rightarrow$	A is fully contained in B (if $x \in A$, then $x \in B$)
$A \subseteq B$	$\rightarrow$	A is contained in or is equivalent to B
$(A \leftrightarrow B)$	$\rightarrow$	$A \subseteq B$ and $B \subseteq A$ (A is equivalent to B)

We define the *null set*, Ø, as the set containing no elements, and the whole set, X, as the set of all elements in the universe. The null set is analogous to an impossible event, and the whole set is analogous to a certain event. All possible sets of X constitute a special set called the *power set*, P(X). For a specific universe X, the power set P(X) is enumerated in the following example.

Example 2.1. We have a universe composed of three elements, $X = \{a, b, c\}$, so the cardinal number is $n_x = 3$. The power set is

$$P(X) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

The cardinality of the power set, denoted $n_{P(X)}$, is found as

$$n_{P(X)} = 2^{nX} = 2^3 = 8.$$

Note that if the cardinality of the universe is infinite, then the cardinality of the power set is also infinity, that is, $n_X = \infty \Rightarrow n_{P(X)} = \infty$.

Operations on Classical Sets

Let A and B be two sets on the universe X. The union between the two sets, denoted $A \cup B$, represents all those elements in the universe that reside in (or belong to) the set A, the set B, or both sets A and B. (This operation is also called the *logical or*; another form of the union is the *exclusive or* operation. The *exclusive or* is described in Chapter 5.) The intersection of the two sets, denoted $A \cap B$, represents all those elements in the universe X that simultaneously reside in (or belong to) both sets A and B. The complement of a set A, denoted $\overline{A}$, is defined as the collection of all elements in the universe that do not reside in the set A. The difference of a set A with respect to B, denoted $A \mid B$, is defined as the collection of all elements in the universe that reside in A and that do not reside in B simultaneously. These operations are shown below in set-theoretic terms.

Union $$A \cup B = \{x | x \in A \text{ or } x \in B\}. \quad (2.1)$$

Intersection $$A \cap B = \{x | x \in A \text{ and } x \in B\}. \quad (2.2)$$

Complement $$\overline{A} = \{x | x \notin A, x \in X\}. \quad (2.3)$$

Difference $$A|B = \{x | x \in A \text{ and } x \notin B\}. \quad (2.4)$$

These four operations are shown in terms of Venn diagrams in Figures 2.2–2.5.

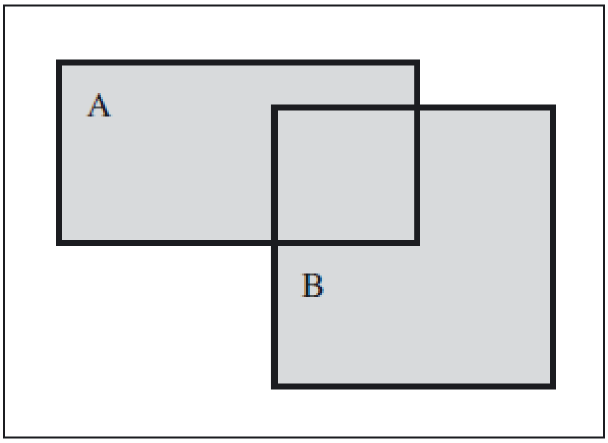

FIGURE 2.2
Union of sets A and B (logical or).

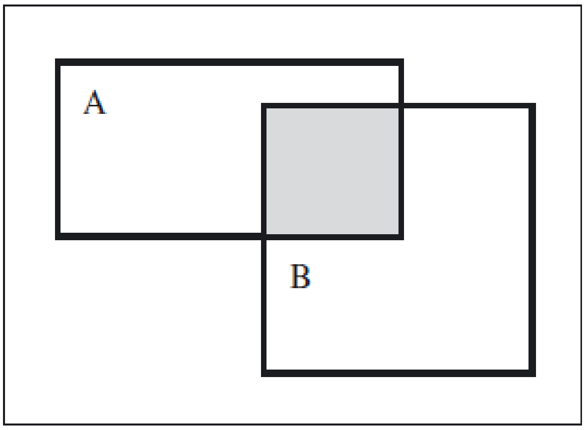

FIGURE 2.3
Intersection of sets A and B.

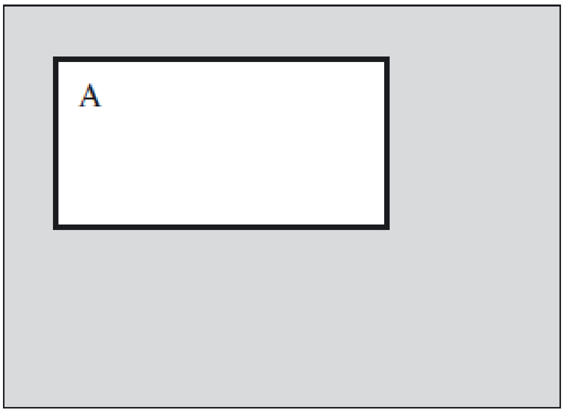

FIGURE 2.4
Complement of set A.

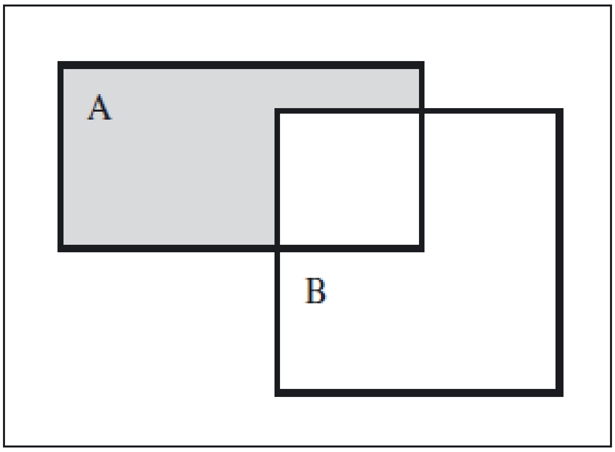

FIGURE 2.5
Difference operation A|B.

Properties of Classical (Crisp) Sets

Certain properties of sets are important because of their influence on the mathematical manipulation of sets. The most appropriate properties for defining classical sets and showing their similarity to fuzzy sets are as follows:

Commutativity
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A.$$

Associativity
$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C.$$

Distributivity
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Idempotency
$$A \cup A = A$$
$$A \cap A = A.$$

Identity
$$A \cup \emptyset = A$$
$$A \cap X = A$$
$$A \cap \emptyset = \emptyset.$$
$$A \cup X = X.$$

Transitivity
$$\text{If } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

Involution
$$\overline{\overline{A}} = A.$$

Axiom of the excluded middle
$$A \cup \overline{A} = X.$$

Axiom of the contradiction
$$A \cap \overline{A} = \emptyset.$$

De Morgan's principles
$$\overline{A \cap B} = \overline{A} \cup \overline{B}.$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

In general, De Morgan's principles can be stated for n sets, as provided here for events, E_i:

$$\overline{E_1 \cup E_2 \cup \cdots \cup E_n} = \overline{E_1} \cap \overline{E_2} \cap \cdots \cap \overline{E_n}.$$
$$\overline{E_1 \cap E_2 \cap \cdots \cap E_n} = \overline{E_1} \cup \overline{E_2} \cup \cdots \cup \overline{E_n}.$$

Mapping of Classical Sets to Functions

As a mapping, the characteristic (indicator) function χ_A is defined as

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases},$$

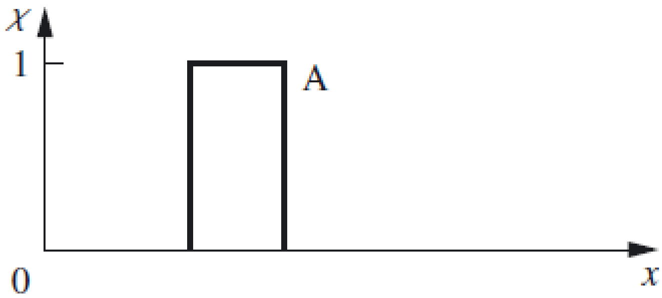

FIGURE 2.10
Membership function is a mapping for crisp set A.

Now, define two sets, A and B, on the universe X. The union of these two sets in terms of function-theoretic terms is given as follows (the symbol $\vee$ is the maximum operator and $\wedge$ is the minimum operator):

Union $\quad A \cup B \longrightarrow \chi_{A\cup B}(x) = \chi_A(x) \vee \chi_B(x) = \max(\chi_A(x), \chi_B(x)).$

The intersection of these two sets in function-theoretic terms is given as follows:

Intersection $\quad A \cap B \longrightarrow \chi_{A\cap B}(x) = \chi_A(x) \wedge \chi_B(x) = \min(\chi_A(x), \chi_B(x)).$

The complement of a single set on universe X, say A, is given as follows:

Complement $\quad \overline{A} \longrightarrow \chi_{\overline{A}}(x) = 1 - \chi_A(x).$

For two sets on the same universe, say A and B, if one set (A) is contained in another set (B), then

Containment $\quad A \subseteq B \longrightarrow \chi_A(x) \le \chi_B(x).$